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Graph product techniques for cordial labeling on cubic root graph structures

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Abstract

Graph labeling is the term used to describe the procedure of giving unique identifiers. Graph attributing is a method for modeling real-world problems in industries such as circuit design, surveillance technology, and telecommunications networks. A graph is characterized as possessing cubic roots cordial labeling if each node is designated with either the value 1, δ or δ^2 or another specified integer. The derived set containing the value 1 or another integer, the mathematical conditions $\eta(k) - \eta(w)| \le 1$ and $|\xi(k) - \xi(w)| \le 1$, $k \neq w$ and $k, w \in \{1, \delta, \delta^2\}$ where $\eta(k)$ and $\xi(k)$ are represents the aggregated vertices and edges assigned in $k \in \{1, \delta, \delta^2\}$. Labeled graphs prove to be immensely beneficial across a variety of mathematical frameworks, including domains such as cyber security and radar code formulation. The application of cordial labeling has been demonstrated to provide significant advantages in the context of signal processing channels and challenges associated with specific code word design. The idea of cubic roots cordial labeling has attracted a lot of academic attention among various labeling techniques, unique structural characteristics and capacity to depict a fair flow of information within linked networks. The investigation of the behavior of cubic roots cordial labeling in relation to graph operations is currently receiving extensive scholarly examination, particularly concerning path graphs, despite its critical relevance. This paper explore the cordial labeling of cubic product graphs generated from the combination of two

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path graphs. The four fundamental graph operations on which we particularly focus on activities, we analyze the specific conditions to possess cubic roots cordial labeling.

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Introduction

Graph theory has an systematic data and followed rules with sufficient necessary conditions which can be correlated deals with the mathematical structures known as graphs, which are used to represent pairwise interactions differ from closed relations. The interconnection nodes can be a graph, which can be enumerated with vertices, sometimes called nodes or points, and edges, sometimes called links or lines. The branch of mathematics known as graph theory studies networks of points joined by lines. The domains of commutative algebra, graph theory, linear algebra, and combinatory are now closely related. Graphs are used in many fields, including as ecology, chemical networks, programming, electrical engineering, sociology, and even medicine. This field is now a potent tool for studying brain networks, having started out innocently enough with games and puzzles. Google Maps is among the most important graph theory applications. Other examples are Facebook and Linked in. OTT services like Netflix use graph theory to enhance their recommendation systems. Mathematics often examines the behaviour of a collection of objects, the creation of new objects from existing ones, and other related topics. Graph products from existing graphs are used to construct new ones. The concept of graph products is vital to graph theory. This study's primary focus is on four popular graph products and some of their characteristics . Adjacency between two vertices (a_1,a_2) and (b_1,b_2) in H is determined by certain conditions based on whether a_1,b_1 in G_1, and a_2,b_2 in G_2, are equal or connected by an edge. In this article, we further explore several classical graph products and investigate various graph parameters associated with them. Graph products are widely used in numerous domains [1, 2]. A structure's configuration must be generated before it can be analyzed [3-5]. The application of graph products is one instance of such a tool. The purpose of product graphs expanded by the development of numerous structural models [6-7]. There are several uses for graph products in computer science [8-10]. This product's next application is to estimate the spectral characteristics of large-scale networks analytically by approximating their structure. The weighted graph products can also be used to create finite element models. A few examples of the extensive and in-depth literature on graph products are dealt [11-13]. The combination of graphs are intend to investigate [17-18]. The findings about graph products are also covered. Throughout, we limit ourselves to undirected graphs. We referred for all fundamental definitions and terminology [14-16].

Preliminaries

Definition 1: The graph $F = (\nu, \mathcal{E})$ allocating the values which represent the function $f: \nu \to \{1, \psi, \psi^2\}$ and the innovated edge labeling $\{1, \psi, \psi^2\}$ are presented in Table 1, is called a cubic roots labeling if $|\ \nu\ (k) - \nu\ (w)| \leq 1$ & $|\ \mathcal{E}\ (k) - \mathcal{E}\ (w)| \leq 1$, $k \neq w$ and $k, w \in \{1, \psi, \psi^2\}$where $\nu\ (k)$ and $\mathcal{E}\ (k)$ are represents the number of vertices and edges assigned by $k \in \{1, \psi, \psi^2\}$, the graph F attains a cordial cubic labeling.

