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Differential games with fractional payoffs

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Abstract

This paper provides a formulation for new differential games with fractional payoffs. By applying the standard argument used in mathematical programming theory, we show that the saddle point function and the equilibrium point exist in these games.

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1. Introduction

Optimization theory under fractional criteria is able to be traced back to Dinkelbach [7], who proposed the parametric method and the so-called Dinkelbach algorithm, and also studied the relation between optimal value in the fractional problem and one in the parametric problem. Von Neumann [18] studied the relation the expanding economic model and optimization theory under fractional criteria.

Besides these, there are many studies on mathematical optimization theory under fractional criteria. For example, Stancu-Minasian [17], Schroeder [16], Aggarwal, Chandrasekaran and Nair [2, 3], Sawasaki, Kimura and Tanaka [15], Lai [11], Fox [8], Iwamoto and Fujita [10], Ren and Krogh [14], Morimoto [12], and Bhatt [6]. The so-called parametric method is well known as the standard method in order to solve fractional optimization problems. For example, Abdelfattah [1].

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On the other hand, deterministic and stochastic differential games studied in this paper, included in the game theory, have been studied bay many authors, for example, Bensoussan and Friedman [4], Bensoussan and Lions [5], Friedman [9], and Nisio [13]. In these papers and monograph, the value of game is characterized by variational inequalities and the viscosity solution of the Issacs equation.

The objective of this paper is to introduce a novel mathematical model combining deterministic differential games and fractional criteria, and to study the the relation between the saddle point function and the equilibrium point in the original fractional games and ones in the parametric games.

The organization of this paper follows : we provide the general formulation of differential game and the definition of the value of game in section 2. In section 3 we give new formulation of fractional differential games without stopping times and introduce their upper game value function, lower value function and the saddle point function. Moreover the properties of these functions are investigated. In section 4 we give new formulation of fractional differential games with deterministic stopping times and introduce their upper game value function, lower game value function and the saddle point function. Moreover the properties of these functions are investigated.

2. Differential game

This paper addresses a differential system m equations

$$\frac{dX}{dt} = f(t, X(t), Y(t), Z(t)) \ (u_0 \le t \le v_0). \tag{1}$$

with an initial condition

$$X(u_0) = X_0. \tag{2}$$

Let $\mathbf{Y}$ and $\mathbf{Z}$ be compact subsets of $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$ respectively. We denote the sets $\mathbf{F}$ and $\mathbf{G}$ of all control functions of each player by the following :

$$\mathbf{F} = \{Y : [u_0, v_0] \to \mathbf{Y} \mid Y : \text{measurable}\},$$
$$\mathbf{G} = \{Z : [u_0, v_0] \to \mathbf{Z} \mid Z : \text{measurable}\}.$$

Now we assume the following :

(A1) The function $f(t, X, Y, Z)$ is continuous with respect to $(t, X, Y, Z) \in [u_0, v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$.

(A2) There is a nonnegative function $k(t)$ with $\int_{u_0}^{v_0} k(t)dt < \infty$, and

$$|f(t, X, Y, Z)| \le k(t)(1 + |x|)$$

for any $(t, X, Y, Z) \in [u_0, v_0] \times \mathbf{R}^m \times \mathbf{Y} \times \mathbf{Z}$.

(A3) For each $r > 0$, there is a nonnegative function $k_r(t)$ with $\int_{u_0}^{v_0} k_r(t)dt < \infty$, and

$$|f(t, X, Y, Z) - f(t, \bar{X}, Y, Z)| \le k_R(t)(1 + |X - \bar{X}|)$$

for any $t \in [u_0, v_0], Y \in \mathbf{Y}, Z \in \mathbf{Z}, |X| \le r, |\bar{X}| \le r$, here $|X|$ denotes the Euclidean norm of $X \in \boldsymbol{R}^m$.

Under the assumptions (A1), (A2) and (A3), for any $Y \in \mathbf{F}, Z \in \mathbf{G}$, we can find one and only one solution solution $X(t)$ of (1) and (2).

Next we introduce the payoff functional

$$P(Y,Z)(X_0) = \mu_{u_0,v_0}(X) + \int_{u_0}^{v_0} h(t,X(t),Y(t),Z(t))dt.$$

Moreover we assume the following :

(B1) Th function $h(t,X,Y,Z)$ is continuous with respect to $(t,X,Y,Z) \in [u_0,v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$.

(B2) Let $\mu_{u_0,v_0}(X)$ be a given functional defined on X_{u_0,v_0}. There is a constant $K > 0$ with

$$|\mu_{u_0,v_0}(X)| \leq K,$$

here X_{u_0,v_0} denotes the collection of all solution $X(t)$ of (1) and (2).

The player y will aim to maximize $P(Y,Z)(X_0)$, while the player z will aim to minimize $P(Y,Z)(X_0)$.

At last we state the important notion in the game theory. The pair $(Y^*,Z^*) \in \mathbf{F} \times \mathbf{G}$ is an equilibrium point of this game if and only if

$$\sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} P(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} P(Y,Z)(X_0) = P(Y^*,Z^*)(X_0),$$

and $P(Y^*,Z^*)(X_0)$ represents this game value. And also the pair $(Y^*,Z^*) \in \mathbf{F} \times \mathbf{G}$ represents a saddle point of this game if and only if

$$P(Y,Z^*)(X_0) \leq P(Y^*,Z^*)(X_0) \leq P(Y^*,Z)(X_0)$$

holds for all $Y \in \mathbf{F}$, $Z \in \mathbf{G}$.

3. Fractional differential game

3.1 *Formulation*

So as to formulate a fractional differential game, we introduce two payoff functionals

$$U(Y,Z)(X_0) = \mu^1_{u_0,v_0}(X) + \int_{u_0}^{v_0} h^1(t,X(t),Y(t),Z(t))dt,$$

$$V(Y,Z)(X_0) = \mu^2_{u_0,v_0}(X) + \int_{u_0}^{v_0} h^2(t,X(t),Y(t),Z(t))dt,$$

and then the fractional payoff functional

$$W(Y,Z)(X_0) = \frac{U(Y,Z)(X_0)}{V(Y,Z)(X_0)}.$$

For this section we suppose (A1), (A2) and (A3) in the previous section and moreover the following :

(C1) The functions $h^1(t,X,Y,Z)$ and $h^2(t,X,Y,Z)$ are continuous in $(t,X,Y,Z) \in [u_0,v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$.

(C2) For any $(t,X,Y,Z) \in [u_0,v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$, $h^2(t,X,Y,Z) > 0$.

The functionals $\mu^1_{u_0,v_0}(X)$ and $\mu^2_{u_0,v_0}(X)$ are defined on X_{u_0,v_0} and there is

a positive constant K such that

$$|\mu^1_{u_0,v_0}(X)| \leq K,$$
$$0 < \mu^2_{u_0,v_0}(X) \leq K$$

for any $X \in X_{u_0,v_0}$.

The player y will aim to maximize $W(Y,Z)(X_0)$, while the player z will aim to minimize $W(Y,Z)(X_0)$. We call this game the fractional differential game (FDG).

We introduce the parameter function :

$$\phi: \boldsymbol{R}^m \to \boldsymbol{R},$$

and denote the upper game value function and lower game value function for (FDG) by

$$\overline{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0),$$
$$\underline{\phi}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0),$$

respectively. It is obvious that $\overline{\phi}(X_0) \geq \underline{\phi}(X_0)$, and the interval $[\underline{\phi}(X_0), \overline{\phi}(X_0)]$ is called the duality gap.

Next, in order to formulate a parameterized differential game, we also introduce the payoff function with parameter $\phi(X_0)$:

$$T_\phi(Y,Z)(X_0) = U(Y,Z)(X_0) - \phi(X_0)V(Y,Z)(X_0),$$

and denote the upper game value function and lower game value function, which is called the differential game with parameter ϕ (DGP_ϕ),

$$\overline{T}_\phi(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_\phi(Y,Z)(X_0),$$
$$\underline{T}_\phi(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_\phi(Y,Z)(X_0).$$

And then the interval $[\underline{T}_\phi(X_0), \overline{T}_\phi(X_0)]$ is the duality gap of (DGP_ϕ).

