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Discrete-time multiparameter fractional optimal stopping problems

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Abstract

We consider fractional, or average reward, optimal stopping problems for multiparameter discrete-time stochastic processes indexed by $\mathbf{N}^d$. We generalize the theory of multiparameter discrete-time optimal stopping problems to include the fractional setting. Moreover we present the characterization of optimal value and optimal stopping point, as well as the efficiency of Denkelbach algorithm for searching an optimal stopping point.

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1. Introduction

This section is devoted to the formulation of the multiparameter optimal stopping problem.

Take d as a natural namber. We consider the set $\mathrm{N}^d \cup \{\infty\}$, where N denotes the set of all nonnegative integers, with the following partial order; for $s = (s_1, s_2, \dots, s_d), t = (t_1, t_2, \dots, t_d) \in \mathrm{N}^d$,

$$s \le t \text{ if and only if } s_i \le t_i \text{ for all } i,$$
$$s < t \text{ if and only if } s \le t \text{ and } s \ne t,$$
$$s < \infty.$$

Let e_i be the vector whose ith coordinate is 1 and all other coordinates are 0. Denote $D(t) = \{t + e_i : i = 1,2, \dots, d\}$, and define $|t| = \sum_{i=1}^{d} t_i$ for $t = (t_1, t_2, \dots, t_d) \in \mathrm{N}^d$.

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Let (Ω, F, P) denote a complete probability space, and $\{F_t, t \in \mathrm{N}^d \cup \{\infty\}\}$ denote a filtration of sub σ-fields of F which fulfills the following conditions;

$$\{\text{every } P-\text{null sets of } F\} \subseteq F_0,$$
$$F_s \subseteq F_t, s \leq t$$
$$F = F_\infty = \sigma(\cup_{t \in N^d} F_t).$$

We call $T: \Omega \to \mathrm{N}^d \cup \{\infty\}$ an $\{F_t\}$-stopping point if $\{T \leq t\} \in F_t$ for any $t \in \mathrm{N}^d$, and a pair $(\{\sigma(n)\}, \tau)$ a tactic if the next conditions are satisfied;

$\sigma(0) = 0$ P-a.e.,
$\sigma(n+1) \in D(\sigma(n))$ P-a.e.,
$\sigma(n+1) is$ $F_{\sigma(n)}$-measurable for any n,
τ is an $\{F_{\sigma(n)}, n \in \mathrm{N}\}$-stopping time,

where $F_T = \{S \in F: S \cap \{T = t\} \in F_t, t \in \mathrm{N}^d\}$ for a stopping point T. $\{\sigma(n)\}$ in the definition of a tactic is also called a predictable increasing path. A stopping point T is accessible if there is a tactic $(\{\sigma(n)\}, \tau)$ for which $T = \sigma(\tau)$ P-a.e.. Let A be the family of all accessible stopping points.

Take $\{R(t), t \in \mathrm{N}^d \cup \{\infty\}\}$ as an integrable stochastic process adapted to the filtration $\{F_t\}$. Then the multiparameter discrete time optimal stopping problem consists of finding a stopping point $T^* \in \mathrm{A}$ (a tactic $(\{\sigma^*(n)\}, \tau^*)$) such that

$$E[R(T^*)] = \sup_{T \in \mathrm{A}} E[R(T)] \ (E[R(\sigma^*(\tau^*))] = \sup_{(\{\sigma(n)\}, \tau)} E[R(\sigma(\tau))]).$$

Several authors have studied optimal stopping problems related to multiparameter discrete-time stochastic processes, for example, Cairoli and Dalang [2], Krengel and Sucheston [4], Lawler and Vanderbei [5], Mandelbaum [6], Mandelbaum and Vanderbei [7], and Mazziotto [8]. Incidentally, Tanaka [13, 14] discuss optimal stopping problems with constraints. Fractional optimization problems have been studied by several authors, for example, Aggarwal, Chandrasekaran and Nair [1], Iwamoto and Fujita [3], Morimoto [9], Ren and Krogh [10], Sawasaki, Kimura and Tanaka [11], Stancu-Minasian [12].