Definition 2: The nodes of two graphs V (*G*) and V (*H*) is represented be V (*G*) × V (*H*)) , is refers the cartesian product of *G* and *H*. Two vertices (*g*, *h*) and (*g*′, *h*′) are linked if either of the following occurs *gg*′ ∈ *E*(*G*) and *h* = *h*′, or *g* = *g*′ and *hh*′ ∈ *E*(*H*). Thus, $V\ (G\times H) = \{(g, h): g \in V\ (G), h \in V\ (H)\}$. The linked parameter are given by the relation, $E(\mathrm{GH}) = (\mathrm{E(G)} \times \mathrm{V\ (H)}) \cup (\mathrm{V\ (G)} \times \mathrm{E(H)}$.

Table 1

Clause for edge labeling

ν	**1**	ψ	ψ^2
1	1	ψ	ψ^2
ψ	ψ	ψ^2	1
ψ^2	ψ^2	1	ψ

Definition 3: The graph *G*∗*H*, is the direct product of *G* and *H*. The nodes of the graphs V (*G*) ×V (*H*), and its (*g*, *h*) and (*g*′, *h*′) are exactly linked, if *gg*′∈*E*(*G*) and *hh*′∈ *E* (*H*). Then, V (*G*∗*H*) = {(*g*, *h*) : *g* ∈ V (*G*) and *h* ∈ V (*H*)}, *E*(*G*∗*H*) = {(*g*, *h*)(*g*′, *h*′) , *gg*′∈*E*(*G*) and *hh*′∈*E*(*H*)}.

Definition 4: The graph *G*⊠*H* is represented by V (*G* ⊠ *H*) = {(*g*, *h*) ∶ *g* ∈ V (*G*) and *h* ∈ V (*H*))}, *E*(*G* ⊠ *H*) = *E* (*G*□*H*) ∪ *E*(*G* × *H*). It is symmetric composition.

Definition 5: The graphs, represented by *G* ∘ *H*, which can be demonstrated by forming a graph that has the vertex set of *V*(*G*) × *V*(*H*). Two vertices, (*g, h*) and (*g*′*, h*′), have connections, *g* is adjacent to *g*′ in *G* or *g* is equal to *g*′ and h is adjacent to *h*′ in *H*. It is not commutative. The order is similar to |*G*∘*H*| , which is equal to |G|×|H|.

Results and discussion

Theorem 1: *The cubic roots cordial labeling preserves* $P_n \times P_m$, $m, n \geq 3$.

Proof: The graph F_1 be the collection of nodes $1 \leq \nu_i \leq n$ and the connecting element $1 \leq \mathcal{E}_i \leq n-1$, F_2 be graph with the collection of nodes $1 \leq \nu_i' \leq m$ and the connecting elements $1 \leq \mathcal{E}_i' \leq m-1$. Then the graphs $F_1 \times F_2$, whose nodes is $\nu(F_1)$ and $\nu(F_2)$, $\nu(F_1) \times \nu(F_2)$. Two vertices (ν_i, ν_i') and (ν_j, ν_j') are precisely if $\nu_i = \nu_j$ and $\nu_i'\nu_j' \in \mathcal{E}(F_2)$, or $\nu_i\nu_j \in \mathcal{E}(F_2)$ and $\nu_i' = \nu_j'$. Thus, $\nu(F_1 \times F_2) = \{(\nu_i, \nu_i'): \nu_i \in \nu(F_1), \nu_i' \in (F_2)\}$. The relation, $\mathcal{E}(F_1F_2) = (\mathcal{E}(F_1) \times \nu(F_2)) \cup (\nu(F_1) \times \mathcal{E}(F_2))$.

Consider the nodes of $F_1 \times F_2$,be labeled from the set $\{1, \psi, \psi^2\}$ such that

$$\nu_{i,j} = \begin{cases} 1 & if\ (i+j) \equiv 2(mod3) \\ \psi & if\ (i+j) \equiv 0(mod3) \\ \psi^2 & if\ (i+j) \equiv 1(mod3) \end{cases}$$

and the innovated edge labeling is represented in Table 2. Where ν (1), ν (ψ) and ν (ψ^2) .