Definition 3.1: If the duality gap of (FDG) equals zero, then (FDG) is said to have the value function

$$\overline{\phi}(X_0) = \underline{\phi}(X_0) = \phi^*(X_0).$$

Furthermore if there is $Y^* \in F$ satisfying

$$\overline{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0),$$

then we call Y^* a maximizer of $W(Y,Z)(X_0)$. Similarly if there exists $Z^* \in G$ such that

$$\underline{\phi}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) = \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0),$$

then we call Z^* a minimizer of $W(Y,Z)(X_0)$.

3.2 *Upper and lower value functions*

Proposition 3.1:

(1) For parameter functions $\phi_1(X_0)$ and $\phi_2(X_0)$, if $\phi_1(X_0) > \phi_2(X_0) \geq 0$, then $\bar{T}_{\phi_1}(X_0) \leq \bar{T}_{\phi_2}(X_0)$.

(2) If $\bar{T}_{\phi}(X_0) < 0$, then $\phi(X_0) \geq \bar{\phi}(X_0)$.

(3) If $\bar{T}_{\phi}(X_0) > 0$, then $\phi(X_0) \leq \bar{\phi}(X_0)$.

(4) If $\phi(X_0) > \bar{\phi}(X_0)$, then $\bar{T}_{\phi}(X_0) \leq 0$.

(5) If $\phi(X_0) < \bar{\phi}(X_0)$, then $\bar{T}_{\phi}(X_0) \geq 0$.

Proof:

(1) If $\phi_1(X_0) > \phi_2(X_0) \geq 0$, since $V(Y,Z)(X_0) > 0$, then for any $(Y,Z) \in \mathbf{F} \times \mathbf{G}$

$$\phi_1(X_0)V(Y,Z)(X_0) > \phi_2(X_0)V(Y,Z)(X_0),$$

and then

$$U(Y,Z)(X_0) - \phi_1(X_0)V(Y,Z)(X_0) < U(Y,Z)(X_0) - \phi_2(X_0)V(Y,Z)(X_0),$$

that is,

$$T_{\phi_1}(Y,Z)(X_0) < T_{\phi_2}(Y,Z)(X_0).$$

Hence we have

$$\bar{T}_{\phi_1}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi_1}(Y,Z)(X_0) \leq \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi_2}(Y,Z)(X_0) = \bar{T}_{\phi_2}(X_0).$$

(2) Since $\bar{T}_{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi}(Y,Z)(X_0) < 0$, we can find $\bar{Z} \in \mathbf{G}$ satisfying

$$\sup_{Y\in\mathbf{F}} T_{\phi}(Y,\bar{Z})(X_0) < 0,$$

that is, for each $Y \in \mathbf{F}$,

$$T_{\phi}(Y,\bar{Z})(X_0) = U(Y,\bar{Z})(X_0) - \phi(X_0)V(Y,\bar{Z})(X_0) < 0.$$

And then for each $Y \in \mathbf{F}$,

$$W(Y,\bar{Z})(X_0) = \frac{U(Y,\bar{Z})(X_0)}{V(Y,\bar{Z})(X_0)} < \phi(X_0),$$

$$\sup_{Y\in\mathbf{F}} W(Y,\bar{Z})(X_0) \leq \phi(X_0),$$

Hence we have

$$\bar{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) \leq \phi(X_0).$$

(3) Since $\bar{T}_{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi}(Y,Z)(X_0) > 0$, for each $Z \in G$,

$$\sup_{Y\in\mathbf{F}} T_{\phi}(Y,Z)(X_0) > 0.$$

There exists $Y_z \in F$ such that

$$T_{\phi}(Y_Z,Z)(X_0) = U(Y_Z,Z)(X_0) - \phi(X_0)V(Y_Z,Z)(X_0) > 0.$$

Hence we have

$$W(Y_Z, Z)(X_0) = \frac{U(Y_Z,Z)(X_0)}{V(Y_Z,Z)(X_0)} > \phi(X_0).$$

And then for $Z \in \mathbf{G}$

$$\sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) \geq W(Y_Z,Z)(X_0) > \phi(X_0),$$

and

$$\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) \geq \phi(X_0).$$

Therefore we obtain

$$\overline{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) \geq \phi(X_0).$$

(4) If $\phi(X_0) > \overline{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0)$, then

$$\phi(X_0) > \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0),$$

or

$$\phi(X_0) > \sup_{Y\in\mathbf{F}} W(Y,\bar{Z})(X_0)$$

for some $\bar{Z} \in \mathbf{G}$. That is, for each $Y \in \mathbf{F}$,

$$\phi(X_0) > W(Y,\bar{Z})(X_0),$$
$$T_\phi(Y,\bar{Z})(X_0) = U(Y,\bar{Z})(X_0) - \phi(X_0)V(Y,\bar{Z})(X_0) < 0,$$
$$\sup_{Y\in\mathbf{F}} T_\phi(Y,\bar{Z})(X_0) \leq 0.$$

Therefore we have

$$0 \geq \sup_{Y\in\mathbf{F}} T_\phi(Y,\bar{Z})(X_0) \geq \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_\phi(Y,Z)(X_0) = \overline{T}_\phi(X_0).$$

(5) If $\overline{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) > \phi(X_0)$, then for each $Z \in \mathbf{G}$

$$\sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) > \phi(X_0),$$

or , there exists $Y_z \in F$ such that

$$W(Y_Z,Z)(X_0) > \phi(X_0).$$

Then

$$T_\phi(Y_Z,Z)(X_0) = U(Y_Z,Z)(X_0) - \phi(X_0)V(Y_Z,Z)(X_0) > 0,$$

that is,

$$\sup_{Y\in\mathbf{F}} T_\phi(Y,Z)(X_0) \geq T_\phi(Y_Z,Z)(X_0) > 0.$$

Hence we have

$$\overline{T}_\phi(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_\phi(Y,Z)(X_0) \geq 0.$$

Proposition 3.2:

(1) For parameter functions $\phi_1(X_0)$ and $\phi_2(X_0)$, if $\phi_1(X_0) > \phi_2(X_0) \geq 0$, then $\underline{T}_{\phi_1}(X_0) \leq \underline{T}_{\phi_2}(X_0)$.

(2) If $\underline{T}_{\phi}(X_0) < 0$, then $\phi(X_0) \geq \underline{\phi}(X_0)$.

(3) If $\underline{T}_{\phi}(X_0) > 0$, then $\phi(X_0) \leq \underline{\phi}(X_0)$.

(4) If $\phi(X_0) > \underline{\phi}(X_0)$, then $\underline{T}_{\phi}(X_0) \leq 0$.

(5) If $\phi(X_0) < \underline{\phi}(X_0)$, then$\underline{T}_{\phi}(X_0) \geq 0$.

Proof:

(1) If $\phi_1(X_0) > \phi_2(X_0) \geq 0$, since $V(Y,Z)(X_0) > 0$, then for any $(Y,Z) \in \mathbf{F} \times \mathbf{G}$

$$\phi_1(X_0)V(Y,Z)(X_0) > \phi_2(X_0)V(Y,Z)(X_0),$$

and then

$$U(Y,Z)(X_0) - \phi_1(X_0)V(Y,Z)(X_0) < U(Y,Z)(X_0) - \phi_2(X_0)V(Y,Z)(X_0),$$

that is,

$$T_{\phi_1}(Y,Z)(X_0) < T_{\phi_2}(Y,Z)(X_0).$$

Hence we have

$$\underline{T}_{\phi_1}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi_1}(Y,Z)(X_0) \leq \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi_2}(Y,Z)(X_0) = \underline{T}_{\phi_2}(X_0).$$

(2) Since $\underline{T}_{\phi}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi}(Y,Z)(X_0) < 0$, for each $Y \in \mathbf{F}$

$$\inf_{Z\in\mathbf{G}} T_{\phi}(Y,Z)(X_0) < 0.$$

We can find $Z_Y \in \mathbf{G}$ satisfying

$$T_{\phi}(Y,Z_Y)(X_0) = U(Y,Z_Y)(X_0) - \phi(X_0)V(Y,Z_Y)(X_0) < 0,$$

and then

$$W(Y,Z_Y)(X_0) = \frac{U(Y,Z_Y)(X_0)}{V(Y,Z_Y)(X_0)} < \phi(X_0).$$

Hence we have for each $Y \in F$

$$\inf_{Z\in G} W(Y,Z)(X_0) \leq W(Y,Z_Y)(X_0) < \phi(X_0),$$

and

$$\sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) \leq \phi(X_0).$$

Therefore we obtain

$$\underline{\phi}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) \leq \phi(X_0).$$

