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Time-delayed optimal control for Helmholtz systems with Robin boundary and conjugation conditions

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Abstract

This work investigates optimal control problems governed by an elliptic system featuring a Helmholtz operator, Robin boundary conditions, and time delays. The aim is to decrease the cost function while complying with the state equation and constraints.

We investigate whether there are solutions and whether they are exclusive to the equation of state, and then verify the existence of optimal controls. Furthermore, we derive necessary optimality conditions using adjoint techniques.

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1. Introduction

Time-delayed optimal control problems arise naturally in various physical and engineering systems where the control action is influenced by past states. These problems are particularly relevant in wave propagation models governed by Helmholtz systems, which describe phenomena such as acoustics, electromagnetics, and quantum mechanics. The presence of Robin boundary conditions allows for a more realistic modeling of wave reflections and impedance effects, while conjugation conditions account for interface continuity in multi-domain settings. We aim to establish the well-posedness of the Helmholtz system under these conditions, formulate

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the associated optimal control problem, and derive necessary optimality conditions.

Furthermore, the impact of time delays on system performance and control strategies is investigated, providing insights into practical implementations in engineering applications in [1-13]. In section two; we take a look at the most advantageous manipulate for N × N optimal distributed control of non cooperative Helmholtz elliptic system with Robin conditions under conjugation conditions. Our first step is to prove that there is a solution for system (1), and then we provide the necessary and sufficient conditions for the control to be optimal for the system. The control problem under conjugation conditions is also studied in section three.

2. Optimization for N×N Non-Cooperative Elliptic Systems involving Helmholtz operators with Robin's Boundary Conditions and time delay

The N×N non cooperative elliptic system is governed by Robin boundary conditions and conjugation conditions, as we consider it.

$$\begin{cases} -\sum_{l=1}^{N}\left[(\Delta + k_l^2)y_l\right] + \sum_{l,m=1}^{N} a_{lm}y_m = \sum_{l=1}^{N}[f_l + \lambda_l y_l(t-\tau)] \text{ in } \Omega \subset \mathbb{R}^N \\ \sum_{l=1}^{N}\left[\beta_l \dfrac{\partial y_l}{\partial n} + \alpha_l y_l\right] = \sum_{l=1}^{N} g_l, \qquad l \neq m \qquad \text{on } \partial\Omega \end{cases} \tag{1}$$

and Conjugation Conditions

$$\begin{cases} R_1\left\{\sum_{l=1}^{N}\dfrac{\partial y_l}{\partial V_A}\right\}^{+} + R_2\left\{\sum_{l=1}^{N}\dfrac{\partial y_l}{\partial V_A}\right\}^{-} = \left[\sum_{l=1}^{N} y_l\right] + \varepsilon \qquad \text{on } \gamma \\ \left[\sum_{l=1}^{N}\dfrac{\partial y_l}{\partial V_A}\right] = \varpi_l, \qquad \text{on } \gamma \end{cases} \tag{*}$$

where $y_i(x,t)$ are the state variable (Solution to the Helmholtz Operator), $k_l^{\,2}$ is the Helmholtz Operator (small relative to the other terms), k_l is the wave number, $\lambda_l > 0$ is the delay coefficient (bounded) , $\tau > 0$ is the delay time, $\alpha_l > 0$ is the Robin coefficient and $\Delta y = \sum_{l=1}^{N}\frac{\partial^2 y}{\partial x_l^2}$ is the Laplacian operator. Our model is defined on $(H^1(\Omega))^2$, $a_{ml} = \begin{cases} 1 & m \geq l \\ -1 & m < l \end{cases}$, $g_m \in H^{\frac{-1}{2}}(\Omega)$, $\frac{\partial y_l}{\partial n} = \Delta y_l . n$ is the Normal Derivative on $\partial\Omega$, n is the outward unit Normal.

$$\int_{\Omega}\left\{-\sum_{l=1}^{N}\left[(\Delta+k_l^2)y_l\right]\Psi_l+\sum_{l,m=1}^{N}a_{lm}y_m\Psi_l\right\}dx=\int_{\Omega}\sum_{l=1}^{N}[f_l+\lambda_l y_l(t-\tau)]\Psi_l\,dx,$$

$$-\sum_{l=1}^{N}\int_{\Omega}(\Delta+k_l^2)y_l\ \Psi_l\,dx=\sum_{l=1}^{N}\left[\int_{\Omega}\nabla y_l\,.\nabla\Psi_l\,dx-\int_{\partial\Omega}\frac{\partial y_l}{\partial n}\Psi_l\ ds\right].$$

For $l,m=1,2,...,N$ and $a_{lm}>0$ for $l\neq m$ systems are considered cooperative. If not, the systems are deemed non- cooperative.