Table 2

Assigining edge parameter

ν	**1**	$\boldsymbol{\psi}$	$\boldsymbol{\psi^2}$
1	1	ψ	ψ^2
$\boldsymbol{\psi}$	ψ	ψ^2	1
$\boldsymbol{\psi^2}$	ψ^2	1	ψ

Case (i) Consider $n \equiv 0(mod\ 3)$, $n = 3\delta$, If $m \equiv 0(mod\ 3)$, $m = 3\delta$, $m,n \geq 3$, $\delta = 1,2,3 \dots$ then $mn \equiv 0(mod\ 3)$, and the innovated edges also from $\{1, \psi, \psi^2\}$. Then, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = 3δ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $2(3\delta^2 - \delta)$. If $m \equiv 1(mod\ 3)$, $m = 3\delta + 1$, $m,n \geq 3$, $\delta = 1,2,3 \dots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $6\delta^2$, $\mathcal{E}$ (ψ^2) = $6\delta^2 - 1$. If $m \equiv 2(mod\ 3)$, $m = 3\delta + 2$, $m,n \geq 3$, $\delta = 1,2,3 \dots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $6\delta^2 + 2\delta$ $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $[6\delta^2 + 2\delta] - 1$.

Case (ii) Consider $n \equiv 1(mod\ 3)$, $n = 3\delta + 1$, If $m \equiv 0(mod\ 3)$, $m = 3\delta$, $m,n \geq 3$, $\delta = 1,2,3 \dots$ then $mn \equiv 0(mod\ 3)$, the number of nodes are assigned 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $6\delta^2$, $\mathcal{E}$ (ψ^2) = $6\delta^2 - 1$. If $m \equiv 1(mod\ 3)$, $m = 3\delta + 1$, $m,n \geq 3$, $\delta = 1,2,3 \dots$ then $mn \equiv 1(mod\ 3)$, the number of nodes assigned 1, ψ and ψ^2 are ν (1) = $[3\delta^2 + 2\delta] + 1$, ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $6\delta^2 + 2\delta$. If $m \equiv 2(mod\ 3)$, $m = 3\delta + 2$, $m,n \geq 3$, $\delta = 1,2,3 \dots$ then $mn \equiv 2(mod\ 3)$, the number of nodes assigned 1, ψ and ψ^2 are ν (1) = ν (ψ) = $[3\delta(\delta + 1)] + 1$ ν (ψ^2) = $[3\delta(\delta + 1)]$ and the linked edges assigned 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ^2) =$6\delta^2 + 4\delta$, $\mathcal{E}$ (ψ) = $[6\delta^2 + 4\delta] + 1$

Case (iii) Consider $n \equiv 2(mod\ 3)$, $n = 3\delta + 2$, If $m \equiv 0(mod\ 3)$, $m = 3\delta$, $m,n \geq 3$, $\delta = 1,2,3 \dots$ then $mn \equiv 0(mod\ 3)$, the number of nodes assigned 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and the linked edges labeled 1, ψ and ψ^2 are $\mathcal{E}(1) = 6\delta^2 + 2\delta$ $\mathcal{E}(\psi) = \mathcal{E}(\psi^2) = [6\delta^2 + 2\delta] - 1$. If $m \equiv 1(mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 1(mod\ 3)$, the number of nodes assigned 1, ψ and ψ^2 are $\nu(1) = \nu(\psi) = [3\delta(\delta + 1)] + 1$ $\nu(\psi^2) = [3\delta(\delta + 1)]$ and the induced edges labeled 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}(\psi^2) = 6\delta^2 + 4\delta$, $\mathcal{E}(\psi) = [6\delta^2 + 4\delta] + 1$. If $m \equiv 2(mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 2(mod\ 3)$, the total number of nodes assigned 1, ψ and ψ^2 are $\nu(1) = \left[\frac{(3\delta+2)^2}{3}\right] + 1$, $\nu(\psi) = \nu(\psi^2) = \left[\frac{(3\delta+2)^2}{3}\right]$ and the innovated edges labeled 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}(\psi^2) = [6\delta^2 + 6\delta + 2] - 1$, $\mathcal{E}(\psi) = 6\delta^2 + 6\delta + 2$.

Hence, for all the above cases, it is satisfying the conditions $|\nu(k) - \nu(w)| \leq 1$ & $|\mathcal{E}(k) - \mathcal{E}(w)| \leq 1$, $k \neq w$ and $k, w \in \{1, \psi, \psi^2\}$. The graphs $F_1 \times F_2$ is a cubic roots cordial labeling.