(3) Since $\underline{T}_{\phi}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi}(Y,Z)(X_0) > 0$, there exists $\bar{Y} \in F$ such that

$$\inf_{Z\in\mathbf{G}} T_{\phi}(\bar{Y},Z)(X_0) > 0.$$

And for each $Z \in \mathbf{G}$

$$T_{\phi}(\bar{Y},Z)(X_0) = U(\bar{Y},Z)(X_0) - \phi(X_0)V(\bar{Y},Z)(X_0) > 0.$$

That is, for each $Z \in \mathbf{G}$

$$W(\bar{Y},Z)(X_0)=\frac{U(\bar{Y},Z)(X_0)}{V(\bar{Y},Z)(X_0)}>\phi(X_0),$$

$$\inf_{Z\in\mathbf{G}} W(\bar{Y},Z)(X_0)\geq\phi(X_0),$$

$$\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} W(Y,Z)(X_0)\geq\inf_{Z\in\mathbf{G}} W(\bar{Y},Z)(X_0)\geq\phi(X_0).$$

Hence we have

$$\underline{\phi}(X_0)=\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} W(Y,Z)(X_0)\geq\phi(X_0).$$

(4) If $\phi(X_0)>\underline{\phi}(X_0)=\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} W(Y,Z)(X_0)$, then for $Y\in\mathbf{F}$

$$\inf_{Z\in\mathbf{G}} W(Y,Z)(X_0)<\phi(X_0).$$

We can find $Z_Y\in\mathbf{G}$ satisfying

$$W(Y,Z_Y)(X_0)<\phi(X_0),$$

and

$$T_\phi(Y,Z_Y)(X_0)=U(Y,Z_Y)(X_0)-\phi(X_0)V(Y,Z_Y)(X_0)<0,$$

that is

$$\inf_{Z\in\mathbf{G}} T_\phi(Y,Z)(X_0)\leq T_\phi(Y,Z_Y)(X_0)<0.$$

Hence we have

$$\underline{T_\phi}(X_0)=\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} T_\phi(Y,Z)(X_0)\geq 0.$$

(5) If $\phi(X_0)<\underline{\phi}(X_0)=\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} W(Y,Z)(X_0)$, then

$$\phi(X_0)<\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} W(Y,Z)(X_0).$$

There exists $\bar{Y}\in\mathbf{F}$ such that

$$\phi(X_0)<\inf_{Z\in\mathbf{G}} W(\bar{Y},Z)(X_0),$$

that is, for each $Z\in\mathbf{G}$

$$\phi(X_0)<W(\bar{Y},Z)(X_0),$$

and

$$T_\phi(\bar{Y},Z)(X_0)=U(\bar{Y},Z)(X_0)-\phi(X_0)V(\bar{Y},Z)(X_0)>0.$$

Furthermore there exists $\bar{Y}\in\mathbf{F}$ such that for each $Z\in\mathbf{G}$

$$\inf_{Z\in\mathbf{G}} T_\phi(\bar{Y},Z)(X_0)\geq 0.$$

Hence we have

$$\underline{T_\phi}(X_0)=\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} T_\phi(Y,Z)(X_0)\geq\inf_{Z\in\mathbf{G}} T_\phi(\bar{Y},Z)(X_0)\geq 0.$$

3.3 *Saddle point function*

Theorem 3.1*: Consider* $Y^*\in\mathbf{F}$ *as a maximizer of the game(FDG).*

(1) $\overline{\phi}(X_0)=\underline{\phi}(X_0)=:\phi^*(X_0)$.

(2) *If* $\bar{T}_{\phi^*}(X_0) \leq 0$, *then* Y^* *is a maximizer of the game* (DGP_{ϕ^*}).

Proof:

(1) By the definitions of $\bar{\phi}(X_0)$ and $\underline{\phi}(X_0)$, we have $\bar{\phi}(X_0) \geq \underline{\phi}(X_0)$. On the contrary, since $Y^* \in \mathbf{F}$ is a maximizer of (FDG), we have

$$\bar{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0)$$
$$\leq \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) = \underline{\phi}(X_0).$$

(2) Since $Y^* \in \mathbf{F}$ is a maximizer, it follows that for any $Z \in \mathbf{G}$

$$\phi^*(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) \leq W(Y^*,Z)(X_0).$$

This implies that for any $Z \in \mathbf{G}$

$$0 \leq T_{\phi^*}(Y^*,Z)(X_0) \leq \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z)(X_0).$$

Hence we have

$$0 \leq \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y^*,Z)(X_0) \leq \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z)(X_0) = \bar{T}_{\phi^*}(X_0) \leq 0,$$

and

$$\inf_{Z\in\mathbf{G}} T_{\phi^*}(Y^*,Z)(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z)(X_0).$$

Therefore Y^* is a maximizer of (DGP_{ϕ^*}).

Corollary 3.1: *Consider* $(Y^*,Z^*) \in \mathbf{F}\times\mathbf{G}$ *as a saddle point of (FDG).*

(1) $T_{\phi^*}(Y^*,Z^*)(X_0) = 0$.

(2) (Y^*,Z^*) *is a saddle point of* (DGP_{ϕ^*}).

Proof:

(1) Let (Y^*,Z^*) be a saddle point of (FDG). Then we have

$$W(Y,Z^*)(X_0) \leq W(Y^*,Z^*)(X_0) \leq W(Y^*,Z)(X_0),$$
$$\sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) \leq \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) \leq \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0)$$
$$= W(Y^*,Z^*)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) \leq \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0).$$

Then we have

$$\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) = W(Y^*,Z^*)(X_0),$$
$$\sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) = \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) = W(Y^*,Z^*)(X_0),$$

from which $Y^* \in \mathbf{F}$ and $Z^* \in \mathbf{G}$ are a maximizer and a minimizer of (FDG) respectively. Hence we get from Theorem 3.1(1)

$$(X_0) = \underline{\phi}(X_0) = \phi^*(X_0),$$
$$\bar{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) = W(Y^*,Z^*)(X_0),$$

that is,

$$\phi^*(X_0) = W(Y^*, Z^*)(X_0).$$

Therefore we obtain

$$T_{\phi^*}(Y^*, Z^*)(X_0) = 0.$$

(2) From Theorem 3.1(1), we have

$$\overline{\phi}(X_0) = \phi^*(X_0),$$

$$\overline{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) = W(Y^*,Z^*)(X_0).$$

That is,

$$\inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) = \phi^*(X_0).$$

And also for any $Z \in \mathbf{G}$,

$$W(Y^*,Z)(X_0) \geq \phi^*(X_0),$$

$$T_{\phi^*}(Y^*,Z)(X_0) \geq 0,$$

that is,

$$\inf_{Z\in\mathbf{G}} T_{\phi^*}(Y^*,Z)(X_0) = 0 = T_{\phi^*}(Y^*,Z^*)(X_0).$$

On the other hand, by Theorem 3.1(1), we get

$$\underline{\phi}(X_0) = \phi^*(X_0),$$

$$\underline{\phi}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) = \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) = W(Y^*,Z^*)(X_0).$$

That is,

$$\sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) = \phi^*(X_0).$$

And so for any $Y \in \mathbf{F}$,

$$W(Y,Z^*)(X_0) \leq \phi^*(X_0),$$

$$T_{\phi^*}(Y,Z^*)(X_0) \leq 0.$$

That is,

$$\sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z^*)(X_0) = 0 = T_{\phi^*}(Y^*,Z^*)(X_0).$$

Therefore we obtain

$$\sup_{Y\in\mathbf{F}} T_{\phi}(Y,Z^*)(X_0) = T_{\phi}(Y^*,Z^*)(X_0) = \inf_{Z\in\mathbf{G}} T_{\phi}(Y^*,Z)(X_0),$$

from which (Y^*, Z^*) is a saddle point of (DGP_{ϕ}).

Theorem 3.2: *Suppose that* $Z^* \in \mathbf{G}$ *is a minimizer of (FDG).*

(1) $\overline{\phi}(X_0) = \underline{\phi}(X_0) =: \phi^*(X_0)$.

(2) *If* $\underline{T_{\phi^*}}(X_0) \geq 0$, *then* Z^* *is a minimizer of* (DGP_{ϕ^*}).

Proof:

(1) The inequality $\underline{\phi}(X_0) \le \bar{\phi}(X_0)$ is clear. On the other hand, since $Z^* \in \mathbf{G}$ is a minimizer of (FDG), we have

$$\underline{\phi}(X_0) = \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) \ge \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \bar{\phi}(X_0).$$

This implies that $\bar{\phi}(X_0) = \underline{\phi}(X_0) = \phi^*(X_0)$.

