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# Intuitionistic fuzzy equivalence relations and ideals on rings under norms

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**Abstract**

In this paper, we investigate norms defined on intuitionistic fuzzy equivalence relations, intuitionistic fuzzy congruence relations, and intuitionistic fuzzy ideals of rings. We explore their properties and structural characteristics, as well as their behavior under ring homomorphisms.

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## 1. Introduction

The concept of a fuzzy relation was first introduced by Zadeh [1]. Later, Goguen [2] and Sanchez [3] examined fuzzy relations in various contexts. Nemitz [4] explored fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali [5] investigated the properties of fuzzy equivalence relations, focusing on their lattice-theoretic characteristics. Samhan [6] characterized fuzzy congruences generated by fuzzy relations on a semigroup and studied the lattice structure of fuzzy congruences on semigroups. In 1982, Liu [7] introduced the concepts of fuzzy rings and fuzzy ideals. The concept of intuitionistic fuzzy sets was introduced by Atanassov [8-10]. Since then, numerous researchers have investigated the properties of intuitionistic fuzzy

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algebraic structures, including BG-subalgebras, BH-subalgebras, and AT-subalgebras, for various applications [11-13]. In [14], fuzzy subrings and ideals were redefined, and the author explored the properties of fuzzy algebraic structures using norms [15-25].

In this paper, we apply norms to introduce intuitionistic fuzzy equivalence relations, intuitionistic fuzzy congruence relations, and intuitionistic fuzzy ideals on rings. We derive new results regarding these concepts. Additionally, we define addition and multiplication for intuitionistic fuzzy congruence relations under norms and introduce the corresponding quotient rings. Finally, we explore ring homomorphisms and isomorphisms between these quotient rings.

## 2. Preliminaries

In this section, we will review some of the fundamental concepts and definitions that are essential for the following discussion.

**Definition 2.1:** Let $X_1, X_2,$ and $X_3$ be sets. A function $f: X_1 \rightarrow \times X_3$ is called a complex mapping if there exist functions $f_1: X_1 \rightarrow X_2$ and $f_2: X_1 \rightarrow X_3$ such that for all $x \in X_1$, $f(x) = (f_1(x), f_2(x))$.

**Definition 2.2:** Let $X$ be a nonempty set. An intuitionistic fuzzy set $(IFS)$ in $X$ is a complex mapping $M = (\mu_M, \nu_M): X \rightarrow [0,1] \times [0,1]$, where $\mu_M: X \rightarrow [0,1]$ represents the degree of membership, $\nu_M: X \rightarrow [0,1]$ represents the degree of non-membership and these functions satisfy the condition: $0 \leq \mu_M(x) + \nu_M(x) \leq 1$ for all $x \in X$. We will denote the set of all $IFSs$ in $X$ as $IFS(X)$.

**Definition 2.3:** An intuitionistic fuzzy relation is an intuitionistic fuzzy subset of $X_1 \times X_2$, that is, is an expression $R$ given by

$$\{\langle (x_1, x_2), \mu_R(x_1, x_2), \nu_R(x_1, x_2) \rangle | x_1 \in X_1, x_2 \in X_2\}$$

where $\mu_R: X_1 \times X_2 \rightarrow [0,1]$ and $\nu_R: X_1 \times X_2 \rightarrow [0,1]$ satisfy the condition $0 \leq \mu_R(x_1, x_2) + \nu_R(x_1, x_2) \leq 1$ for any $(x_1, x_2) \in X_1 \times X_2$.

**Definition 2.4:** A $t$-norm $T^{norm}$ is a function $T^{norm}: [0,1] \times [0,1] \rightarrow [0,1]$ such that:

(T1) $T^{norm}(x_1, 1) = x_1$

(T2) $T^{norm}(x_1, x_2) \leq T^{norm}(x_1, x_3)$ if $x_2 \leq x_3$

(T3) $T^{norm}(x_1, x_2) = T^{norm}(x_2, x_1)$

(T4) $T^{norm}(x_1, T^{norm}(x_2, x_3)) = T^{norm}(T^{norm}(x_1, x_2), x_3)$,

for all $x_1, x_2, x_3 \in [0,1]$.