Consider $F_1 = P_6$ and $F_2 = P_8$, Here $n \equiv 0(mod\ 3)$, $n = 3\delta$ and $m \equiv 2(mod\ 3)$, $m = 3\delta + 2$, $\delta = 2$, then $mn \equiv 0(mod\ 3), and$ 1, ψ and ψ^2 are $\nu(1) = \nu(\psi) = \nu(\psi^2) = 16$ and1, ψ and ψ^2 are $\mathcal{E}(1) = 28$, $\mathcal{E}(\psi) = \mathcal{E}(\psi^2) = 27$. Hence, $|\nu(1) - \nu(\psi)| = |\nu(1) - \nu(\psi^2)| = |\nu(\psi^2) - \nu(\psi)| = 0$, $|\mathcal{E}(1) - \mathcal{E}(\psi)| = 1$, $|\mathcal{E}(1) - \mathcal{E}(\psi^2)| = 1$, $|\mathcal{E}(\psi^2) - \mathcal{E}(\psi)| = 0$. So, it is satisfying the conditions $|\nu(k) - \nu(w)| \leq 1$ & $|\mathcal{E}(k) - \mathcal{E}(w)| \leq 1$, $k \neq w$ and $k, w \in \{1, \psi, \psi^2\}$. The cartesian product attains cubic roots cordial labeling.

Theorem 2: *The cordial labeling preserves for $P_n * P_m$, $m, n \geq 3$.*

Proof: The path P_n and P_m is $P_n * P_m$ and the nodes be V $(P_n) \times$ V (P_m), The vertices of $P_n * P_m$, all possible ordered pairs of vertices from P_n and P_m. If P_n has vertices {1,2,3,…..n} and P_m has vertices {1,2,3,…..m}, then vertices of the product graph will be the set of all pairs (i , j) where $1 \leq i \leq n$ and $1 \leq j \leq m$. Consider the vertices of $P_n * P_m$, be labeled from the set $\{1, \psi, \psi^2\}$ such that

$$v_{i,j} = \begin{cases} 1 & if\ (i+j) \equiv 2(mod3) \\ \psi & if\ (i+j) \equiv 0(mod3) \\ \psi^2 & if\ (i+j) \equiv 1(mod3) \end{cases}$$

and the induced edge labeling are $\nu(1)$, $\nu(\psi)$ and $\nu(\psi^2)$. $\mathcal{E}(1)$, $\mathcal{E}(\psi)$ and $\mathcal{E}(\psi^2)$, edges labeled 1, ψ and ψ^2 respectively.

Case (i) Consider $n \equiv 0(mod\ 3)$, $n = 3\delta$, If $m \equiv 0(mod\ 3)$, $m = 3\delta$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are $\nu(1) = \nu(\psi) = \nu(\psi^2) = 3\delta^2$ and1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}(\psi) = 6\delta^2 - 4n + 1$, $\mathcal{E}(\psi^2) = 6\delta^2 - 4n$. If $m \equiv 1(mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are $\nu(1) = \nu(\psi) = \nu(\psi^2) = 3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}(\psi) = \mathcal{E}(\psi^2) = 6\delta^2 - 2\delta$. If $m \equiv 2(mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are $\nu(1)$

$= \nu\ (\psi) = \nu\ (\psi^2) = 3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = 6\delta^2 - 1$, $\mathcal{E}\ (\psi)$ $=6\delta^2$, $\mathcal{E}\ (\psi^2) = 6\delta^2 - 1$.

Case (ii) Consider $n \equiv 1(\, mod\ 3),\ n = 3\delta + 1$.If $m \equiv 0(\, mod\ 3)$, $m = 3\delta$, $m,n \geq 3$, $\delta = 1,2,3$ ... then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are $\nu\ (1) = \nu\ (\psi) = \nu\ (\psi^2) = 3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}\ (\psi) = \mathcal{E}\ (\psi^2) = 6\delta^2 - 2\delta$. If $m \equiv 1(\, mod\ 3)$, $m = 3\delta + 1$, $m,n \geq 3$, $\delta = 1,2,3$ ... then $mn \equiv 1(mod\ 3)$, 1, ψ and ψ^2 are $\nu\ (1) = [3\delta^2 + 2\delta] + 1$, $\nu\ (\psi) = \nu\ (\psi^2) = 3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}\ (\psi) = \mathcal{E}\ (\psi^2) = 6\delta^2$. If $m \equiv 2(\, mod\ 3)$, $m = 3\delta + 2$, $m,n \geq 3$, $\delta = 1,2,3$ ... then $mn \equiv 2(mod\ 3)$, t1, ψ and ψ^2 are $\nu\ (1) = [3\delta(\delta + 1)] + 1$, $\nu\ (\psi) = [3\delta(\delta + 1)]$ $\nu\ (\psi^2) = [3\delta(\delta + 1)] + 1$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}\ (\psi^2) = \mathcal{E}\ (\psi) = [6\delta^2 + 2\delta]$.