(2) Since $Z^* \in \mathbf{G}$ is a minimizer of (FDG), it follows that for any $Y \in \mathbf{F}$,

$$\phi^*(X_0) = \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) \ge W(Y,Z^*)(X_0).$$

And also

$$U(Y,Z^*)(X_0) - \phi^*(X_0)V(Y,Z^*)(X_0) \le 0,$$

$$0 \ge \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z^*)(X_0) \ge \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y,Z)(X_0) = \underline{T}_{\phi^*}(X_0) \ge 0.$$

Hence we have

$$\sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z^*)(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y,Z)(X_0) = \underline{T}_{\phi^*}(X_0),$$

from which Z^* is a minimizer of (DGP_{ϕ^*}).

Theorem 3.3: *Suppose that* $\bar{\phi}(X_0) = \underline{\phi}(X_0) =: \phi^*(X_0)$.

(1) *If* $Y^* \in \mathbf{F}$ *is a maximizer of* (DGP_{ϕ^*}) *satisfing*

$$\inf_{Z\in\mathbf{G}} T_{\phi^*}(Y^*,Z)(X_0) = \bar{T}_{\phi^*}(X_0) \ge 0,$$

then Y^* *is a maximizer of (FDG).*

(2) *If* $Z^* \in \mathbf{G}$ *is a minimizer of* (DGP_{ϕ^*}) *satisfying*

$$\sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z^*)(X_0) = \underline{T}_{\phi^*}(X_0) \le 0,$$

then Z^* *is a minimizer of (FDG).*

Proof:

(1) By the assumptions, $\bar{T}_{\phi^*}(X_0) \ge 0$ and Y^* is a maximizer of (DGP_{ϕ^*}), it follows that for any $Z \in \mathbf{G}$

$$0 \le \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y^*,Z)(X_0) \le T_{\phi^*}(Y^*,Z)(X_0).$$

This shows that for any $Z \in \mathbf{G}$

$$\phi^*(X_0) \le W(Y^*,Z)(X_0) \le \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0).$$

Hence

$$\phi^*(X_0) \le \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) \le \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \bar{\phi}(X_0).$$

This shows that Y^* is a maximizer of (FDG).

(2) By the assumptions, $\underline{T}_{\phi^*}(X_0) \le 0$ and Z^* is a minimizer of (DGP_{ϕ^*}), it shows that for any $Y \in \mathbf{F}$

$$T_{\phi^*}(Y,Z^*)(X_0) \le \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z^*)(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y,Z)(X_0) \le 0.$$

This implies that for each $Y \in \mathbf{F}$

$$\inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) \le W(Y,Z^*)(X_0) \le \phi^*(X_0).$$

Hence

$$\underline{\phi}(X_0) \le \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) \le \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) = \phi^*(X_0).$$

This shows that Z^* is a minimizer of (FDG).

Theorem 3.4: *We assume that* $\bar{\phi}(X_0) = \underline{\phi}(X_0) = \phi^*(X_0)$ *and that* $(Y^*,Z^*) \in \boldsymbol{F}\times\boldsymbol{G}$ *is a saddle point of* (DGP_{ϕ^*}) *satisfying* $T_{\phi^*}(Y^*,Z^*)(X_0) = 0$. *Then* (Y^*,Z^*) *is a saddle point of (FDG).*

Proof: Because (Y^*,Z^*) is a saddle point of (DGP_{ϕ^*}), $Y^* \in \mathbf{F}$ is a maximizer of (DGP_{ϕ^*}). That is,

$$\bar{T}_{\phi^*}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y^*,Z)(X_0).$$

On the other hand, since (Y^*,Z^*) is a saddle point of (DGP_{ϕ^*}), $Z^* \in \mathbf{G}$ is a minimizer of (DGP_{ϕ^*}). That is,

$$\underline{T}_{\phi^*}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y,Z)(X_0) = \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z^*)(X_0).$$

Moreover we have

$$\underline{T}_{\phi^*}(X_0) = \sup_{Y\in\mathbf{F}} T_{\phi^*}(Y,Z^*)(X_0) = T_{\phi^*}(Y^*,Z^*)(X_0) = \inf_{Z\in\mathbf{G}} T_{\phi^*}(Y^*,Z)(X_0) = \bar{T}_{\phi^*}(X_0).$$

Since $T_{\phi^*}(Y^*,Z^*)(X_0) = 0$, we get $\bar{T}_{\phi^*}(X_0) = \underline{T}_{\phi^*}(X_0) = 0$.

Furthermore by Theorem 3.3 (1) and (2), $Y^* \in \mathbf{F}$ is a maximizer of (FDG). That is,

$$\bar{\phi}(X_0) = \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} W(Y,Z)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0) \le W(Y^*,Z^*)(X_0).$$

On the other hand, since (Y^*,Z^*) is a saddle point of (FDG), $Z^* \in \mathbf{G}$ is a minimizer of (FDG). That is,

$$\underline{\phi}(X_0) = \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} W(Y,Z)(X_0) = \sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) \ge W(Y^*,Z^*)(X_0).$$

From $\bar{\phi}(X_0) = \underline{\phi}(X_0)$,

$$\sup_{Y\in\mathbf{F}} W(Y,Z^*)(X_0) = W(Y^*,Z^*)(X_0) = \inf_{Z\in\mathbf{G}} W(Y^*,Z)(X_0).$$

Hence (Y^*,Z^*) is a saddle point of (FDG).

3.4 *Examples*

This section provides two examples related to $T_\phi(Y,Z)(X_0)$.

Example 3.1: We consider the following :

$$m = 1, u_0 = 0, v_0 = 1, -1 < X_0 < 1, \mathbf{Y} = \mathbf{Z} = [-1,1],$$

$$f(t,X,Y,Z) = (Y-Z)^2,$$
$$U(Y,Z)(X_0) = X(1), V(Y,Z)(X_0) = X(1)+3,$$
$$T_\phi(Y,Z)(X_0) = (1-\phi(X_0))X(u_0) - 3\phi(X_0).$$

A first we calculate $\inf\limits_{Z\in\mathbf{G}} \sup\limits_{Y\in\mathbf{F}} T_\phi(Y,Z)(X_0)$. If $\phi < 1$, then

$$\begin{aligned}\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} \{(1-\phi)X(1) - 3\phi\} &= (1-\phi)\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} X(1) - 3\phi \\ &= (1-\phi)\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} \{X_0 + \int_0^1 (Y(t)-Z(t))^2 dt\} - 3\phi \\ &= (1-\phi)\{X_0 + \inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} \int_0^1 (Y(t)-Z(t))^2 dt\} - 3\phi\end{aligned}$$

For any $Z(t)$,

$$\sup_{Y\in\mathbf{F}} \int_0^1 (Y(t)-Z(t))^2 dt = \int_0^1 (\bar{Y}_Z(t)-Z(t))^2 dt,$$

where

$$\bar{Y}_Z(t) = \begin{cases} -1 & (Z(t)\geq 0) \\ 1 & (Z(t)<0). \end{cases}$$

If $Z(t) \geq 0$,

$$\bar{Y}_Z(t) - Z(t) = -1 - Z(t) = -1 - |Z(t)| = -(1+|Z(t)|),$$

otherwise

$$\bar{Y}_Z(t) - Z(t) = 1 - Z(t) = 1 + |Z(t)|.$$

That is,

$$\begin{aligned}\sup_{Y\in F} \int_0^1 (Y(t)-Z(t))^2 dt &= \int_0^1 (\bar{Y}_Z(t)-Z(t))^2 dt \\ &= \int_0^1 (1+|Z(t)|)^2 dt.\end{aligned}$$

Then

$$\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} \int_0^1 (Y(t)-Z(t))^2 dt = \inf_{Z\in\mathbf{G}} \int_0^1 (1+|Z(t)|)^2 dt = 1.$$

Hence

$$\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} \{(1-\phi)X(1) - 3\phi\} = (1-\phi)(X_0+1) - 3\phi.$$

If $\phi > 1$, then

$$\begin{aligned}\inf_{Z\in\mathbf{G}} \sup_{Y\in\mathbf{F}} \{(1-\phi)X(1) - 3\phi\} &= (1-\phi)\sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} X(1) - 3\phi \\ &= (1-\phi)\sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} \{X_0 + \int_0^1 (Y(t)-Z(t))^2 dt\} - 3\phi \\ &= (1-\phi)\{X_0 + \sup_{Y\in\mathbf{F}} \inf_{Z\in\mathbf{G}} \int_0^1 (Y(t)-Z(t))^2 dt\} - 3\phi\end{aligned}$$