We shall consider fractional optimal stopping problems for multiparameter discerete time stochastic processes indexed by N^d. Moreover we shall discuss the characterization of optimal value and optimal stopping point, as well as the efficiency of Denkelbach algorithm for searching an optimal stopping point.

2. Formulation and Snell envelopes of parametric problem

Assume that $\{Z(t), t \in \mathrm{N}^d \cup \{\infty\}\}$ and $\{W(t), t \in \mathrm{N}^d \cup \{\infty\}\}$ are $\{F_t\}$-adapted integrable multiparameter stochastic processes satisfying the following conditions :

(1) $E[\sup_{t \in \mathrm{N}^d \cup \{\infty\}} Z^+(t)] < \infty$,

(2) There is $K > 0$ for which $W(t) \geq K$ for all $t \in \mathrm{N}^d \cup \{\infty\}$,

(3) $E[\sup_{t \in N^d \cup \{\infty\}} W(t)] < \infty$.

Then we find $T^* \in$ A, which is an optimal stopping point, for which

$$\frac{E[Z(T^*)]}{E[W(T^*)]} = \sup_{T \in \mathrm{A}} \frac{E[Z(T)]}{E[W(T)]}.$$

With the aim of discussing an optimal stopping point for our problem, we introduce the following parametric problem: find, for $\theta \in R$, $T^* \in \mathrm{A}$ such that

$$E[Z(T^*) - \theta W(T^*)] = \sup_{T \in \mathrm{A}} E[Z(T) - \theta W(T)].$$

Now we define new stochastic process, which is the Snell envelope of the parametric problem, by

$$S^\theta(t) = \operatorname*{ess\,sup}_{U \in \mathrm{A}^t} E[Z(U) - \theta W(U) | F_t],$$

$$S^\theta(\infty) = Z(\infty) - \theta W(\infty),$$

where we set $\mathrm{A}^t = \{U \in \mathrm{A} | U \geq t\}$ for $t \in \mathrm{N}^d$.

In the rest of this section we state properties of the Snell envelope and an optimal stopping point for the parametric problem. We refer Cairoli and Dalang [2].

Proposition 2.1: For $\theta \in \mathrm{R}$, $\{S^\theta(t), t \in \mathrm{N}^d \cup \{\infty\}\}$ is the smallest d-parameter supermartingale that dominates $\{Z(t) - \theta W(t), t \in \mathrm{N}^d \cup \{\infty\}\}$. Moreover, we have for $t \in \mathrm{N}^d$

$$S^\theta(t) = \max\{Z(t) - \theta W(t), \max_{u \in D(t)} E[S^\theta(u) | F_t]\}.$$

Proposition 2.2: For $\theta \in \mathrm{R}$, $\{S^\theta(t), t \in \mathrm{N}^d \cup \{\infty\}\}$ is of class D_{A}, that is, $\{S^\theta(T), T \in \mathrm{A}\}$ is uniformly integrable.

Proposition 2.3: Suppose

$$\limsup_{t \to \infty} Z(t) \leq Z(\infty),$$
$$\limsup_{t \to \infty} W(t) = \liminf_{t \to \infty} W(t) = W(\infty).$$

If $\{T_n, n \in \mathrm{N}\} \subseteq \mathrm{A}$ and $T = \lim_{n\to\infty} \uparrow T_n$, then $E[S^\theta(T)] = \lim_{n\to\infty} E[S^\theta(T_n)]$. Specifically, when $\{\Gamma(n), n \in \mathrm{N}\}$ is a predictable increasing path, it follows that $E[S^\theta(\Gamma(\infty-))] = E[S^\theta(\Gamma(\infty))]$ and $S^\theta(\Gamma(\infty-)) = E[S^\theta(\Gamma(\infty)) | \sigma(\cup_{n \in \mathrm{N}} F_{\Gamma(n)})]$, where $S^\theta(\Gamma(\infty-)) = \lim_{n\to\infty} S^\theta(\Gamma(n))$.