**Corollary 2.5:** *Let $T^{norm}$ be a $t$-norm. Then*

(i) $T^{norm}(x_1, 0) = 0$.

(ii) $T^{norm}(0,0) = 0$.

*for all* $x_1 \in [0,1]$.

**Definition 2.6:** A $t$-conorm $C^{norm}$ is a function $C^{norm}: [0,1] \times [0,1] \rightarrow [0,1]$ such that:
(C1) $C^{norm}(x_1, 0) = x$
(C2) $C^{norm}(x_1, x_2) \leq C^{norm}(x_1, x_3)$ if $x_2 \leq x_3$
(C3) $C^{norm}(x_1, x_2) = C^{norm}(x_2, x_1)$
(C4) $C^{norm}(x_1, C^{norm}(x_2, x_3)) = C^{norm}(C^{norm}(x_1, x_2), x_3)$ ,
for all $x_1, x_2, x_3 \in [0,1]$.

**Corollary 2.7:** *Let $C^{norm}$ be a $t$-conorm. Then*
(i) $C^{norm}(x_1, 1) = 1$.
(ii) $C^{norm}(0,0) = 0$.
*for all* $x_1 \in [0,1]$.

Recall that $t$-norm $T^{norm}$ ( $t$-conorm $C^{norm}$) is idempotent if for all $x_1 \in [0,1]$, $T^{norm}(x_1, x_1) = x_1(C^{norm}(x_1, x_1) = x_1)$.

**Lemma 2.8:** (See [26]) Let $T^{norm}$ be a $t$-norm and $C^{norm}$ be a $t$-conorm. Then

$$(T^{norm}(T^{norm}(x_1, x_2), T^{norm}(x_3, x_4)) = T^{norm}(T^{norm}(x_1, x_3), T^{norm}(x_2, x_4))$$

*and*

$$C^{norm}(C^{norm}(x_1, x_2), C^{norm}(x_3, x_4)) = C^{norm}(C^{norm}(x_1, x_3), C^{norm}(x_2, x_4)),$$

*for all* $x_1, x_2, x_3, x_4 \in [0,1]$.

## 3. Main results

**Definition 3.1:** Let $M = (\mu_M, \nu_M) \in IFS(R \times R)$ an called an intuitionistic fuzzy equivalence relation on $R \times R$ under norms ($T^{norm}$ and $C^{norm}$) if the following conditions are satisfied:
(1) $\mu_M(r, r) = 1$,
(2) $\mu_M(r, s) = \mu_M(s, r)$,
(3) $\mu_M(r, t) \geq T^{norm}(\mu_M(r, s), \mu_M(s, t))$,
(4) $\nu_M(r, r) = 0$,
(5) $\nu_M(r, s) = \nu_M(s, r)$,
(6) $\nu_M(r, t) \leq C^{norm}(\nu_M(r, s), \nu_M(s, t))$,
for all $r, s, t \in R$.

**Corollary 3.2:** *Definition 3.1 is equivalent that $M = (\mu_M, \nu_M) \in IFS(R \times R)$ is called an intuitionistic fuzzy equivalence relation on $R \times R$ under norms ($T^{norm}$ and $C$) if the following conditions are satisfied:*
(i) $M(r, r) = (1,0)$,
(ii) $M(r, s) = M(s, r)$,
(iii) $M(r, t) \supseteq (T^{norm}(\mu_M(r, s), \mu_M(s, t), C^{norm}(\nu_M(r, s), \nu_M(s, t)))$,
*for all* $r, s, t \in R$.