For any $Z(t)$,

$$\inf_{Y\in\mathbf{F}} \int_0^1 (Y(t)-Z(t))^2 dt = \int_0^1 (\bar{Y}_Z(t)-Z(t))^2 dt = 0,$$

where $\bar{Y}_Z(t) = Z(t)$. That is,

$$\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\int_0^1 (Y(t)-Z(t))^2dt = 0.$$

Hence,

$$\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}}\{(1-\phi)X(1)-3\phi\} = (1-\phi)X_0 - 3\phi.$$

Next we calculate $\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} T_\phi(Y,Z)(X_0)$. If $\phi < 1$, then

$$\begin{aligned}\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{(1-\phi)X(1)-3\phi\} &= (1-\phi)\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}X(1) - 3\phi\\ &= (1-\phi)\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{X_0+\int_0^1 (Y(t)-Z(t))^2dt\} - 3\phi\\ &= (1-\phi)\{X_0+\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\int_0^1 (Y(t)-Z(t))^2dt\} - 3\phi.\end{aligned}$$

For any $Y(t)$,

$$\inf_{Z\in\mathbf{G}}\int_0^1 (Y(t)-Z(t))^2dt = \int_0^1 (Y(t)-\bar{Z}_Y(t))^2dt = 0,$$

where $\bar{Z}_Y(t) = Y(t)$. That is,

$$\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\int_{u_0}^{u_0} (Y(t)-Z(t))^2dt = 0.$$

Hence

$$\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{(1-\phi)x(u_0)-3\phi\} = (1-\phi)X_0 - 3\phi.$$

If $\phi > 1$, then

$$\begin{aligned}\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{(1-\phi)X(1)-3\phi\} &= (1-\phi)\inf_{Y\in\mathbf{F}}\sup_{Z\in\mathbf{G}}X(1) - 3\phi\\ &= (1-\phi)\inf_{Y\in\mathbf{F}}\sup_{Z\in\mathbf{G}}\{X_0+\int_0^1 (Y(t)-Z(t))^2dt\} - 3\phi\\ &= (1-\phi)\{X_0+\inf_{Y\in\mathbf{F}}\sup_{Z\in\mathbf{G}}\int_0^1 (Y(t)-Z(t))^2dt\} - 3\phi\end{aligned}$$

For any $Y(t)$,

$$\sup_{Z\in\mathbf{G}}\int_0^1 (Y(t)-Z(t))^2dt = \int_0^1 (Y(t)-\bar{Z}_Y(t))^2dt,$$

where

$$\bar{Z}_Y(t) = \begin{cases}-1 & (Y(t)\geq 0)\\ 1 & (Y(t)<0).\end{cases}$$

If $Y(t)\geq 0$,

$$Y(t)-\bar{Z}_Y(t) = Y(t)+1 = |Y(t)|+1$$

otherwise

$$Y(t)-\bar{Z}_Y(t) = Y(t)-1 = -|Y(t)|-1 = -(|Y(t)|+1).$$

That is,

$$\begin{aligned}\sup_{Z\in\mathbf{G}}\int_0^1 (Y(t)-Z(t))^2dt &= \int_0^1 (Y(t)-\bar{Z}_Y(t))^2dt\\ &= \int_0^1 (|Y(t)|+1)^2dt.\end{aligned}$$

Then

$$\inf_{Y\in\mathbf{F}}\sup_{Z\in\mathbf{G}}\int_0^1 (Y(t)-Z(t))^2dt = \inf_{Y\in\mathbf{F}}\int_0^1 (|Y(t)|+1)^2dt = 1.$$

Hence

$$\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{(1-\phi)X(1)-3\phi\} = (1-\phi)(X_0+1)-3\phi.$$

Therefore

$$\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}} T_\phi(Y,Z)(X_0) = \begin{cases}(1-\phi)(X_0+1)-3\phi & (\phi<1)\\ (1-\phi)X_0-3\phi & (\phi>1),\end{cases}$$

and

$$\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} T_\phi(Y,Z)(X_0) = \begin{cases}(1-\phi)X_0-3\phi & (\phi<1)\\ (1-\phi)(X_0+1)-3\phi & (\phi>1).\end{cases}$$

Example 3.2: We consider the following :

$$m=1, u_0=0, v_0=1, -1<X_0<1, \mathbf{Y}=\mathbf{Z}=[-1,1],$$
$$f(t,X,Y,Z)=y+z,$$
$$U(Y,Z)(X_0)=X(1), V(Y,Z)(X_0)=X(1)+3,$$
$$T_\phi(Y,Z)(X_0)=(1-\phi(X_0))X(1)-3\phi(X_0).$$

At first we calculate $\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}} T_\phi(Y,Z)(X_0)$. If $\phi<1$, then

$$\begin{aligned}\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}}\{(1-\phi)X(1)-3\phi\} &= (1-\phi)\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}}X(1)-3\phi\\ &= (1-\phi)\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}}\{X_0+\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}}\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\sup_{Y\in\mathbf{F}}\int_0^1 Y(t)dt+\inf_{Z\in\mathbf{G}}\int_0^1 Z(t)dt\}-3\phi\\ &= (1-\phi)\{X_0+1-1\}-3\phi\\ &= (1-\phi)X_0-3\phi.\end{aligned}$$

If $\phi>1$, then

$$\begin{aligned}\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}}\{(1-\phi)X(1)-3\phi\} &= (1-\phi)\sup_{Z\in\mathbf{G}}\inf_{Y\in\mathbf{F}}X(1)-3\phi\\ &= (1-\phi)\sup_{Z\in\mathbf{G}}\inf_{Y\in\mathbf{F}}\{X_0+\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\sup_{Z\in\mathbf{G}}\inf_{Y\in\mathbf{F}}\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\inf_{Y\in\mathbf{F}}\int_0^1 Y(t)dt+\sup_{Z\in\mathbf{G}}\int_0^1 Z(t)dt\}-3\phi\\ &= (1-\phi)\{X_0-1+1\}-3\phi\\ &= (1-\phi)X_0-3\phi.\end{aligned}$$

Next we calculate $\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}} T_\phi(Y,Z)(X_0)$. If $\phi<1$, then

$$\begin{aligned}\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{(1-\phi)X(1)-3\phi\} &= (1-\phi)\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}X(1)-3\phi\\ &= (1-\phi)\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{X_0+\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\sup_{Y\in\mathbf{F}}\int_0^1 Y(t)dt+\inf_{Z\in\mathbf{G}}\int_0^1 Z(t)dt\}-3\phi\\ &= (1-\phi)\{X_0+1-1\}-3\phi\\ &= (1-\phi)X_0-3\phi.\end{aligned}$$

If $\phi>1$, then

$$\begin{aligned}\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}\{(1-\phi)X(1)-3\phi\} &= (1-\phi)\inf_{Y\in\mathbf{F}}\sup_{Z\in\mathbf{G}}X(1)-3\phi\\ &= (1-\phi)\inf_{Y\in\mathbf{F}}\sup_{Z\in\mathbf{G}}\{X_0+\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\inf_{Y\in\mathbf{F}}\sup_{Z\in\mathbf{G}}\int_0^1 (Y(t)+Z(t))dt\}-3\phi\\ &= (1-\phi)\{X_0+\inf_{Y\in\mathbf{F}}\int_0^1 Y(t)dt+\sup_{Z\in\mathbf{G}}\int_0^1 Z(t)dt\}-3\phi\\ &= (1-\phi)\{X_0-1+1\}-3\phi\\ &= (1-\phi)X_0-3\phi.\end{aligned}$$

Therefore we have

$$\inf_{Z\in\mathbf{G}}\sup_{Y\in\mathbf{F}}T_\phi(Y,Z)(X_0)=(1-\phi)X_0-3\phi=\sup_{Y\in\mathbf{F}}\inf_{Z\in\mathbf{G}}T_\phi(Y,Z)(X_0),$$

and

$$\begin{cases}\text{if } \phi<1, & y^*(t)=1, z^*(t)=-1 \text{ is an equilibrium point,}\\ \text{if } \phi=1, & \text{any } Y(t), Z(t) \text{ are equilibrium points.}\\ \text{if } \phi>1, & y^*(t)=-1, z^*(t)=1 \text{ is an equilibrium point.}\end{cases}$$