Proposition 2.4: We set

$$S^{\theta,(0)}(t) = Z(t) - \theta W(t),$$

$$S^{\theta,(n+1)}(t) = \begin{cases} \max\{Z(t) - \theta W(t), E[Z(\infty) - \theta W(\infty) | F_t], \max_{u \in D(t)} E[S^{\theta,(n)}(u) | F_t]\}, & t \in \mathrm{N}^d \\ Z(\infty) - \theta W(\infty), & t = \infty. \end{cases}$$

Then we have for $t \in \mathrm{N}^d \cup \{\infty\}$

$$\lim_{n \to \infty} S^{\theta,(n)}(t) = S^\theta(t).$$

Proposition 2.5: For $\theta \in \mathrm{R}$, $T \in \mathrm{A}$ being optimal is equivalent to

$$S^{\theta}(T) = Z(t) - \theta W(t) \; and \; E[S^{\theta}(T)] = E[S^{\theta}(0)].$$

Proposition 2.6: Let $T \in \mathrm{A}$ be optimal and $S \in \mathrm{A}$ satisfy $S \leq T$. S being optimal is equivalent to $S^{\theta}(S) = X(S) - \theta Y(S)$. Particularly, for every predictable increasing path $\{\Gamma(n), n \in \mathrm{N}\}$ with $\Gamma(|T|) = T$, $\Gamma(\tau)$ is optimal, where $\tau = \inf\{n | n \in \mathrm{N} \cup \{\infty\}, \Gamma(n) \in \{t | t \in \mathrm{N}^d \cup \{\infty\}, S^{\theta}(t) = Z(t) - \theta W(t)\}\}$.

Theorem 2.1: *Suppose*

$$\limsup_{t\to\infty} Z(t) \leq Z(\infty),$$
$$\limsup_{t\to\infty} W(t) = \liminf_{t\to\infty} W(t) = W(\infty).$$

Let $\{\Gamma(n), n \in \mathrm{N}\}$ be an optimizing increasing path, that is,

$$E[S^{\theta}(\Gamma(n+1)) | F_{\Gamma(n)}] = \max_{u \in D(\Gamma(n))} E[S^{\theta}(u) | F_{\Gamma(n)}].$$

Set

$$\xi = \inf\{n \in \mathrm{N} \cup \{\infty\} | \Gamma(n) \in \{t | S^{\theta}(t) = Z(t) - \theta W(t)\}\},$$
$$S = \Gamma(\xi),$$
$$\tau = \inf\{n \in \mathrm{N} | S^{\theta}(\Gamma(n)) > E[S^{\theta}(\Gamma(n+1)) | F_{\Gamma(n)}]\}\},$$
$$T = \Gamma(\tau).$$

Then S and T are optimal.

Theorem 2.2: *Suppose that*

$$E[\sup_{t\in\mathrm{N}^d} Z^+(t)] < \infty,$$
$$W(t) \geq K \text{ for all } t \in \mathrm{N}^d,$$
$$E[\sup_{t\in\mathrm{N}^d} W(t)] < \infty.$$

Let $\{\Gamma(n), n \in \mathrm{N}\}$ be an optimizing tactic and

$$\xi = \inf\{n \in \mathrm{N} \cup \{\infty\} | \Gamma(n) \in \{t | t \in \mathrm{N}^d \cup \{\infty\}, S^{\theta}(t) = Z(t) - \theta W(t)\}\},$$
$$S = \Gamma(\xi).$$

If $P(S < \infty) = 1$, then S is optimal in the class $\mathrm{A}_f := \{T \in \mathrm{A} | P(T < \infty) = 1\}$. In particular, if

$$\limsup_{t\to\infty} Z(t) = -\infty$$

then $P(S < \infty) = 1$. Moreover if $P(\tau < \infty) = 1$ where τ is defined in the previous theorem, then $\Gamma(\tau)$ is optimal in A_f.