**Definition 3.3:** *Let* $M = (\mu_M, \nu_M) \in IFS(R \times R)$ *an called an intuitionistic fuzzy congruence relation on* $R \times R$ *if the following conditions are satisfied:*

(1) $\mu_M(r,r) = 1$,
(2) $\mu_M(r,s) = \mu_M(s,r)$,
(3) $\mu_M(r,t) \geq T^{norm}(\mu_M(r,s), \mu_M(s,t))$,
(4) $\mu_M(r+t, s+u) \geq T^{norm}(\mu_M(r,s), \mu_M(t,u))$,
(5) $\mu_M(rt, su) \geq T^{norm}(\mu_M(r,s), \mu_M(t,u))$,
(6) $\nu_M(r,r) = 0$,
(7) $\nu_M(r,s) = \nu_M(s,r)$,
(8) $\nu_M(r,t) \leq C^{norm}(\nu_M(r,s), \nu_M(s,r))$,
(9) $\nu_M(r+t, s+u) \leq C^{norm}(\nu_M(r,s), \nu_M(t,u))$,
(10) $\nu_M(rt, su) \leq C^{norm}(\nu_M(r,s), \nu_M(t,u))$,

*for all* $x, y, z, t \in R$.

We denote the set of all intuitionistic fuzzy congruence relations on $R \times R$ under norms ($T^{norm}$ and $C^{norm}$) by $IFSCN(R)$.

**Corollary 3.4:** *Definition 3.3 is equivalent that* $M = (\mu_M, \nu_M) \in IFSCN(R)$ *if the following conditions are satisfied:*

(i) $M(r,r) = (1,0)$,
(ii) $M(r,s) = M(s,r)$,
(iii) $M(r,t) \supseteq (T^{norm}(\mu_M(r,s), \mu_M(s,t), C^{norm}(\nu_M(r,s), \nu_M(s,t)))$,
(iv) $M(r+t, s+u) \supseteq (T^{norm}(\mu_M(r,s), \mu_M(t,u)), C^{norm}(\nu_M(r,s), \nu_M(t,u)))$,
(v) $M(rt, su) \supseteq (T^{norm}(\mu_M(r,s), \mu_M(s,u)), C^{norm}(\nu_M(r,s), \nu_M(t,u)))$,

*for all* $r, s, t, u \in R$.

**Example 3.5:** Let $R = (\mathbb{Z}, +, .)$ be a ring of integer numbers. Define

$$M = (\mu_M, \nu_M) : \mathbb{Z} \times \mathbb{Z} \to [0,1]$$

by

$$\mu_M(r,s) = \begin{cases} 1 & \text{if } r = s \\ 0.45 & \text{otherwise} \end{cases}$$

and

$$\nu_M(r,s) = \begin{cases} 0 & \text{if } r = s \\ 0.35 & \text{otherwise.} \end{cases}$$

If $T^{norm}(r,s) = T_m^{norm}(r,s) = rs$ and $C^{norm}(r,s) = C_p^{norm}(r,s) = r + s - rs$ for all $r, s \in [0,1]$, then $M = (\mu_M, \nu_M) \in IFSCN(R)$.

**Proposition 3.6:** Let $T^{norm}$ and $C^{norm}$ are idempotent t-norms and t-conorms, respectively, and $M = (\mu_M, \nu_M) \in IFSCN(R)$, then the following assertions hold:

(1) $\mu_M(r,s) \geq T^{norm}(\mu_M(rt, st), \mu_M(tr, ts))$.
(2) $\mu_M(r^{-1}, s^{-1}) = \mu_M(r,s)$.

(3) $\mu_M(r,s) \geq T^{norm}(\mu_M(r+t,s+t), T^{norm}(\mu_M(rt,st), \mu_M(tr,ts)))$.
(4) $\mu_M(-r,-s) = \mu_M(r,s)$.
(5) $\nu_M(r,s) \leq C^{norm}(\nu_M(rt,st), \nu_M(tr,ts))$.
(6) $\nu_M(r^{-1},s^{-1}) = \nu_M(r,s)$.
(7) $\nu_M(r,s) \leq C^{norm}(\nu_M(r+t,s+t), C^{norm}(\nu_M(rt,st), \nu_M(tr,ts)))$.
(8) $\nu_M(-r,-s) = \nu_M(r,s)$,
*for all* $r,s,t \in R$.