4. Fractional differential game with stopping time

4.1 *Formulation*

In this section we refer the formulation of Bensoussan and Friedman [4]. When the player y selects the function $Y(t)$ (which is called control) and the deterministic stopping time $S\in[u_0,v_0]$, and the player z selects $Z(t)$ and $T\in[u_0,v_0]$, we take the following functionals :

$$\begin{aligned}M(Y,S;Z,T)(X_0) &= \int_{u_0}^{S\wedge T} h^1(t,X(t),Y(t),Z(t))\exp[\int_{u_0}^t k^1(s,X(s),Y(s),Z(s))ds]dt\\ &+\chi_{S\le T}g_1^1(X(S),S)\exp[\int_{u_0}^S k^1(s,X(s),Y(s),Z(s))ds]\\ &+\chi_{S>T}g_2^1(X(T),T)\exp[\int_{u_0}^T k^1(s,X(s),Y(s),Z(s))ds],\\ N(Y,S;Z,T)(X_0) &= \int_{u_0}^{S\wedge T} h^2(t,X(t),Y(t),Z(t))\exp[\int_{u_0}^t k^2(s,X(s),Y(s),Z(s))ds]dt\\ &+\chi_{S\le T}g_1^2(X(S),S)\exp[\int_{u_0}^S k^2(s,X(s),Y(s),Z(s))ds]\\ &+\chi_{S>T}g_2^2(X(T),T)\exp[\int_{u_0}^T k^2(s,X(s),Y(s),Z(s))ds],\end{aligned}$$

where $S \wedge T = \min(S, T)$,

$$\chi_{S \le T} = \begin{cases} 1 & (S \le T) \\ 0 & (S > T) \end{cases}, \chi_{S > T} = \begin{cases} 1 & (S > T) \\ 0 & (S \le T) \end{cases}'$$

and the following criterion :

$$O(Y, S; Z, T)(X_0) = \frac{M(Y,S;Z,T)(X_0)}{N(Y,S;Z,T)(X_0)}.$$

The following conditions are assumed from now on :

(E1) The function $f(t, X, Y, Z)$ is continuous with respect to $(t, X, Y, Z) \in [u_0, v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$.

(E2) There is a nonnegative function $k(t)$ with $\int_{u_0}^{v_0} k(t)dt < \infty$, such that

$$X \cdot f(t, X, Y, Z) \le k(t)(1 + |X|^2)$$

for any $(t, X, Y, Z) \in [u_0, v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$.

(E3) For each $R > 0$, there is a nonnegative function $k_R(t)$ with $\int_{u_0}^{v_0} k_R(t)dt < \infty$, such that

$$|f(t, X, Y, Z) - f(t, \bar{X}, Y, Z)| \le k_R(t)|X - \bar{X}|$$

for any $t \in [u_0, v_0], \boldsymbol{Y} \in \mathbf{Y}, Z \in \mathbf{Z}, |X| \le R, |\bar{X}| \le R$.

(F1) The functions $h^1(t, X, Y, Z)$ and $h^2(t, X, Y, Z)$ are continuous with respect to $(t, X, Y, Z) \in [u_0, v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$.

(F2) For each $(t, X, Y, Z) \in [u_0, v_0] \times \boldsymbol{R}^m \times \mathbf{Y} \times \mathbf{Z}$, $h^2(t, X, Y, Z) > 0$.

(G1) The functions $g_1^1(X, t)$, $g_2^1(X, t)$, $g_1^2(X, t)$, and $g_2^2(X, t)$ are continuous with respect to $(X, t) \in \mathbf{R}^m \times [u_0, v_0]$.

(G2) For each $(X, t) \in \boldsymbol{R}^m \times [u_0, v_0]$, $g_1^2(X, t) > 0$ and $g_2^2(X, t) > 0$.

Furthermore we define the control sets:

$$\mathcal{F} = \{(Y, S) | Y : [u_0, v_0] \to \mathbf{Y} : \text{measurable}, S \in [u_0, v_0]\}$$
$$\mathcal{G} = \{(Z, T) | Z : [u_0, v_0] \to \mathbf{Z} : \text{measurable}, T \in [u_0, v_0]\}$$

The player y will aim to maximize $O(Y, S; Z, T)(X_0)$, while the player z will aim to minimize $O(Y, S; Z, T)(X_0)$. We call this game the fractional differential game with stopping time (FDGST).

We introduce the parameter function : $\psi : \boldsymbol{R}^m \to \boldsymbol{R}$, and denote the upper game value function and lower game value function for (FDGST) by

$$\bar{\psi}(X_0) = \inf_{(Z,T) \in \mathcal{G}} \sup_{(Y,S) \in \mathcal{F}} O(Y, S; Z, T)(X_0),$$
$$\underline{\psi}(X_0) = \sup_{(Y,S) \in \mathcal{F}} \inf_{(Z,T) \in \mathcal{G}} O(Y, S; Z, T)(X_0)$$

respectively. It is obvious that $\bar{\psi}(X_0) \ge \underline{\psi}(X_0)$, and the interval $[\underline{\psi}(X_0), \bar{\psi}(X_0)]$ is called the duality gap of (FDGST).

Next, in order to formulate a parameterized differential game, we also introduce the payoff function with parameter $\psi(X_0)$:

$$T_\psi(Y,S;Z,T)(X_0) = M(Y,S;Z,T)(X_0) - \psi(X_0)N(Y,S;Z,T)(X_0)$$

and denote the upper game value function and lower game value function, which is called the differential game with stopping time and parameter ψ (DGSTP$_\psi$),

$$\bar{T}_\psi(X_0) = \inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} T_\psi(Y,S;Z,T)(X_0),$$

$$\underline{T}_\psi(X_0) = \sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} T_\psi(Y,S;Z,T)(X_0).$$

And then the interval $[\underline{T}_\psi(X_0), \bar{T}_\psi(X_0)]$ is the duality gap of (DGSTP$_\psi$).

Definition 4.1: If the duality gap of (FDGST) equals zero, then (FDGST) is said to have the value function

$$\bar{\psi}(X_0) = \underline{\psi}(X_0) = \psi^*(X_0)$$

Furthermore if there is $(Y^*,S^*) \in \mathcal{F}$ satisfying

$$\bar{\psi}(X_0) = \inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} O(Y,S;Z,T)(X_0) = \inf_{(Z,T)\in\mathcal{G}} O(Y^*,S^*;Z,T)(X_0)$$

then we call (Y^*,S^*) a maximizer of $O(Y,S;Z,T)(X_0)$. Similarly if there is $(Z^*,T^*) \in \mathcal{G}$ satisfying

$$\underline{\psi}(X_0) = \sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} O(Y,S;Z,T)(X_0) = \sup_{(Y,S)\in\mathcal{F}} O(y,S;z^*,T^*)(X_0)$$

then we call (Z^*,T^*) a minimizer of $O(Y,S;Z,T)(X_0)$.

4.2 *Upper and lower value functions*

In this and next sections, we state the results, which proofs is the same as in sections 3.2 and 3.3.

Proposition 4.1:

(1) For parameter functions $\psi_1(X_0)$ and $\psi_2(X_0)$, if $\psi_1(X_0) > \psi_2(X_0) \geq 0$, then $\bar{T}_{\psi_1}(X_0) \leq \bar{T}_{\psi_2}(X_0)$.

(2) If $\bar{T}_\psi(X_0) < 0$, then $\psi(X_0) \geq \bar{\psi}(X_0)$.

(3) If $\bar{T}_\psi(X_0) > 0$, then $\psi(X_0) \leq \bar{\psi}(X_0)$.

(4) If $\psi(X_0) > \bar{\psi}(X_0)$, then $\bar{T}_\psi(X_0) \leq 0$.

(5) If $\psi(X_0) < \bar{\psi}(X_0)$, then $\bar{T}_\psi(X_0) \geq 0$.

Proposition 4.2:

(1) For parameter functions $\psi_1(X_0)$ and $\psi_2(X_0)$, if $\psi_1(X_0) > \psi_2(X_0) \geq 0$, then $\underline{T}_{\psi_1}(X_0) \leq \underline{T}_{\psi_2}(X_0)$.

(2) If $\underline{T}_\psi(X_0) < 0$, then $\psi(X_0) \geq \underline{\psi}(X_0)$.