Case (iii) Consider $n \equiv 2(\, mod\ 3),\ n = 3\delta + 2$. If $m \equiv 0(\, mod\ 3)$, $m = 3\delta$, $m,n \geq 3$, $\delta = 1,2,3$ ... then $mn \equiv 0(mod\ 3)$, the vertices labeled 1, ψ and ψ^2 are $\nu\ (1) = \nu\ (\psi) = \nu\ (\psi^2) = 3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = 6\delta^2 - 1$, $\mathcal{E}\ (\psi) = 6\delta^2$, $\mathcal{E}\ (\psi^2) = 6\delta^2 - 1$. If $m \equiv 1(\, mod\ 3)$, $m = 3\delta + 1$, $m,n \geq 3$, $\delta = 1,2,3$ ... then $mn \equiv 1(mod\ 3)$, 1, ψ and ψ^2 are $\nu\ (1) = [3\delta(\delta + 1)] + 1$, $\nu\ (\psi) = [3\delta(\delta + 1)]$ $\nu\ (\psi^2) = [3\delta(\delta + 1)] + 1$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}\ (\psi^2) = \mathcal{E}\ (\psi) = [6\delta^2 + 2\delta]$. If $m \equiv 2(\, mod\ 3)$, $m = 3\delta + 2$, $m,n \geq 3$, $\delta = 1,2,3$ ... then $mn \equiv 2(mod\ 3)$, and 1, ψ and ψ^2 are $\nu\ (1) = 3\delta^2 + 4\delta + 2$, $\nu\ (\psi) = \nu\ (\psi^2) = 3\delta^2 + 4\delta + 1$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}\ (\psi) = \mathcal{E}\ (\psi^2) = [6\delta^2 + 2\delta]$. Hence, $|\ \nu\ (k) - \nu\ (w)| \leq 1$ & $|\ \mathcal{E}\ (k) - \mathcal{E}\ (w)| \leq 1$, $k \neq w$ and $k, w \in \{1,\ \psi,\ \psi^2\}$. The graph $P_n * P_m$ is a cubic roots cordial labeling.

Theorem 3: *The cordial labeling preserves for* $P_n \boxtimes P_m$, $m, n \geq 3$.

Proof: The graphs $P_n \boxtimes P_m$ results in a grid-like graph. A vertex (u,v) is connected to the vertex (u',v'), if the vertices are in the same row or column and their corresponding indices are the closest. A vertex (u,v) is closer to a vertex (u',v') if one of the following conditions is satisfied $u = u'$ in the same row and v is neighbor to v' in the path graph, V = v′ and u is neighbour to u′ in the path graph , u is neighbor to u' in the path graph, and v is adjacent to v' in the path graph. Consider the vertices of $P_n \boxtimes P_m$, be labeled from the set $\{1,\ \psi,\ \psi^2\}$ such that

$$v_{i,j} = \begin{cases} 1 & if\ \ (i+j) \equiv 2(mod3) \\ \psi & if\ (i+j) \equiv 0(mod3) \\ \psi^2 & if\ (i+j) \equiv 1(mod3) \end{cases}$$

and the induced edge labelings are $\nu\ (1), \nu\ (\psi)$ and $\nu\ (\psi^2)$ labeled 1, ψ and ψ^2 respectively. Similarly, $\mathcal{E}(1)$, $\mathcal{E}\ (\psi)$ and $\mathcal{E}\ (\psi^2)$.

