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More identities on *r*-Pell and *r*-Pell hybrid numbers

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Abstract

Regarding *r*-Pell numbers, we provide some identities. The *r*-Pell numbers with negative subscripts are shown below. The binomial summations for the *r*-Pell numbers are obtained. Several identities pertaining to *r*-Pell hybrid numbers are provided. We provide the negative subscripted *r*-Pell hybrid numbers. We develop various Cassini identities for the conjugate of the *r*-Pell hybrid numbers in accordance with two different determinant definitions of a hybrid square matrix (whose entries are hybrid numbers).

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1. Introduction

The Pell numbers for $n \geq 2$ are defined as follows: $P_n = 2P_{n-1} + P_{n-2}, P_0 = 0, P_1 = 1$. Over time, Pell numbers have been the subject of numerous generalizations [1], [6], [12], [16]. As an illustration, the r-Pell number is defined as

$$P(r,n) = 2^r P(r,n-1) + 2^{r-1} P(r,n-2), \tag{1.1}$$

with initial conditions $P(r,0) = 2,\ P(r,1) = 1 + 2^{r+1}$ and $n \geq 2,\ r \geq 1$ in [1]. Additionally, the writers discovered the following Binet formula for these numbers:

$$P(r,n) = C_1 r_1^n + C_2 r_2^n \tag{1.2}$$

where $n \geq 0,\ r \geq 1,\ r_{1,2} = \frac{1}{2}\left(2^r \mp \sqrt{4^r + 2^{r+1}}\right)$ and $C_{1,2} = 1 \mp \frac{2^r+1}{\sqrt{4^r+2^{r+1}}}$.

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The authors in [2] defined the following matrix

$$Q_n(r) = \begin{pmatrix} P(r,n+1) & P(r,n) \\ P(r,n) & P(r,n-1) \end{pmatrix} \tag{1.3}$$

and obtained the equality $Q_n(r) = Q_1(r)U(r)^{n-1}$, where

$$Q_1(r) = \begin{pmatrix} 2^{r+1} + 2.4^r & 1 + 2^{r+1} \\ 1 + 2^{r+1} & 2 \end{pmatrix} \text{ and } U(r) = \begin{pmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{pmatrix}. \tag{1.4}$$

Two dimensional number systems that are well-known include dual, hyperbolic, and complex numbers. Numerous scholars have focused on the geometric and practical uses of these numbers, particularly in the last century. One way to conceptualize a complex, hyperbolic, dual number is as a generalization of a hybrid number. The set of hybrid numbers in [15], $\mathbb{K}$, is defined as

$$\mathbb{K} = \{x + y\mathbf{i} + w\boldsymbol{\varepsilon} + t\mathbf{h} \colon \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1, x, y, w, t \in \mathbb{R}\}.$$

For any two hybrid numbers $Z_1 = x_1 + y_1\mathbf{i} + w_1\boldsymbol{\varepsilon} + t_1\mathbf{h}$ and $Z_2 = x_2 + y_2\mathbf{i} + w_2\boldsymbol{\varepsilon} + t_2\mathbf{h}$, it write

- $Z_1 = Z_2 \text{only if} x_1 = x_2, y_1 = y_2, w_1 = w_2, t_1 = t_2$ (equality),
- $Z_1 + Z_2 = (x_1 + x_2) + (y_1 + y_2)\mathbf{i} + (w_1 + w_2)\boldsymbol{\varepsilon} + (t_1 + t_2)\mathbf{h}$ (addition),
- $Z_1 - Z_2 = (x_1 - x_2) + (y_1 - y_2)\mathbf{i} + (w_1 - w_2)\boldsymbol{\varepsilon} + (t_1 - t_2)\mathbf{h}$ (substraction),
- $kZ_1 = kx_1 + ky_1\mathbf{i} + kw_1\boldsymbol{\varepsilon} + kt_1\mathbf{h}$ (the multiplication by scalar),
- $\overline{Z_1} = x_1 - y_1\mathbf{i} - w_1\boldsymbol{\varepsilon} - t_1\mathbf{h}$ (the conjugate of a hybrid number).

The addition operation is both commutative and associative in the hybrid numbers. Zero is the null element. Regarding the addition operation, $-Z = -x - y\mathbf{i} - w\boldsymbol{\varepsilon} - t\mathbf{h}$ is the inverse element of Z. This proves that $(\mathbb{K}, +)$ is an Abelian group. Although it is not commutative, hybrid number multiplication possesses the associativity characteristic.