(3) If $\underline{T}_\psi(X_0) > 0$, then $\psi(X_0) \leq \underline{\psi}(X_0)$.

(4) If $\psi(X_0) > \underline{\psi}(X_0)$, then $\underline{T}_\psi(X_0) \leq 0$.

(5) If $\psi(X_0) < \underline{\psi}(X_0)$, then $\underline{T}_\psi(X_0) \geq 0$.

4.3 *Saddle point function*

Theorem 4.1: *Consider* $(Y^*, S^*) \in \mathcal{F}$ *as a maximizer of (FDGST).*

(1) $\bar{\psi}(X_0) = \underline{\psi}(X_0) =: \psi^*(X_0)$.

(2) *If* $\bar{T}_{\psi^*}(X_0) \leq 0$, *then* (Y^*, S^*) *is a maximizer of* $(DGSTP_{\psi^*})$.

Corollary 4.2: *Consider* $((Y^*, S^*), (Z^*, T^*)) \in \mathcal{F} \times \mathcal{G}$ *as a saddle point of (FDGST).*

(1) $T_{\psi^*}(Y^*, S^*; Z^*, T^*)(X_0) = 0$.

(2) $(Y^*, S^*; Z^*, T^*)$ *is a saddle point of* $(DGSTP_{\psi})$.

Theorem 4.2: *Suppose that* $(Z^*, T^*) \in \mathcal{G}$ *is a minimizer of (FDGST).*

(1) $\bar{\psi}(X_0) = \underline{\psi}(X_0) =: \psi^*(X_0)$.

(2) *If* $\underline{T}_{\psi^*}(X_0) \geq 0$, *then* (Z^*, T^*) *is a minimizer of* $(DGSTP_{\psi^*})$.

Theorem 4.3: *Suppose that* $\bar{\psi}(X_0) = \underline{\psi}(X_0) =: \psi^*(X_0)$.

(1) *If* $(Y^*, S^*) \in \mathcal{F}$ *is a maximizer of* $(DGSTP_{\psi^*})$ *satisfying*

$$\inf_{(Z,T)\in\mathcal{G}} T_{\psi^*}(Y^*, S^*; Z, T)(X_0) = \bar{T}_{\psi^*}(X_0) \geq 0,$$

then (Y^*, S^*) *is a maximizer of (FDGST).*

(2) *If* $(Z^*, T^*) \in \mathcal{G}$ *is a minimizer of* $(DGSTP_{\psi^*})$ *satisfying*

$$\sup_{(Y,S)\in\mathcal{F}} T_{\psi^*}(Y, S; Z^*, T^*)(X_0) = \underline{T}_{\psi^*}(X_0) \leq 0,$$

then (Z^*, T^*) *is a minimizer of (FDGST).*

Theorem 4.4: *Suppose that* $\bar{\psi}(X_0) = \underline{\psi}(X_0) =: \psi^*(X_0)$ *and that* $((Y^*, S^*), (Z^*, T^*)) \in \mathcal{F} \times \mathcal{G}$ *is a saddle point of* $(DGSTP_{\psi^*})$ *satisfying* $T_{\psi^*}(Y^*, S^*; Z^*, T^*)(X_0) = 0$. *Then* $((Y^*, S^*), (Z^*, T^*))$ *is a saddle point of (FDGST).*

4.4 *Examples*

This section provides two examples related to $T_{\psi}(Y, S; Z, T)(X_0)$.

Example 4.1: We consider the following :

$$m = 1, u_0 = 0, v_0 = 1, -1 < X_0 < 1, \mathbf{Y} = \mathbf{Z} = [-1,1],$$
$$f(t, X, Y, Z) = (Y - Z)^2,$$
$$h^1 = h^2 = 0, k^1 = k^2 = 0, g_1^1 = g_2^1 = x, g_1^2 = g_2^2 = X + 3,$$
$$T_{\psi}(Y, S; Z, T)(X_0) = (1 - \psi(X_0))X(S \wedge T) - 3\phi(X_0).$$

A first we calculate $\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} T_{\psi}(Y, S; Z, T)(X_0)$. If $\psi < 1$, then

$$\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} \{(1 - \psi)X(S \wedge T) - 3\psi\} = (1 - \psi) \inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} X(S \wedge T) - 3\psi$$

$$= (1-\psi)\inf_{(Z,T)\in\mathcal{G}}\sup_{(Y,S)\in\mathcal{F}}\{X_0+\int_0^{S\wedge T}(Y(t)-Z(t))^2dt\}-3\psi$$

$$= (1-\psi)\{X_0+\inf_{(Z,T)\in\mathcal{G}}\sup_{(Y,S)\in\mathcal{F}}\int_0^{S\wedge T}(Y(t)-Z(t))^2dt\}-3\psi$$

For each $(Z(t),T)$,

$$\sup_{(Y,S)\in\mathcal{F}}\int_0^{S\wedge T}(Y(t)-Z(t))^2dt=\int_0^{S\wedge T}(\bar{Y}_Z(t)-Z(t))^2dt=\int_0^{T}(1+|Z(t)|)^2dt,$$

where

$$\bar{Y}_Z(t)=\begin{cases}-1 & (Z(t)\geq 0)\\ 1 & (Z(t)<0),\end{cases}\qquad S=1$$

Then

$$\inf_{(Z,T)\in\mathcal{G}}\sup_{(Y,S)\in\mathcal{F}}\int_0^{S\wedge T}(Y(t)-Z(t))^2dt=\inf_{(Z,T)\in\mathcal{G}}\int_0^{T}(1+|Z(t)|)^2dt=0.$$

Hence

$$\inf_{(Z,T)\in\mathcal{G}}\sup_{(Y,S)\in\mathcal{F}}\{(1-\psi)X(S\wedge T)-3\psi\}=(1-\psi)X_0-3\psi.$$

If $\psi>1$, then

$$\inf_{(Z,T)\in\mathcal{G}}\sup_{(Y,S)\in\mathcal{F}}\{(1-\psi)X(S\wedge T)-3\psi\}=(1-\psi)\sup_{(Z,T)\in\mathcal{G}}\inf_{(Y,S)\in\mathcal{F}}X(S\wedge T)-3\psi$$

$$= (1-\psi)\sup_{(Z,T)\in\mathcal{G}}\inf_{(Y,S)\in\mathcal{F}}\{X_0+\int_0^{S\wedge T}(Y(t)-Z(t))^2dt\}-3\psi$$

$$= (1-\psi)\{X_0+\sup_{(Z,T)\in\mathcal{G}}\inf_{(Y,S)\in\mathcal{F}}\int_0^{S\wedge T}(Y(t)-Z(t))^2dt\}-3\psi$$

For any $(Z(t),T)$,

$$\inf_{(Y,S)\in\mathcal{F}}\int_0^{S\wedge T}(Y(t)-Z(t))^2dt=\int_0^{S\wedge T}(\bar{Y}_Z(t)-Z(t))^2dt=0,$$

where $\bar{Y}_Z(t)=Z(t)$ and S is any stopping time, or, $Y(t)$ is any control function and $S=0$. That is,

$$\sup_{(Z,T)\in\mathcal{G}}\inf_{(Y,S)\in\mathcal{F}}\int_0^{S\wedge T}(Y(t)-Z(t))^2dt=0.$$

Hence,

$$\inf_{Z\in G}\sup_{Y\in F}\{(1-\psi)X(S\wedge T)-3\psi\}=(1-\psi)X_0-3\psi.$$

Next we calculate $\sup_{(Y,S)\in\mathcal{F}}\inf_{(Z,T)\in\mathcal{G}}T_\psi(Y,S;Z,T)(X_0)$. If $\psi<1$, then

$$\sup_{(Y,S)\in\mathcal{F}}\inf_{(Z,T)\in\mathcal{G}}\{(1-\psi)x(S\wedge T)-3\psi\}=(1-\psi)\sup_{(Y,S)\in\mathcal{F}}\inf_{(Z,T)\in\mathcal{G}}x(S\wedge T)-3\psi$$

$$= (1-\psi)\sup_{(Y,S)\in\mathcal{F}}\inf_{(Z,T)\in\mathcal{G}}\{X_0+\int_0^{S\wedge T}(Y(t)-Z(t))^2dt\}-3\psi$$

$$= (1-\psi)\{X_0+\sup_{(Y,S)\in\mathcal{F}}\inf_{(Z,T)\in\mathcal{G}}\int_0^{S\wedge T}(Y(t)-Z(t))^2dt\}-3\psi.$$