Case (i) Consider $n \equiv 0(\, mod\ 3),\ n = 3\delta$. If $m \equiv 0(\, mod\ 3)$, $m = 3\delta$, $m,n \geq 3$, $\delta = 1,2,3$ ... then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are $\nu\ (1) = \nu\ (\psi)$ $= \nu\ (\psi^2) = 3\delta^2$ and 1, ψ and ψ^2 are $\mathcal{E}(1) = \mathcal{E}\ (\psi) = 12\delta^2 - 6n + 1$, $\mathcal{E}\ (\psi^2) =$

$12\delta^2 - 6n$. If $m \equiv 1(\,mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $12\delta^2 - 2\delta$. If $m \equiv 2(\,mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) =$12\delta^2 + 2\delta - 1$, $\mathcal{E}$ (ψ^2) = $12\delta^2 + 2\delta - 2$.

Case (ii) Consider $n \equiv 1(\,mod\ 3)$, $n = 3\delta + 1$. If $m \equiv 0(\,mod\ 3)$, $m = 3\delta$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $12\delta^2 - 2\delta$. If $m \equiv 1(\,mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 1(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = $[3\delta^2 + 2\delta] + 1$, ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $12\delta^2 + 2\delta$. If $m \equiv 2(\,mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 2(mod\ 3)$, 1, ψ and ψ^2 are ν (1) =$[3\delta(\delta + 1)] + 1$, ν (ψ) = $[3\delta(\delta + 1)]$ ν (ψ^2) = $[3\delta(\delta + 1)] + 1$ 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $12\delta^2 + 2\delta$, $\mathcal{E}$ (ψ^2) = $12\delta^2 + 2\delta$, $\mathcal{E}$ (ψ) = $12\delta^2 + 2\delta + 1$.

Case (iii) Consider $n \equiv 2(\,mod\ 3)$, $n = 3\delta + 2$. If $m \equiv 0(\,mod\ 3)$, $m = 3\delta$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) =$12\delta^2 + 2\delta - 1$, $\mathcal{E}$ (ψ^2) = $12\delta^2 + 2\delta - 2$. If $m \equiv 1(\,mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 1(mod\ 3)$, 1, ψ and ψ^2 are ν (1) =$[3\delta(\delta + 1)] + 1$, ν (ψ) = $[3\delta(\delta + 1)]$ ν (ψ^2) = $[3\delta(\delta + 1)] + 1$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $12\delta^2 + 2\delta$, $\mathcal{E}$ (ψ^2) = $12\delta^2 + 2\delta$, $\mathcal{E}$ (ψ) = $12\delta^2 + 2\delta + 1$. If $m \equiv 2(\,mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 2(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = $3\delta^2 + 4\delta + 2$, ν (ψ) = ν (ψ^2) = $3\delta^2 + 4\delta + 1$ and are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) =$[12\delta^2 + 10\delta + 2]$. Hence, | ν (k) − ν (w)| ≤ 1 & | $\mathcal{E}$ (k) − $\mathcal{E}$ (w)| ≤ 1, $k \neq w$ and $k, w \in \{1,\ \psi,\ \psi^2\}$. The product $P_n \boxtimes P_m$ is a cubic roots cordial labeling.

Theorem 4: *Cubic-root cordial is invariant under the lexicographic graph product for for $P_n \circ P_m$, $m, n \geq 3$.*

Proof: The graph $P_n \circ P_m$ exists between two vertices u, and $(u',')$, if either the first vertices are adjacent in the path graph P_n , u is adjacent to u' in P_n or they are the same vertex and the second vertices are adjacent in the path graph P_m , u = u′ and v is adjacent to v' in P_m. The resulting graph is a more complex structure where each vertex of P_n modified by P_m.

Consider the vertices of $P_n \circ P_m$, , be labeled from the set $\{1,\ \psi,\ \psi^2\}$ such that

$$v_{i,j} = \begin{cases} 1 & if\ \ (i+j) \equiv 2(mod3) \\ \psi & if\ (i+j) \equiv 0(mod3) \\ \psi^2 & if\ (i+j) \equiv 1(mod3) \end{cases}$$

and the induced edge labeling are ν (1), ν (ψ) and ν (ψ^2). Similarly, $\mathcal{E}(1)$, $\mathcal{E}$ (ψ) and $\mathcal{E}$ (ψ^2) denote the edges labeled 1, ψ and ψ^2 respectively. From the following cases the lexicographic product of graphs $P_n \circ P_m$,

Case (i) Consider $n \equiv 0(\,mod\ 3)$, $n = 3\delta$. If $m \equiv 0(\,mod\ 3)$, $m = 3\delta$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, the vertices labeled 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $54\delta^2 - 100n + 54$. If $m \equiv 1(\,mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ^2) = $60\delta^2 - 100n + 54$, $\mathcal{E}$ (ψ) = $60\delta^2 - 100n + 53$.If $m \equiv 2(\,mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and labeled 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) =$66\delta^2 - 98n + 53$, $\mathcal{E}$ (ψ^2) = $66\delta^2 - 98n + 52$.