**Definition 3.7:** Let $M = (\mu_M, \nu_M) \in IFSCN(R)$ and $a \in R$. Define a fuzzy subset $M_a = (\mu_{M_a}, \nu_{M_a})$ on $R$ as

$$M_a(r) = (\mu_{M_a}(r), \nu_{M_a}(r)) = (\mu_M(a,r), \nu_M(a,r)) = M(a,r)$$

for all $r \in R$. We denote the set of fuzzy subsets $M_a = (\mu_{M_a}, \nu_{M_a})$ on $R$ by $\frac{R}{M}$.

**Proposition 3.8:** Let $M = (\mu_M, \nu_M) \in IFSCN(R)$ and $u \in R$, then $M_a(r) = M_{o_R}(r-u)$ for all $r \in R$.

***Proof:*** Let $u, r \in R$ then

$$\begin{aligned}
\mu_{M_{0_R}}(r-u) &= \mu_M(0_R, r-u) \\
&= \mu_M(u-u, u-r) \\
&\geq T^{norm}(\mu_M(u,r), \mu_M(-u,-u)) \\
&= T^{norm}(\mu_M(u,r), \mu_M(u,u)) \\
&= T^{norm}(\mu_M(u,r), 1) \\
&= \mu_M(u,r) \\
&= \mu_M(u+0_R, r-u+u) \\
&\geq T^{norm}(\mu_M(0_R, r-u), \mu_M(u,u)) \\
&= T^{norm}(\mu_M(0_R, r-u), 1) \\
&= \mu_M(0_R, r-u) \\
&= \mu_{M_{0_R}}(r-u).
\end{aligned}$$

Thus $\mu_{M_{0_R}}(r-u) = \mu_M(r,u) = \mu_{M_u}(r)$.
Also

$$\begin{aligned}
\nu_{M_{o_R}}(r-u) &= \nu_M(0_R, r-u) \\
&= \nu_M(u-u, r-u) \\
&\leq C^{norm}(\nu_M(u,r), \nu_M(-u,-u)) \\
&= C^{norm}(\nu_M(r,u), \nu_M(u,u)) \\
&= C^{norm}(\nu_M(r,u), 0) \\
&= \nu_M(r,u) \\
&= \nu_M(u+0_R, r-u+u) \\
&\leq C^{norm}(\nu_M(0_R, r-u), \nu_M(u,u)) \\
&= C^{norm}(\nu_M(0_R, r-u), 0)
\end{aligned}$$

$$= \nu_M(0_R, r-u)$$
$$= \nu_{M_{0_R}}(r-u).$$

Thus $\nu_{M_{0_R}}(r-u) = \nu_M(u,r) = \nu_{M_u}(r)$. Therefore

$$M_u(r) = (\mu_{M_u}(r), \nu_{M_u}(r)) = (\mu_{M_{0_R}}(r-u), \nu_{M_{0_R}}(r-u)) = M_{0_R}(r-u).$$

**Definition 3.9:** Let $M = (\mu_M, \nu_M)$ be an $IFS$ on a ring $R$. $M$ is called an intuitionistic fuzzy ideal of $R$ under norms ($T^{norm}$ and $C^{norm}$) if the following conditions are satisfied for all $r, s \in R$:

(1) $\mu_M(r+s) \geq T^{norm}(\mu_M(r), \mu_M(s))$,

(2) $\mu_M(rs) \geq (\mu_M(r), \mu_M(s))$,

(3) $\mu_M(-r) = \mu_M(r)$,

(4) $\mu_M(0_R) = 1$,

(5) $\mu_M(rs) \geq \mu_M(r)$ and $\mu_M(rs) \geq \mu_M(s)$,

(6) $\nu_M(r+s) \leq C^{norm}(\nu_M(r), \nu_M(s))$,

(7) $\nu_M(rs) \leq C^{norm}(\nu_M(r), \nu_M(s))$,

(8) $\nu_M(-r) = \nu_M(r)$,

(9) $\nu_M(0_R) = 0$,

(10) $\nu_M(rs) \leq \nu_M(r)$ and $\nu_M(rs) \leq \nu_M(s)$,

We denote the set of all intuitionistic fuzzy ideals of $R$ under norms ($T^{norm}$ and $C^{norm}$) by $IFSIN(R)$.