The hybrid number base's product table is found in Table 1.

Table 1

The product table for the basis of $\mathbb{K}$

.	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
1	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	i	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}+1$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-\boldsymbol{\varepsilon}-\mathbf{i}$	$\boldsymbol{\varepsilon}$	1

Additionally, we have $Z\overline{Z} = \overline{Z}Z$ based on the hybridian product. The real number

$$Z\overline{Z} = \overline{Z}Z = x^2 + (y-w)^2 - w^2 - t^2 \tag{1.5}$$

is called the character of the hybrid number $Z = x + y\mathbf{i} + w\boldsymbol{\varepsilon} + t\mathbf{h}$.

Fibonacci, Lucas, Pell and Jacobsthal hybrid numbers have recently attracted a great deal of attention in contemporary mathematics as they can be found in the extensive works of [3], [5], [7], [8], [10], [11], [13], [17], [18], [19], [20], [21], [22], [23]. For example in [10], [11], the generalizations of the Lucas and Fibonacci hybrid numbers were provided by K z lates. The Horadam hybrid numbers were introduced by Szynal-Liana in [17]. The authors also looked at the Jacobsthal and Jacobsthal-Lucas hybrid numbers in [19]. According to [20], the Pell hybrid number is

$$PH_n = P_n + P_{n+1}\mathbf{i} + P_{n+2}\boldsymbol{\varepsilon} + P_{n+3}\mathbf{h}. \tag{1.6}$$

in [20]. In [4], the r-Pell hybrid number is defined as

$$PH_n^r = P(r,n) + P(r,n+1)\mathbf{i} + P(r,n+2)\boldsymbol{\varepsilon} + P(r,n+3)\mathbf{h}, \tag{1.7}$$

where $P(r,n)$ is given by (1.1). And the closed form of these numbers is

$$PH_n^r = C_1\underline{r_1}r_1^n + C_2\underline{r_2}r_2^n, \tag{1.8}$$

where $\underline{r_1} = 1 + r_1\mathbf{i} + r_1^2\boldsymbol{\varepsilon} + r_1^3\mathbf{h}$, $\underline{r_2} = 1 + r_2\mathbf{i} + r_2^2\boldsymbol{\varepsilon} + r_2^3\mathbf{h}$, $C_{1,2}$ and $r_{1,2}$ are given by (1.2).

Based on [9] and [14] research, we use the determinant of a hybrid matrix, which is equivalent to the determinant of a quaternion matrix with quaternions as entries. We define the determinant of a hybrid matrix since the set of all hybrid integers is non-commutative. Let A be a square hybrid matrix. Indicate A by,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $a_{ij} \in \mathbb{K}$ for $i = 1,2$ and $j = 1,2$. By definition, $detA = a_{11}a_{22} - a_{12}a_{21}$ defines the determinant of A. The term "multiplication from above to down below" refers to the definition given above. Another product direction can be established because the set of all hybrids is not commutative. More specifically, the formula $detA = a_{22}a_{11} - a_{21}a_{12}$ is known as the "multiplication from down below to above" in this definition.

We present the novel properties of the r-Pell and the r-Pell hybrid numbers in this work. Additionally, we define these numbers with negative subscripts. Using two definitions of a hybrid square matrix as determinants, we derive alternative Cassini identities for the conjugate of these numbers.

2. The r-Pell numbers

Definition 2.1: The r-Pell number with negative subscripts is defined as

$$P(r,-n-1) = 2^{1-r}P(r,-n+1) - 2P(r,-n), \tag{2.1}$$

where $n \geq 0$, $r \geq 1$, $P(r,0) = 2$ and $P(r,1) = 1 + 2^{r+1}$.

The relation between r-Pell number with negative subscripts and classic Pell number with negative subscripts is $P(1,-n) = P_{-n+2}$. The first five terms of the r-Pell number with negative subscripts are as follows:

$P(r,-1) = 2 \cdot 2^{-r}$,
$P(r,-2) = 0$,
$P(r,-3) = 4 \cdot 4^{-r}$,

$P(r,-4) = -8 \cdot 4^{-r}$,
$P(r,-5) = 8 \cdot 8^{-r} + 16 \cdot 4^{-r}$.