Case (ii) Consider $n \equiv 1(\,mod\ 3)$, $n = 3\delta + 1$, If $m \equiv 0(\,mod\ 3)$, $m = 3\delta$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + \delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $54\delta^2 - 100n + 54$. If $m \equiv 1(\,mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 1(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = $[3\delta^2 + 2\delta] + 1$, ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $63\delta^2 - 97n + 54$. If $m \equiv 2(\,mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 2(mod\ 3)$, 1, ψ and ψ^2 are ν (1) =$[3\delta(\delta + 1)] + 1$, ν (ψ) = $[3\delta(\delta + 1)]$ ν (ψ^2) = $[3\delta(\delta + 1)] + 1$ and the induced edges labeled 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $71\delta^2 - 101\delta + 60$, $\mathcal{E}$ (ψ^2) = $71\delta^2 - 101\delta + 60$, $\mathcal{E}$ (ψ) = $71\delta^2 - 101\delta + 61$.

Case (iii) Consider $n \equiv 2(\,mod\ 3)$, $n = 3\delta + 2$. If $m \equiv 0(\,mod\ 3)$, $m = 3\delta$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$then $mn \equiv 0(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = ν (ψ) = ν (ψ^2) = $3\delta^2 + 2\delta$ and 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) =$66\delta^2 - 98n + 53$, $\mathcal{E}$ (ψ^2) = $66\delta^2 - 98n + 52$ .If $m \equiv 1(\,mod\ 3)$, $m = 3\delta + 1$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$ then $mn \equiv 1(mod\ 3)$, 1, ψ and ψ^2 are ν (1) =$[3\delta(\delta + 1)] + 1$, ν (ψ) = $[3\delta(\delta + 1)]$ ν (ψ^2) = $[3\delta(\delta + 1)] + 1$ 1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) = $63\delta^2 - 97n + 54$. If $m \equiv 2(\,mod\ 3)$, $m = 3\delta + 2$, $m, n \geq 3$, $\delta = 1,2,3 \ldots$then $mn \equiv 2(mod\ 3)$, 1, ψ and ψ^2 are ν (1) = $3\delta^2 + 4\delta + 2$, ν (ψ) = ν (ψ^2) = $3\delta^2 + 4\delta + 1$ and1, ψ and ψ^2 are $\mathcal{E}(1)$ = $\mathcal{E}$ (ψ) = $\mathcal{E}$ (ψ^2) =$[72\delta^2 - 88\delta + 56]$. Hence, for all the above cases,| ν $(k) - \nu$ $(w)| \leq 1$ & | $\mathcal{E}$ $(k) - \mathcal{E}$ $(w)| \leq 1$, $k \neq w$ and $k, w \in \{1,\ \psi,\ \psi^2\}$. The lexicographic product of $P_n \circ P_m$, is a cubic roots cordial labeling.

Conclusion

The cubic-root cordial labelings of path graphs under four standard graph-product operations. Our investigation combines constructive labeling schemes, structural analysis of edge interactions, and counting arguments to determine when a cubic-root cordial labeling exists for product graphs formed from two path graphs. For each product type, we provide explicit constructions that extend known cubic-root cordial labelings of the factor paths to their corresponding product graphs. These constructions demonstrate that many product graphs inherit cordiality from their factors under mild parity and size conditions. In particular, when the parity and residue constraints on mmm and nnn are satisfied, our labeling assignments ensure that the induced edge-label distribution meets the cubic-root cordial balance requirements. The four product operations exhibit distinct behaviors. The Cartesian and lexicographic products are generally more amenable to preserving cubic-root cordiality, as their edge-formation rules allow local label patterns on the factor graphs to combine additively. The direct product is the most restrictive, since edges arise only from simultaneous adjacency in both factors, whereas the strong product behaves intermediately between these extremes. These algorithms are practical to implement and yield deterministic labelings whenever our derived sufficient conditions hold. Product graphs frequently model grids, layered networks, and hierarchical compositions, determining when cubic-root cordial labelings exist provides valuable insight for designing label-based encodings and balanced edge-partition schemes in applications such as network addressing, parallel-processing grids, and layered machine-learning architectures. Balanced labeling may help distribute resources or signals more uniformly across edges in such systems.