3. Parametric method

For $\theta \in R$, we set $d(\theta) = \sup_{T\in\mathrm{A}} E[Z(t) - \theta W(t)]$.

Proposition 3.1: The function $\theta \mapsto d(\theta)$ is convex.

Proof: For each $\theta_1 < \theta_2,\ \lambda \in [0,1]$, we have

$$\begin{aligned} d(\lambda\theta_1 + (1-\lambda)\theta_2) &= \sup_{T\in A} E[Z(T) - (\lambda\theta_1 + (1-\lambda)\theta_2)W(T)] \\ &= \sup_{T\in A} E[\lambda(Z(T) - \theta_1 W(T)) + (1-\lambda)(Z(T) - \theta_2 W(T))] \\ &\leq \sup_{T\in A} E[\lambda(Z(T) - \theta_1 W(T))] + \sup_{T\in A} E[(1-\lambda)(Z(T) - \theta_2 W(T))] \\ &= \lambda d(\theta_1) + (1-\lambda) d(\theta_2). \end{aligned}$$

Proposition 3.2: The function $\theta \mapsto d(\theta)$ is strictly decreasing.

Proof: For any $\theta_1 < \theta_2$ and $T \in A$, we have

$$\begin{aligned} Z(T) - \theta_1 W(T) &= Z(T) - \theta_2 W(T) + (\theta_2 - \theta_1) W(T) \\ &\geq Z(T) - \theta_2 W(T) + (\theta_2 - \theta_1) K \\ E[Z(T) - \theta_1 W(T)] &\geq E[Z(T) - \theta_2 W(T)] + (\theta_2 - \theta_1) K. \end{aligned}$$

Therefore we obtain $d(\theta_1) \geq d(\theta_2) + (\theta_2 - \theta_1)K > d(\theta_2)$. W

Proposition 3.3: $\lim_{\theta\to\infty} d(\theta) = -\infty$ *and* $\lim_{\theta\to-\infty} d(\theta) = \infty$.

Proof: We may assume that $\theta > 0$.

$$\begin{aligned} d(\theta) &= \sup_{T\in A} E[Z(T) - \theta W(T)] \leq E[\sup_{t\in N\cup\{\infty\}} Z^+(T) - \theta K] \\ &\to -\infty\ (\theta \to \infty). \end{aligned}$$

Next we may assume that $\theta < 0$.

$$\begin{aligned} d(\theta) &= \sup_{T\in A} E[Z(T) - \theta W(T)] \geq \sup_{T\in A} E[Z(T) - \theta K] \\ &= E[Z(T)] - \theta K \\ &\to \infty\ (\theta \to \infty). \end{aligned}$$

Proposition 3.4: There exists only one $\overline{\theta}$ such that $d(\overline{\theta}) = 0$.

Proposition 3.5: The function $\theta \mapsto d(\theta)$ is Lipschitz continuous.

Proof: For any θ_1, θ_2, we have

$$\begin{aligned} |d(\theta_1) - d(\theta_2)| &= |\sup_{T\in A} E[Z(T) - \theta_1 W(T)] - \sup_{T\in A} E[Z(T) - \theta_2 W(T)]| \\ &\leq \sup_{T\in A} |E[Z(T) - \theta_1 W(T)] - E[Z(T) - \theta_2 W(T)]| \\ &= \sup_{T\in A} |\theta_1 - \theta_2| E[W(T)] \\ &\leq |\theta_1 - \theta_2| E[\sup_{t\in N^d\cup\{\infty\}} W(t)]. \end{aligned}$$

Proposition 3.6: We set $\theta = \frac{E[Z(T)]}{E[W(T)]}, T \in A$. Then $d(\theta) \geq 0$.