Theorem 2.1: *From $n \geq 4$, we have*

$$P(r,-n) = (-1)^{n-1}2^{(n-2)(1-r)}P(r,n-4). \tag{2.2}$$

Proof: Let us use the principle of mathematical induction on n. For $n = 4$, it is easy to see that

$$\begin{aligned} P(r,-4) &= -2^{2(1-r)}P(r,0) \\ &= -8 \cdot 4^{-r}. \end{aligned}$$

Let us suppose that it is true for all positive integers $k \leq n$, as is customary in inductions. Stated differently, $P(r,-k) = (-1)^{k-1}2^{(k-2)(1-r)}P(r,k-4)$.

Finally, we need to show that it is true for $k+1$. Stated differently, using the notion of r-Pell numbers, we have

$$\begin{aligned} (-1)^k 2^{(k-1)(1-r)}P(r,k-3) = {} & (-1)^k 2^{(k-1)(1-r)}[2^r P(r,k-4) \\ & +2^{r-1}P(r,k-5)]. \end{aligned}$$

Based on its assumption and equation (2.1), we write

$$\begin{aligned} (-1)^k 2^{(k-1)(1-r)}P(r,k-3) &= -2P(r,-k) + 2^{1-r}P(r,-k+1) \\ &= P(r,-k-1). \end{aligned}$$

Thus, the evidence is finished.

In the following theorem, we give the Binet formula of the r-Pell numbers with negative indices.

Theorem 2.2: *For every $n \geq 0$ integers, we can write the Binet formula for the negative indices r-Pell numbers as the form*

$$P(r,-n) = C_1 r_1^{-n} + C_2 r_2^{-n},$$

such that r_1, r_2 and C_1, C_2 are defined in equation (1.2).

Proof: The equation (2.1) will be examined. The characteristic equation of (2.1) is assumed to have r_1 and r_2 as its roots. Therefore, the general answer to it is provided by

$$P(r,-n) = C_1 r_1^{-n} + C_2 r_2^{-n}$$

It is evident that C_1 and C_2 are as in equation (1.2) by using the beginning conditions in Definition 2.1 and using basic algebraic operations.

We provide the generating function and summation formula for the r-Pell numbers with negative indices in the subsequent theorem.

Theorem 2.3: *For every $n \geq 0$ integers, we have*

i) $\sum_{k=0}^{n-1} P(r,-k) = \frac{P(r,-n+1)+P(r,-n)-3\cdot 2^r-1}{1-2^r-2^{r-1}}$,

ii) $\sum_{n=0}^{\infty} P(r,-n)t^n = \frac{2-2^{1-r}t}{1-2^{1-r}t+2t^2}$.

Proof:

i) By considering Theorem 2.2, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} P(r,-k) &= \sum_{k=0}^{n-1} C_1 r_1^{-k} + C_2 r_2^{-k} \\ &= C_1 \sum_{k=0}^{n-1} r_1^{-k} + C_2 \sum_{k=0}^{n-1} r_2^{-k} \\ &= C_1 r_1^{1-n} \frac{1-r_1^n}{1-r_1} + C_2 r_2^{1-n} \frac{1-r_2^n}{1-r_2}.\end{aligned}$$

If necessary operations are done, we have

$$\sum_{k=0}^{n-1} P(r,-k) = \frac{P(r,-n+1)+P(r,-n)-3\cdot 2^r-1}{1-2^r-2^{r-1}}.$$

ii) Let $\sum_{n=0}^{\infty} P(r,-n)t^n = f(t)$ be the generating function for r-Pell numbers with negative indices. Then, we have

$$\begin{aligned}f(t)(1-2^{1-r}t+2t^2) &= \sum_{n=0}^{\infty} P(r,-n)t^n - 2^{1-r}\sum_{n=0}^{\infty} P(r,-n)t^{n+1} \\ &\quad +2\sum_{n=0}^{\infty} P(r,-n)t^{n+2} \\ &= 2-2^{1-r}t.\end{aligned}$$

The proof is completed.

In the following theorem, we derive the binomial summations of the r-Pell numbers.

Theorem 2.4: *The identities are hold:*

i) $2^{(r-1)n}\sum_{k=0}^{n} \binom{n}{k} 2^k P(r,k) = P(r,2n)$,

ii) $2^{(r-1)n}\sum_{k=0}^{n} \binom{n}{k} 2^k P(r,k+1) = P(r,2n+1)$,

iii) $\sum_{k=0}^{n} \binom{n}{k} (-2^{r-1})^{ik} P(r,(n-2k)i) = 2(r_1^i + r_2^i)^n$,

iv) $\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^{n-rk} P(r,k) = P(r,-n)$,

v) $P(r,n) = \frac{1}{2^{n-1}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n}{2k} 2^{r(n-2k)}(4^r+2^{r+1})^k +$
$\frac{1}{2^{n-1}}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n}{2k+1} 2^{r(n-2k)}(4^r+2^{r+1})^k(1+2^{-r})$.