References

[1] R. Hammack, W. Imrich, and S. Klavžar, *Handbook of Product Graphs*. Boca Raton, FL, USA: Taylor & Francis, (2011).

[2] W. Imrich, S. Klavžar, and D. F. Rall, *Topics in Graph Theory: Graphs and Their Cartesian Product*. Natick, MA, USA: A K Peters, (2008).

[3] A. Kaveh and H. Rahami, "An efficient method for decomposition of regular structures using graph products," *Int. J. Numer. Methods Eng.*, vol. 61, no. 11, pp. 1797–1808, (2004).

[4] A. Kaveh and H. Rahami, "A unified method for eigen decomposition of graph products," *Commun. Numer. Methods Eng.*, vol. 21, no. 7, pp. 377–388, (2005).

[5] A. Kaveh, M. Nikbakht, and H. Rahami, "Improved group theoretic method using graph products for the analysis of symmetric-regular structures," *Acta Mech.*, vol. 210, no. 3–4, pp. 265–289, (2010).

[6] W. Imrich and H. Izbicki, "Associative products of graphs," *Monatsh. Math.*, vol. 80, pp. 277–281, (1975).

[7] G. Sabidussi, "Graph multiplication," *Math. Z.*, vol. 72, pp. 446–457, (1960).

[8] N. G. Gidheeba and A. Vijayan, “Root cube mean cordial labeling of some standard graphs,” *Biosci. Biotechnol. Res. Commun.*, vol. 13, no. 15, pp. 330–337, (2020).

[9] C. Barrientos and S. Minion, “Broader families of cordial graphs,” *Indones. J. Combinatorics*, vol. 5, no. 1, pp. 46–62, (2021).

[10] S. S. Narayanan and M. Vijayaragavan, “Some 3-divisor cordial graphs derived from path,” *Jordan J. Math. Stat.*, vol. 14, no. 2, pp. 335–350, (2021).

[11] S. Nada, A. Abd El-hay, and A. Elrokh, “Total cordial labeling of corona product of paths and second power of fan graph,” *Turk. J. Comput. Math. Educ.*, vol. 13, no. 13, pp. 681–690, (2022).

[12] A. Abd El-hay, A. A. Khalid, and A. Elrokh, “A novel problem and algorithm for solving cordial labeling of some fifth powers of graphs,” *Eur. J. Pure Appl. Math.*, vol. 18, no. 1, pp. 1–14, (2025).

[13] S. Nada, A. Elrokh, and A. Abd El-hay, “On signed product cordial of cone graph and its second power,” *Turk. J. Comput. Math. Educ.*, vol. 13, pp. 597–606, (2022).

[14] A. Abd El-hay and A. Rabie, “Signed product cordial labeling of corona product between paths and second power of fan graphs,” *Ital. J. Pure Appl. Math.*, vol. 48, pp. 287–294, (2022).

[15] W. Zhao, M. Naeem, and I. Ahmad, “Prime cordial labeling of generalized Petersen graph under some graph operations,” *Symmetry*, vol. 14, no. 4, pp. 1–22, (2022).

[16] M. Domingo and A. Racca, “Product cordial graph in the context of some graph operations on crown, helm, and wheel graph,” *Int. Math. Forum*, vol. 17, no. 2, pp. 89–104, (2022).

[17] A. D. Akwu and D. O. A. Ajayi, “Totally antimagic total labeling of ladders, prisms and generalised Petersen graphs,” *J. Discrete Math. Sci. Cryptogr.*, vol. 27, no. 1, pp. 31–44, (2024).

[18] R. Boonklurb, N. Ruamkaew, and S. Singhun, “Directed edge-graceful labeling of digraph consisting of c cycles of the same size,” *J. Discrete Math. Sci. Cryptogr.*, vol. 25, no. 1, pp. 53–72, (2022).

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