Proof: For any $T \in A$, we have

$$d(\theta) = \sup_{T \in \mathrm{A}} E[Z(T) - \theta W(T)] \geq E[Z(T) - \theta W(T)] = 0.$$

4. Optimal value

Theorem 4.1: *Suppose*

$$\limsup_{t \to \infty} Z(t) \leq Z(\infty),$$
$$\limsup_{t \to \infty} W(t) = \liminf_{t \to \infty} W(t) = W(\infty).$$

Let $\{\Gamma(n), n \in \mathrm{N}\}$ be an optimizing tactic for the Snell envelope $\{Z^{\overline{\theta}}(t), t \in \mathrm{N}^d \cup \{\infty\}\}$ and

$$\xi = \inf\{n \in \mathrm{N} \cup \{\infty\} | \Gamma(n) \in \{t \in \mathrm{N}^d \cup \{\infty\} | Z^{\overline{\theta}}(t) = Z(t) - \overline{\theta} W(t)\}\},$$
$$S = \Gamma(\xi).$$

Then we obtain

$$\overline{\theta} = \frac{E[Z(S)]}{E[W(S)]} = \sup_{T \in \mathrm{A}} \frac{E[Z(T)]}{E[W(T)]}.$$

Proof: For any $T \in \mathrm{A}$, we have

$$0 = d(\overline{\theta}) = E[X(S) - \overline{\theta} Y(S)] \geq E[Z(t) - \overline{\theta} W(t)].$$

Then we obtain

$$\overline{\theta} = \frac{E[X(S)]}{E[Y(S)]} \geq \frac{E[Z(t)]}{E[W(t)]}.$$

Remark 4.1: Assume that $\{Z(t), t \in \mathrm{N}^d\}$ and $\{W(t), t \in \mathrm{N}^d\}$ are integrable $\{F_t\}$-adapted stochastic processes which fulfills the next conditions

(X 2.1) $E[\sup_{t \in \mathrm{N}^d} Z^+(t)] < \infty$.

(X 2.2) There is $K > 0$ for whcih $W(T) \geq K$ for each $t \in \mathrm{N}^d$.

(X 2.3) $E[\sup_{t \in \mathrm{N}^d} W(t)] < \infty$.

Then we consider two problems: finding $T^* \in \mathrm{A}_f$ for whcih

$$\frac{E[Z(T^*)]}{E[W(T^*)]} = \sup_{T \in \mathrm{A}_f} \frac{E[Z(T)]}{E[W(T)]};$$

finding $T^* \in \mathrm{A}_f$ for whcih

$$E[Z(T^*) - \theta W(T^*)] = \sup_{T \in \mathrm{A}_f} E[Z(T) - \theta W(T)].$$

Now we define the Snell envelope of the parametric problem by

$$S^\theta(t) = \operatorname*{ess\,sup}_{U \in \mathrm{A}_f^t} E[X(U) - \theta Y(U) | F_t],$$

where we set $\mathrm{A}_f^t = \{U \in \mathrm{A}_f | U \geq t\}$ for $t \in \mathrm{N}^d$.

By replacing A by A_f, the propositions in this section hold true.

Theorem 4.2: *Let* $\{\Gamma(n), n \in \mathrm{N}\}$ *be an optimizing tactic for the Snell envelope* $\{Z^{\overline{\theta}}(t), t \in \mathrm{N}^d\}$ *and*

$\xi = \inf\{n \in \mathrm{N} \cup \{\infty\} | \Gamma(n) \in \{t \in \mathrm{N}^d \cup \{\infty\} | Z^{\overline{\theta}}(t) = Z(t) - \overline{\theta}W(t)\}\}$,
$S = \Gamma(\xi)$.

When $\limsup_{t\to\infty} Z(t) = -\infty$, it follows that

$$\overline{\theta} = \frac{E[X(S)]}{E[Y(S)]} = \sup_{T\in \mathrm{A}_f} \frac{E[Z(T)]}{E[W(T)]}.$$

5. Dinkelbach algorithm

We consider the following algorithm. Let $\delta > 0$ be fixed.