Proof:

i) From the equation (1.2), we write

$$\begin{aligned}2^{(r-1)n}\sum_{k=0}^{n} \binom{n}{k} 2^k P(r,k) &= 2^{(r-1)n}\sum_{k=0}^{n} \binom{n}{k} 2^k (C_1 r_1^k + C_2 r_2^k) \\ &= 2^{(r-1)n}\left[C_1 \sum_{k=0}^{n} \binom{n}{k} (2r_1)^k + C_2 \sum_{k=0}^{n} \binom{n}{k} (2r_2)^k\right] \\ &= 2^{(r-1)n}[C_1(2r_1+1)^n + C_2(2r_2+1)^n].\end{aligned}$$

By considering the equalities $2r_1+1 = \frac{r_1^2}{2^{r-1}}$, $2r_2+1 = \frac{r_2^2}{2^{r-1}}$ and again the equation (1.2), we obtain claimed result.

iii) By the equations (1.2) and $r_1r_2=-2^{r-1}$, we have

$$\sum_{k=0}^{n}\binom{n}{k}(-2^{r-1})^{ik}P(r,(n-2k)i)=\sum_{k=0}^{n}\binom{n}{k}(-2^{r-1})^{ik}(C_1r_1^{(n-2k)i}+C_2r_2^{(n-2k)i})$$

$$=C_1\sum_{k=0}^{n}\binom{n}{k}r_1^{ni-ki}r_2^{ki}+C_2\sum_{k=0}^{n}\binom{n}{k}r_1^{ki}r_2^{ni-ki}$$

$$=C_1(r_1^i+r_2^i)^n+C_2(r_1^i+r_2^i)^n.$$

By taking account the equality $C_{1,2}=1\pm\frac{2^r+1}{\sqrt{4^r+2^{r+1}}}$, we obtain claimed result.

The remaining partions of the theorem can also be proven in this manner.

3. The r-Pell hybrid numbers

First, the negative subscripted r-Pell hybrid numbers are provided by the following definition.

Definition 3.1: The negative subscripted r-Pell hybrid numbers are defined as

$$PH_{-n}^{r}=P(r,-n)+P(r,-n+1)\mathbf{i}+P(r,-n+2)\boldsymbol{\varepsilon}+P(r,-n+3)\mathbf{h}. \quad (3.1)$$

The following are the first five terms, with negative subscripts, of the r-Pell hybrid number:

$PH_{-1}^{r}=2\cdot 2^{-r}+2\mathbf{i}+(1+2^{r+1})\boldsymbol{\varepsilon}+(2^{r+1}+2\cdot 4^{r})\mathbf{h}$,
$PH_{-2}^{r}=2\cdot 2^{-r}\mathbf{i}+2\boldsymbol{\varepsilon}+(1+2^{r+1})\mathbf{h}$,
$PH_{-3}^{r}=4\cdot 4^{-r}+2\cdot 2^{-r}\boldsymbol{\varepsilon}+2\mathbf{h}$,
$PH_{-4}^{r}=-8\cdot 4^{-r}+4\cdot 4^{-r}\mathbf{i}+2\cdot 2^{-r}\mathbf{h}$,
$PH_{-5}^{r}=8\cdot 8^{-r}(1+2^{r+1})-8\cdot 4^{-r}\mathbf{i}++4\cdot 4^{-r}\boldsymbol{\varepsilon}$,

From the equalities $P(r,-n)=(-1)^{n-1}2^{(n-2)(1-r)}P(r,n-4)$, we have

$$PH_{-n}^{r}=\begin{cases}2^{(2-n)(1-r)}X, n \text{ is even}\\ -2^{(2-n)(1-r)}X, n \text{ is odd}\end{cases} \quad (3.2)$$

where $X=-P(r,n-4)+2^{r-1}P(r,n-5)\mathbf{i}-2^{2r-2}P(r,n-6)\boldsymbol{\varepsilon}+2^{3r-3}P(r,n-7)\mathbf{h}$.

In the following theorem, we derive the binomial summations of the r-Pell hybrid number.