**Example 3.10:** Let $R = (\mathbb{R}, +, .)$ be a ring of real numbers and $M = (\mu_M, \nu_M) \in IFS(R)$. Define $\mu_M: \mathbb{Z} \to [0,1]$ by

$$\mu_M(r) = \begin{cases} 1 & \text{if } r = 0_R \\ 0.25 & \text{otherwise} \end{cases}$$

and

$\nu_M: \mathbb{Z} \to [0,1]$ by

$$\nu_M(r) = \begin{cases} 0 & \text{if } r = 0_R \\ 0.55 & \text{otherwise.} \end{cases}$$

If $T^{norm}(r,s) = T_m^{norm}(r,s) = rs$ and $C^{norm}(r,s) = C_p^{norm}(r,s) = r + s - rs$ for all $r, s \in [0,1]$, then $M = (\mu_M, \nu_M) \in IFSIN(R)$.

**Proposition 3.11:** Let $M = (\mu_M, \nu_M) \in IFSCN(R)$ such that $T^{norm}$ and $C^{norm}$ be idempotent. Then $M_{0_R} = (\mu_{M_{0_R}}, \nu_{M_{0_R}}) \in IFSIN(R)$.

**Definition 3.12:** Given an intuitionistic fuzzy set $M = (\mu_M, \nu_M) \in IFS(R)$, we define a new intuitionistic fuzzy set $C^{norm}(M) = (C^{norm}(\mu_M), C^{norm}(\nu_M)) \in IFS(R \times R)$ such that for all $(r,s) \in R \times R$ we have

$$\begin{aligned} C^{norm}(M)(r,s) &= (C^{norm}(\mu_M)(r,s), C^{norm}(\nu_M)(r,s)) \\ &= (\mu_M(r-s), \nu_M(r-s)) \\ &= M(r-s). \end{aligned}$$

We call that $C^{norm}(M)$ is an intuitionistic fuzzy relation induced by $M = (\mu_M, \nu_M)$.

**Proposition 3.13:** Let $M = (\mu_M, \nu_M) \in IFSIN(R)$, then $C^{norm}(M) = (C^{norm}(\mu_M), C^{norm}(\nu_M)) \in IFSCN(R)$.

***Proof:*** Let $r, s, t, u \in R$ and $M = (\mu_M, \nu_M) \in IFSIN(R)$ , then

(1) $C^{norm}(\mu_M)(r,r) = \mu_M(r-r) = \mu_M(0_R) = 1$.

(2) $C^{norm}(\mu_M)(r,s) = \mu_M(r-s) = \mu_M(-(r-s)) = \mu_M(s-r) = C^{norm}(\mu_M)(s,r)$.

(3)
$$\begin{aligned} C^{norm}(\mu_M)(r,s) &= \mu_M(r-s) \\ &= \mu_M(r-t+t-s) \\ &\geq T^{norm}(\mu_M(r-t), \mu_M(t-s)) \\ &= T^{norm}(C^{norm}(\mu_M)(r,t), C^{norm}(\mu_M)(t,s)). \end{aligned}$$

So $C^{norm}(\mu_M)(r,s) \geq T^{norm}(C^{norm}(\mu_M)(r,t), C^{norm}(\mu_M)(t,s))$.

(4)
$$\begin{aligned} C^{norm}(\mu_M)(r+t, s+u) &= \mu_M(r+t-(s+u)) \\ &= \mu_M(r-s+t-u) \\ &\geq T^{norm}(\mu_M(r-s), \mu_M(t-u)) \\ &= T^{norm}(C^{norm}(\mu_M)(r,s), C^{norm}(\mu_M)(t,u)) \end{aligned}$$

then

$$C^{norm}(\mu_M)(r+t, s+u) \geq T^{norm}(C^{norm}(\mu_M)(r,s), C^{norm}(\mu_M)(t,u)).$$

(5)
$$\begin{aligned} C^{norm}(\mu_M)(rt, su) &= \mu_M(rt-(su)) \\ &= \mu_M((r-s)t + s(t-u)) \\ &\geq T^{norm}(\mu_M((r-s)t), \mu_M(s(t-u))) \\ &\geq T^{norm}(\mu_M(r-s), \mu_M(t-u)) \\ &= T^{norm}(C^{norm}(\mu_M)(r.s), C^{norm}(\mu_M)(t,u)). \end{aligned}$$

Then $C^{norm}(\mu_M)(rt, su) \geq T^{norm}(C^{norm}(\mu_M)(r,s), C^{norm}(\mu_M)(t,u))$.