(1) For $T_1 \in \mathrm{A}(\mathrm{A}_f)$ such that $\theta_1 = \frac{E[Z(T_1)]}{E[W(T_1)]}$, we have $d(\theta_1) \geq 0$.
 (i) If $d(\theta_1) < \delta$, then stop. In particular, T_1 is exactly optimal if $d(\theta_1) = 0$.
 (ii) If $d(\theta_1) \geq \delta$, then set $\theta_2 = \theta_1 = \frac{E[Z(T_1)]}{E[W(T_1)]}$ and go to (2).

(2) Find $T_2 \in \mathrm{A}(\mathrm{A}_f)$ such that $d(\theta_2) = \sup_{T\in \mathrm{A}(\mathrm{A}_f)} E[Z(t) - \theta W(t)] = E[Z(T_2) - \theta_2 W(T_2)]$.
 (i) If $d(\theta_2) < \delta$, then stop. In particular, T_2 is exactly optimal if $d(\theta_2) = 0$.
 (ii) If $d(\theta_2) \geq \delta$, then set $\theta_3 = \frac{E[Z(T_2)]}{E[W(T_2)]}$ and go to (3).

(3) Find $T_3 \in A(A_f)$ such that $d(\theta_3) = \sup_{T\in \mathrm{A}(\mathrm{A}_f)} E[Z(t) - \theta W(t)] = E[Z(T_3) - \theta_3 W(T_3)]$.
 (i) If $d(\theta_3) < \delta$, then stop. In particular, T_3 is exactly optimal if $d(\theta_3) = 0$.
 (ii) If $d(\theta_3) \geq \delta$, then set $\theta_4 = \frac{E[Z(T_3)]}{E[W(T_3)]}$ and go to (k).

(k) Find $T_k \in \mathrm{A}(\mathrm{A}_f)$ such that $d(\theta_k) = \sup_{T\in \mathrm{A}(\mathrm{A}_f)} E[Z(t) - \theta W(t)] = E[Z(T_k) - \theta_k W(T_k)]$.
 (i) If $d(\theta_k) < \delta$, then stop. In particular, T_k is exactly optimal if $d(\theta_k) = 0$.
 (ii) If $d(\theta_k) \geq \delta$, then set $\theta_{k+1} = \frac{E[Z(T_k)]}{E[W(T_K)]}$, $k := k + 1$ and go to (k).

Proposition 5.1: For any k such that $d(\theta_k) \geq \delta$, we have $\theta_{k+1} > \theta_k$.

Proof:

$$0 < d(\theta_k) = E[Z(T_k) - \theta_k W(T_k)] = \theta_{k+1} E[W(T_k)] - E[\theta_k W(T_k)]$$
$$= (\theta_{k+1} - \theta_k) E[W(T_k)],$$

from which we obtain $\theta_{k+1} > \theta_k$. W

Proposition 5.2: $\lim_{k\to\infty} \theta_k = \overline{\theta}$.

Proof: By Proposition 5.1, there exists $\theta := \lim_{k\to\infty}\theta_k$. Suppose that $\theta < \overline{\theta}$. From the proof of Proposition 5.1, we have

$$0 < d(\theta_k) = E[Z(T_k) - \theta_k W(T_k)] = \theta_{k+1}E[W(T_k)] - E[\theta_k W(T_k)]$$
$$= (\theta_{k+1} - \theta_k)E[\sup_{t\in N^d \cup\{\infty\}} W(t)],$$

and then $\lim_{k\to\infty} d(\theta_k) = 0$. Therefore we obtain $d(\theta) = 0$. On the other hand, from $\theta < \overline{\theta}$, we have $d(\theta) < d(\overline{\theta})$. This is a contradiction. By the same arguments, we can conclude $\theta = \overline{\theta}$. W

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