Theorem 3.1: *The identities are hold:*

i) $\sum_{k=0}^{n}\binom{n}{k}2^{rn-n+k}PH_k^r=PH_{2n}^r$,

ii) $\sum_{k=0}^{n}\binom{n}{k}2^{rn-n+k}PH_{k+1}^r=PH_{2n+1}^r$,

iii) $\sum_{k=0}^{n}\binom{n}{k}2^{n-rk}(-1)^{n-k}PH_k^r=PH_{-n}^r$,

iv) $PH_n^r=\frac{PH_0^r}{2^{n(1-r)}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(1+2^{1-r})^k+\frac{C_1\underline{r_1}-C_2r_2}{2^{n(1-r)}}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}(1+2^{1-r})^{k+1/2}$.

Proof:

i) From the equation (1.8), we write

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}2^{rn-n+k}PH_k^r &= 2^{n(r-1)}\sum_{k=0}^{n}\binom{n}{k}\left(C_1\underline{r_1}r_1^k+C_2\underline{r_2}r_2^k\right)2^k\\ &= 2^{n(r-1)}\left[C_1\underline{r_1}\sum_{k=0}^{n}\binom{n}{k}(2r_1)^k+C_2\underline{r_2}\sum_{k=0}^{n}\binom{n}{k}(2r_2)^k\right]\\ &= 2^{n(r-1)}\left[C_1\underline{r_1}(1+2r_1)^n+C_2\underline{r_2}(1+2r_2)^n\right].\end{aligned}$$

The equalities $1+2r_1=\frac{r_1^2}{2^{r-1}}$ and $1+2r_2=\frac{r_2^2}{2^{r-1}}$ are taken into consideration. And the equation (1.8) once more, we get the intended result.

iii) Taking into account the equation (1.8), we obtain

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}2^{n-rk}(-1)^{n-k}PH_k^r &= (-2)^n\sum_{k=0}^{n}\binom{n}{k}2^{-rk}(-1)^{-k}\\ &\qquad\left(C_1\underline{r_1}r_1^k+C_2\underline{r_2}r_2^k\right)\\ &= (-2)^n\left[C_1\underline{r_1}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{r_1}{-2^r}\right)^k+C_2\underline{r_2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{r_2}{-2^r}\right)^k\right]\\ &= (-2)^n\left[C_1\underline{r_1}\left(1-\frac{r_1}{2^r}\right)^n+C_2\underline{r_2}\left(1-\frac{r_1}{2^r}\right)^n\right].\end{aligned}$$

By taking account the equalities $r_1+r_2=2^r$, $r_1r_2=-2^{r-1}$ and Definition 3.1, we obtain

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}2^{n-rk}(-1)^{n-k}PH_k^r &= C_1\underline{r_1}r_1^{-n}+C_2\underline{r_2}r_2^{-n}\\ &= PH_{-n}.\end{aligned}$$

The theorem's other components can also be proven in this manner.

The $QH_n(r)$ matrix can be defined in the same way as the $Q_n(r)$ matrix. The matrix is

$$QH_n(r)=\begin{pmatrix}PH_{n+1}^r & PH_n^r\\ PH_n^r & PH_{n-1}^r\end{pmatrix}. \tag{3.3}$$

We obtain the equality

$$QH_n(r)=QH_1(r)U(r)^{n-1}, \tag{3.4}$$

where

$$QH_1(r)=\begin{pmatrix}PH_2^r & PH_1^r\\ PH_1^r & PH_0^r\end{pmatrix} \tag{3.5}$$

and $U(r)$ is defined as in (1.4). The proof of (3.4) is easily seen by the principle of mathematical induction on n. Now, we give the two types of Cassini equalities for the r-Pell hybrid numbers. One of these equations was previously given in [4], but here we easily obtain it from the determinant(for hybrid numbers) of the $QH_n(r)$ matrix.

The identities are satisfied:

i) $PH_{n+1}^rPH_{n-1}^r-(PH_n^r)^2=[PH_2^rPH_0^r-(PH_1^r)^2](-2^{r-1})^{n-1}$,

ii) $PH_{n-1}^rPH_{n+1}^r-(PH_n^r)^2=[PH_0^rPH_2^r-(PH_1^r)^2](-2^{r-1})^{n-1}$,

where $n>0$ integer.