(6)
$$C^{norm}(\nu_M)(r,r) = \nu_M(r-r) = \nu_M(0_R) = 0.$$

(7)
$$\begin{aligned} C^{norm}(\nu_M)(r,s) &= \nu_M(r-s) = \nu_M(-(r-s)) = \nu_M(s-r) \\ &= C^{norm}(\nu_M)(s,r). \end{aligned}$$

(8)
$$\begin{aligned} C^{norm}(\nu_M)(r,s) &= \nu_M(r-s) \\ &= \nu_M(r-t+t-s) \\ &\leq C^{norm}(\nu_M(r-t), \nu_M(t-s)) \\ &= C^{norm}(C^{norm}(\nu_M)(r,t), C^{norm}(\nu_M)(t,u)). \end{aligned}$$

Then $C^{norm}(\nu_M)(r,s) \leq C^{norm}(C^{norm}(\nu_M)(r,t), C^{norm}(\nu_M)(t,s))$.

(9)

$$\begin{aligned} C^{norm}(\nu_M)(r+t, s+u) &= \nu_M(r+t-(s+u)) \\ &= \nu_M(r-s+t-u) \\ &\leq C^{norm}(\nu_M(r-s), \nu_M(t-u)) \\ &= C^{norm}(C^{norm}(\nu_M)(r,s), C^{norm}(\nu_M)(t,u)). \end{aligned}$$

Then

$$C^{norm}(\nu_M)(r+t, s+u) \leq C^{norm}(C^{norm}(\nu_M)(r,s), C^{norm}(\nu_M)(t,u)).$$

(10)

$$\begin{aligned} C^{norm}(\nu_M)(rt, su) &= \nu_M(rt-(su)) \\ &= \nu_M((r-s)t + y(t-u)) \\ &\leq C^{norm}(\nu_M((r-s)t), \nu_M(s(t-u))) \\ &\leq C^{norm}(\nu_M(r-s), \nu_M(t-u)) \\ &= C^{norm}(C^{norm}(\nu_M)(r,s), C^{norm}(\nu_M)(t,u)). \end{aligned}$$

So $C^{norm}(\nu_M)(rt, su) \leq C^{norm}(C^{norm}(\nu_M)(r,s), C^{norm}(\nu_M)(t,u))$. Therefore $C^{norm}(M) = (C^{norm}(\mu_M), C^{norm}(\nu_M)) \in IFSCN(R)$.

**Proposition 3.14:** Let $M = (\mu_M, \nu_M) \in IFSCN(R)$ and $u, v \in R$. Then $M_u = M_v$ if and only if $M_{0_R}(u-v) = (1,0)$.

**Corollary 3.15:** *Let $M = (\mu_M, \nu_M) \in IFSCN(R)$ and $u, v \in R$. Then $M_u = M_v$ if and only if $M(u,v) = (1,0)$.*

**Definition 3.16:** Let $N = (\mu_N, \nu_M) \in IFSCN(R)$ and $u, v \in R$. Define addition

$$N_u \oplus N_v = (\mu_{N_u}, \nu_{N_u}) \oplus (\mu_{N_v}, \nu_{N_v}) = (\mu_{N_u} \oplus \mu_{N_v}, \nu_{N_v} \oplus \nu_{N_v}) : R \to [0,1]$$

by

$$(\mu_{N_u} \oplus \mu_{N_v})(r) = \begin{cases} \sup_{r=s+t} T^{nom}(\mu_{N_u}(s), \mu_{N_v}(t)) & \text{if } r = s+t \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\nu_{N_u} \oplus \nu_{N_u})(r) = \begin{cases} \inf_{r=s+t} C^{norm}(\nu_{N_u}(s), \nu_{N_v}(t)) & \text{if } r = s+t \\ 0 & \text{otherwise.} \end{cases}$$