For any $(Y(t),S)$,

$$\inf_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)-Z(t))^2 dt = \int_0^{S\wedge T} (Y(t)-\bar{Z}_Y(t))^2 dt = 0,$$

here $\bar{Z}_Y(t) = Y(t)$ and T is any stopping time, or, $Z(t)$ is any and $T = 0$. That is,

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)-Z(t))^2 dt = 0.$$

Hence

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \{(1-\psi)X(S\wedge T) - 3\psi\} = (1-\phi)X_0 - 3\psi.$$

If $\psi > 1$, then

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \{(1-\psi)X(S\wedge T) - 3\psi\} = (1-\phi) \inf_{(Y,S)\in\mathcal{F}} \sup_{(Z,T)\in\mathcal{G}} X(S\wedge T) - 3\psi$$
$$= (1-\psi) \inf_{(Y,S)\in\mathcal{F}} \sup_{(Z,T)\in\mathcal{G}} \{X_0 + \int_0^{S\wedge T} (Y(t)-Z(t))^2 dt\} - 3\psi$$
$$= (1-\psi)\{X_0 + \inf_{(Y,S)\in\mathcal{F}} \sup_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)-Z(t))^2 dt\} - 3\psi$$

For any $(Y(t), S)$,

$$\sup_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)-Z(t))^2 dt = \int_0^{S} (Y(t)-\bar{Z}_Y(t))^2 dt,$$

here

$$\bar{Z}_Y(t) = \begin{cases} -1 & (Y(t)\geq 0) \\ 1 & (Y(t)<0), \end{cases}$$

and $T = 1$. Then

$$\inf_{(Y,S)\in\mathcal{F}} \sup_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)-Z(t))^2 dt = \int_0^{S} (Y(t)-\bar{Z}_Y(t))^2 dt$$
$$= \int_0^1 (|Y(t)|+1)^2 dt = 0.$$

Hence

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \{(1-\phi)X(S\wedge T) - 3\psi\} = (1-\psi)X_0 - 3\psi.$$

Therefore we obtain

$$\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} T_\psi(Y,S;Z,T)(X_0) = (1-\psi)X_0 - 3\psi$$
$$= \sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} T_\psi(Y,S;Z,T)(X_0),$$

and

$$\begin{cases} \text{if } \psi<1, & Y^*(t) = \begin{cases} -1 & Z^*(t)\geq 0 \\ 1 & Z^*(t)<0, \end{cases} S^* = 1, Z^*(t) \text{ is any}, T^* = 0 \text{ is an equilibrium point,} \\ \text{if } \psi = 1, & \text{any } Y(t), S, Z(t), T \text{ are equilibrium points,} \\ \text{if } \psi>1, & Y^*(t), S^* = 0, Z^*(t) = \begin{cases} -1 & Y^*(t)\geq 0 \\ 1 & Y^*(t)<0, \end{cases} T^* = 1 \text{ is an equilibrium point.} \end{cases}$$

Example 4.2: We consider the following :

$$m = 1, u_0 = 0, v_0 = 1, -1 < X_0 < 1, \mathbf{Y} = \mathbf{Z} = [-1,1],$$
$$f(t, X, Y, Z) = Y + Z,$$
$$h^1 = h^2 = 0, k^1 = k^2 = 0, g_1^1 = g_2^1 = x, g_1^2 = g_2^2 = X + 3,$$
$$T_\psi(Y, S; Z, T)(X_0) = (1 - \psi(X_0))X(S \wedge T) - 3\phi(X_0).$$

At first we calculate $\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} T_\psi(Y, S; Z, T)(X_0)$. If $\psi < 1$, then

$$\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} \{(1 - \psi)X(S \wedge T) - 3\psi\} =$$
$$(1 - \psi)\{X_0 + \inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} \int_0^{S\wedge T} (Y(t) + Z(t))dt\} - 3\psi$$

For any $(Z(t), T)$,

$$\sup_{(Y,S)\in\mathcal{F}} \int_0^{S\wedge T} (Y(t) + Z(t))dt = \int_0^{S\wedge T} (1 + Z(t))dt = \int_0^{1\wedge T} (1 + Z(t))dt,$$

where

$$\bar{Y}(t) = 1 \text{ and } S = 1.$$

Then

$$\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} \int_0^{S\wedge T} (Y(t) + Z(t))dt = \inf_{(Z,T)\in\mathcal{G}} \int_0^{1\wedge T} (1 + Z(t))dt = 0.$$

Hence

$$\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} \{(1 - \psi)X(S \wedge T) - 3\psi\} = (1 - \psi)X_0 - 3\psi.$$

If $\psi > 1$, then

$$\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} \{(1 - \psi)x(S \wedge T) - 3\psi\} =$$
$$(1 - \psi)\{X_0 + \sup_{(Z,T)\in\mathcal{G}} \inf_{(Y,S)\in\mathcal{F}} \int_0^{S\wedge T} (Y(t) + Z(t))dt\} - 3\psi$$

For any $(Z(t), T)$,

$$\inf_{(Y,S)\in\mathcal{F}} \int_0^{S\wedge T} (Y(t) + Z(t))dt = \int_0^{1\wedge T} (-1 + Z(t))dt,$$

where $\bar{Y}_Z(t) = -1$and $S = 1$. That is,

$$\sup_{(Z,T)\in\mathcal{G}} \inf_{(Y,S)\in\mathcal{F}} \int_0^{S\wedge T} (-1 + Z(t))dt = 0.$$

where $Z(t) = 1$ and T is any stopping time. Hence,

$$\inf_{z\in\mathcal{G}} \sup_{y\in\mathcal{F}} \{(1 - \psi)X(S \wedge T) - 2\psi\} = (1 - \psi)X_0 - 3\psi.$$

Next we calculate $\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} T_\psi(Y, S; Z, T)(X_0)$. If $\psi < 1$, then

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \{(1 - \psi)X(S \wedge T) - 3\psi\} =$$
$$(1 - \psi)\{X_0 + \sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t) + Z(t))dt\} - 3\psi.$$

For each $(Y(t), S)$,

$$\inf_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)+Z(t))dt = \int_0^{S\wedge 1} (Y(t)+\bar{Z}_Y(t))dt = 0,$$

where $\bar{Z}_Y(t) = -1$ and $T = 1$. That is,

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)+Z(t))dt = 0,$$

where $Y(t) = 1$ and S is any stopping time, or, $Y(t)$ is any and $S = 0$. Hence

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \{(1-\psi)X(S\wedge T) - 3\psi\} = (1-\phi)X_0 - 3\psi.$$

If $\psi > 1$, then

$$\operatorname{Sup}_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \{(1-\psi)X(S\wedge T) - 3\psi\} =$$

$$(1-\psi)\{X_0 + \inf_{(Y,S)\in\mathcal{F}} \sup_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)+Z(t))dt\} - 3\psi$$

For each $(Y(t), S)$,

$$\sup_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)+Z(t))dt = \int_0^{S\wedge 1} (Y(t)+1)dt,$$

where $Z(t) = 1$ and $T = 1$.

Then

$$\inf_{(Y,S)\in\mathcal{F}} \sup_{(Z,T)\in\mathcal{G}} \int_0^{S\wedge T} (Y(t)+Z(t))dt = \int_0^{S} (Y(t)+1)^2 dt = 0,$$

where $Y(t) = -1$ and S is any stopping time, or, $Y(t)$ is any and $S = 0$.

Hence

$$\sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} \{(1-\phi)X(S\wedge T) - 3\psi\} = (1-\psi)X_0 - 3\psi.$$

Therefore

$$\inf_{(Z,T)\in\mathcal{G}} \sup_{(Y,S)\in\mathcal{F}} T_\psi(Y,S;Z,T)(X_0) = (1-\psi)X_0 - 3\psi = \sup_{(Y,S)\in\mathcal{F}} \inf_{(Z,T)\in\mathcal{G}} T_\psi(Y,S;Z,T)(X_0),$$

and

$$\begin{cases} \text{if } \psi < 1, & Y^*(t) = 1, S^* = 1, Z^*(t) = -1, T^* = 1 \text{ is an equilibrium point,} \\ \text{if } \phi = 1, & \text{any } Y(t), S, Z(t), T \text{ are equilibrium points,} \\ \text{if } \phi > 1, & Y^*(t) = -1, S* = 1, Z^*(t) = 1, T^* = 1 \text{ is an equilibrium point.} \end{cases}$$

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