We now investigate Cassini identities for the r-Pell hybrid numbers conjugates. Because of this, we give the matrix's conjugate in the equation (3.5)

$$\overline{QH_1(r)} = \begin{pmatrix} \overline{PH_2^r} & \overline{PH_1^r} \\ \overline{PH_1^r} & \overline{PH_0^r} \end{pmatrix}. \tag{3.6}$$

Theorem 3.2: *The identities are satisfied:*

i) $\overline{PH}_{n+1}^{r}\overline{PH}_{n-1}^{r} - \left(\overline{PH}_{n}^{r}\right)^2 = \left[\overline{PH}_{2}^{r}\overline{PH}_{0}^{r} - \left(\overline{PH}_{1}^{r}\right)^2\right](-2^{r-1})^{n-1}$,

ii) $\overline{PH}_{n-1}^{r}\overline{PH}_{n+1}^{r} - \left(\overline{PH}_{n}^{r}\right)^2 = \left[\overline{PH}_{0}^{r}\overline{PH}_{2}^{r} - \left(\overline{PH}_{1}^{r}\right)^2\right](-2^{r-1})^{n-1}$,

where $n > 0$ integer.

Proof: Using the matrices in (1.4) and (3.6) and the equation (1.7), we obtain

$$\overline{QH_n(r)} = \overline{QH_1(r)}U(r)^{n-1}. \tag{3.7}$$

i) After obtaining the determinant under each of the matrices in (3.6) and (3.7), we obtain the identity by using the "multiplication from above to down below" techniques that were discussed in the introductory section. In other words,

$$\overline{PH}_{n+1}^{r}\overline{PH}_{n-1}^{r} - \left(\overline{PH}_{n}^{r}\right)^2 = \left[\overline{PH}_{2}^{r}\overline{PH}_{0}^{r} - \left(\overline{PH}_{1}^{r}\right)^2\right](-2^{r-1})^{n-1}.$$

ii) After obtaining the determinant under of the matrices in (3.6) and (3.7) and applying the "multiplication from down below to above" criteria that were discussed in the introductory section, we are able to achieve the identity.

The relationships between the r-Pell hybrid numbers and their conjugates are provided by the following theorem.

Theorem 3.3: *We have for* $m, n \geq 0$ *integers*

i) $\overline{PH}_{m+n}^{r} = 2^{r-1}\left[\overline{PH}_{m+1}^{r}PH_{n-2}^{r} + 2^{r-1}\overline{PH}_{m}^{r}PH_{n-3}^{r}\right]$,

ii) $\overline{PH}_{m+n}^{r} = 2^{r-1}\left[PH_{m+1}^{r}\overline{PH}_{n-2}^{r} + 2^{r-1}PH_{m}^{r}\overline{PH}_{n-3}^{r}\right]$.

Proof:

i) From (3.7) equation, we can write

$$\begin{aligned} \overline{QH_{m+n-1}(r)} &= \overline{QH_1(r)}U(r)^{m+n-2} \\ &= \overline{QH_m(r)}QH_1(r)^{-1}QH_n(r) \\ &= \overline{QH_m(r)}2^{r-1}\begin{pmatrix} PH_{n-2}^r & PH_{n-3}^r \\ 2^{r-1}PH_{n-3}^r & 2^{r-1}PH_{n-4}^r \end{pmatrix}. \end{aligned} \tag{3.8}$$

Equating the pivot elements of the matrices (3.8) equation yields the desired identity.

ii) Similar to (3.7) equation, we can write

$$\overline{QH_{m+n-1}(r)} = U(r)^{m+n-2}\overline{QH_1(r)}. \tag{3.9}$$

Equating the pivot elements of the matrices (3.9) equation yields the desired identity.

The following outcomes are obtained by taking $m = n$ from Theorem 3.3.

Corollary 3.1: *For* $n \geq 0$ *integers, we have*

i) $\overline{PH}_{2n}^{r} = 2^{r-1}\left[\overline{PH}_{n+1}^{r}PH_{n-2}^{r} + 2^{r-1}\overline{PH}_{n}^{r}PH_{n-3}^{r}\right]$,

ii) $\overline{PH}_{2n}^{r} = 2^{r-1}\left[PH_{n+1}^{r}\overline{PH}_{n-2}^{r} + 2^{r-1}PH_{n}^{r}\overline{PH}_{n-3}^{r}\right]$.

4. Conclusion

The r-Pell and r-Pell hybrid numbers have been studied in this study. Simple matrix algebra and elementary algebraic operations prove several of these numbers' properties. In fact, the findings reported here may spur additional research on the topic of r-Pell-Lucas hybrid numbers.

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