Also define product

$$N_u \odot N_v = (\mu_{N_u}, \nu_{N_v}) \mathrm{e} (\mu_{N_v}, \nu_{N_v}) = (\mu_{-N_u} \mathrm{e} \mu_{N_v}, \nu_{N_u} \mathrm{e} \nu_{N_v}) : R \to [0,1]$$

by

$$(\mu_{N_u} \odot \mu_{N_v})(r) = \begin{cases} \sup_{r=st} T^{norm}(\mu_{N_u}(s), \mu_{N_v}(t)) & \text{if } r = st \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\nu_{N_u} \odot \nu_{N_v})(r) = \begin{cases} \inf_{r=st} C^{norm}(\nu_{N_u}(s), \nu_{N_v}(t)) & \text{if } r = st \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 3.17:** Let $N = (\mu_N, \nu_N) \in IFSCN(R)$ and $u, v \in R$. Then $N_u \oplus N_v = N_{u+v}$.

**Proposition 3.18:** Let $N = (\mu_N, \nu_N) \in IFSCN(R)$ and $u, v \in R$. Then $N_u \odot N_v = N_{uv}$.

**Proposition 3.19:** If $N = (\mu_N, \nu_N) \in IFSCN(R)$, then $\left(\frac{R}{N}, \oplus, \odot\right)$ will be a ring.

***Proof:*** Let $u, v, w \in R$. As Propositions 3.17 and 3.18 we get that $N_u \oplus N_v \in \frac{R}{N}$ and $N_u \odot N_v \in \frac{R}{N}$. It is easy to prove that $\left(\frac{R}{N}, \oplus\right)$ is an abelian group. Now

(1) $N_u \odot (N_v \odot N_w) = N_u \odot N_{vw} = N_{u(vw)} = N_{(uv)w} = (N_u \odot N_v) \odot N_w$ (associative multiplication).

(2) $N_u \odot (N_v \oplus N_w) = N_u \odot N_{v+w} = N_{u(v+w)} = N_{uv+uw} = N_{uv} \oplus N_{uw} = (N_u \odot N_v) \oplus (N_u \odot N_w)$ (distributive laws).

(3) $(N_v \oplus N_w) \odot N_u = N_{v+w} \odot N_u = N_{(v+w)u} = N_{vu+vw} = N_{vu} \oplus N_{wu} = (N_v \odot N_u) \oplus (N_w \odot N_u)$ (distributive laws).

Then $\left(\frac{R}{N}, \oplus, \odot\right)$ will be a ring.

If $R$ be a commutative ring, then $N_u \odot N_v = N_{uv} = N_{vu} = N_v \odot N_u$ and so $\left(\frac{R}{N}, \oplus, \odot\right)$ will be a commutative ring.

If $R$ be a ring with identity $1_R$, then $N_u \odot N_{1_{\mathrm{R}}} = N_{u1_R} = N_u$ and thus $\left(\frac{R}{N}, \oplus, \odot\right)$ will be a ring with identity $N_{1_{\mathrm{R}}}$.

**Proposition 3.20:** Let $N = (\mu_N, \nu_N) \in IFSCN(R)$ and define

$$N^{-1}(\alpha, \beta) = \{(r, s) | N(r, s) = (\alpha, \beta)\}$$

for all $\alpha, \beta, r, s \in R$. Then

(i) $N^{-1}(1,0)$ will be an intuitionistic fuzzy equivalence relation on $R$ under norms($T^{norm}$ and $C^{norm}$).

(ii) $N^{-1}(1,0)$ will be a congruence relation on $R$.

**Definition 3.21:** Let $R$ and $S$ be two rings and $f: R \to S$ be a ring homomorphism. Define $ker(f) = \{(r, s) | f(r) = f(s)\}$ for all $r, s \in R$.

**Proposition 3.22:** Let $\chi_{ker(f)}: R \times R \to \{0,1\}$ be characteristic function of $ker(f)$ such that defined by

$$\chi_{\mathrm{ker}(f)}(r, s) = \begin{cases} 1 & \text{if } (r, s) \in \mathrm{ker}(f) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } f(r) = f(s) \\ 0 & \text{otherwise} \end{cases}$$

and $\chi'_{\mathrm{ker}(f)}: R \times R \to \{0,1\}$ be anti characteristic function of $\mathrm{ker}(f)$ such that defined by

$$\chi'_{\ker(f)}(r,s) = \begin{cases} 0 & \text{if } (r,s) \in \ker(f) \\ 1 & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } f(r) = f(s) \\ 1 & \text{otherwise.} \end{cases}$$

Then $\chi = (\chi_{\ker(f)}, \chi'_{\ker(f)}) \in IFSCN(R)$.

**Lemma 3.23:** *Let $N = (\mu_N, \nu_M) \in IFSCN(R)$ and $(\frac{R}{N}, \oplus, \odot)$ be a ring. Define $N^\star: R \to \frac{R}{N}$ as $N^\star(u) = (\mu_{N_u}, \nu_{N_u})$ for all $u \in R$. Then $N^\star$ is a ring homomorphism.*

**Proposition 3.24:** Let $R$ and $S$ be two rings and $f: R \to S$ be a ring homomorphism and $\chi = (\chi_{ker(f)}, \chi'_{ker(f)})$ such that $\chi^\star: R \to \frac{R}{\chi}$ as $\chi^\star(u) = (\chi_{ker(f)_u}, \chi'_{ker(f)_u})$ for all $u \in R$. Then there is a homomorphism $g: \frac{R}{\chi} \to S$ such that $f = go\chi^\star$.

***Proof:*** Define $g: \frac{R}{\chi} \to S$ by $(g(\chi^\star(u)) = g(\chi_{\ker(f)_u}, \chi'_{\ker(f)_u}) = f(u)$ for all $u \in R$. At first we show that $g$ is well-defined. Let $u, v \in R$ and $\chi_{\ker(f)_u}, \chi'_{\ker(f)_u}) = (\chi_{\ker(f)_v}, \chi'_{\ker(f)_v})$. Then $\chi_{\ker(f)_u} = \chi_{\ker(f)_v}$ and $\chi'_{\ker(f)_u}) = \chi'_{\ker(f)_v})$ then $\chi_{\ker(f)}(u,v) = 1$ and $\chi'_{\ker(f)})(u,v) = 0$ which mean that $(u,v) \in \ker(f)$ and $f(u) = f(v)$ as wanted. Now

$$g(\chi_u \oplus \chi_v) = g(\chi_{u+v}) = f(u+v) = f(u) + f(v) = g(\chi_u) \oplus g(\chi_v).$$

Also

$$g(\chi_u \odot \chi_v) = g(\chi_{uv}) = f(uv) = f(u)f(v) = g(\chi_u) \odot g(\chi_v).$$

Thus $g$ will be a homomorphism. Since $g(\chi^\star(u)) = g(\chi_{\ker(f)_u}, \chi'_{\ker(f)_u}) = f(u)$ so $f = go\chi^\star$.

**Proposition 3.25:** Let $N = (\mu_N, \nu_M) \in IFSCN(R)$ and $L = (\mu_L, \nu_L) \in IFSCN(R)$ such that $N \subseteq L$. Let $N^\star: R \to \frac{R}{N}$ as $N^\star(u) = (\mu_{N_u}, \nu_{N_u})$ and $L^\star: R \to \frac{R}{L}$ as $L^\star(u) = (\mu_{L_u}, \nu_{L_u})$ for all $u \in R$. Then there is an unique homomorphism $g: \frac{R}{N} \to \frac{R}{L}$ such that $goN^\star = L^\star$ and $\frac{\frac{R}{N}}{\chi_{ker(g)}}$ is isomorphic to $\frac{R}{L}$.

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