1.1 *Existence and uniqueness of Solution*

The chains we have are as follows:

$$\underbrace{(H^1(\Omega)\times...\times H^1(\Omega))}_{N-times}\subseteq\underbrace{(L^2(\Omega)\times...\times L^2(\Omega))}_{N-times}\subseteq\underbrace{(W^{-1}(\Omega)\times...\times W^{-1}(\Omega))}_{N-times},$$

we define the following continuous bilinear form on $(H^1(\Omega))^N$

$$b(y,\xi)=\int_{\Omega}\left[\sum_{l=1}^{N}\nabla y_l.\nabla\xi_l-k_l^2\,y_l\zeta_l+\sum_{l,m=1}^{N}a_{lm}y_m\xi_l\right]dx$$
$$+\int_{\partial\Omega}\sum_{l=1}^{N}\left[\frac{\alpha_l}{\beta}y_l\right]\xi_l ds+\int_{\gamma}\sum_{l=1}^{N}\left[\frac{[y_l][\xi_l]}{R_1+R_2}\right]d\gamma.$$

The Linear Functional

$$L(\xi_l)=\int_{\Omega}\sum_{l=1}^{N}[f_l+\lambda_l y_l(t-\tau)]\xi_l\ dx-\int_{\partial\Omega}\sum_{l=1}^{N}\left[\frac{g_l}{\beta}\right]\xi_l\,ds$$
$$+\int_{\gamma}\sum_{l=1}^{N}\frac{(R_2\varpi-\varepsilon)}{R_1+R_2}[\xi_l]d\gamma-\int_{\gamma}\varpi_l\xi_1^{+}d\gamma \qquad \text{for all}$$

$$\Psi_i\in\underbrace{(H^1(\Omega)\times...\times H^1(\Omega))}_{N-times}.$$

For $y=(y_1,y_2,y_3,...,y_N)\in\underbrace{(H^1(\Omega)\times...\times H^1(\Omega))}_{N-times}$ and by lax- Milligram lemma can be used to prove that:-

Theorem 1 : *For* $f_i=(f_1,f_2,f_3,...,f_N)\in\underbrace{(H^1(\Omega)\times...\times H^1(\Omega))}_{N-times}$ *, there is a unique solution* $y_l\in\underbrace{(H^1(\Omega)\times...\times H^1(\Omega))}_{N-times}$ *of system (1).*

Proof: The bilinear form can be translated as

$$b(y,\xi)=\int_{\Omega}\left[\sum_{l=1}^{N}\nabla y_l.\nabla\xi_l-k_l^2\,y_l\xi_l+\sum_{l.m=1}^{N}a_{lm}y_m\xi_l\right]dx$$
$$+\int_{\partial\Omega}\sum_{l=1}^{N}\left[\frac{\alpha_l}{\beta}y_l\right]\xi_l\,ds+\int_{\gamma}\sum_{l=1}^{N}\left[\frac{[y_l][\xi_l]}{R_1+R_2}\right]d\gamma.$$

$$b(y,y)\geq\int_{\Omega}\left[\sum_{l=1}^{N}|\nabla y_l|^2-k_l^2\,y_l^2\right]dx+\int_{\partial\Omega}\sum_{l=1}^{N}\frac{\alpha_l}{\beta}|y_l|^2\,ds$$
$$+\int_{\gamma}\sum_{l=1}^{N}\frac{[y_l]^2}{R_1+R_2}d\gamma,\ |\lambda_l|\leq\lambda$$

small enough to maintain stability

$$|b(y,y)|\geq\sum_{l=1}^{N}\left\|\nabla y_l(x)\right\|^2_{L^2(\Omega)}-\sum_{l=1}^{N}k_l^2\left\|y_l(x)\right\|^2_{L^2(\Omega)}\quad+\sum_{l=1}^{N}\frac{\alpha_l}{\beta}\left\|y_l(x)\right\|^2_{L^2(\partial\Omega)}$$
$$+\frac{1}{R_1+R_2}\sum_{l=1}^{N}\left\|[y_l(x)]\right\|^2_{L^2(\gamma)}$$

For sufficiently large $\alpha_l>0$ this term stabilizes the system further and counteracts any negative contributions from the Helmholtz term, $k_l^2\leq c,c$ where is a constant depending on the domain and boundary coefficients.

$b(y,y)\geq C\sum_{l=1}^{N}\left\|y_l(x)\right\|^2_{(H^1(\Omega))^N}$ [Coerciveness] $\forall y_l\in(H^1(\Omega))^N$. (2) According to the Cauchy Schwartz inequality, the linear form is continuous on $(H^1(\Omega))^2$

$$\left|\int_{\Omega}\sum_{l=1}^{N}y_l(t-\tau)\xi_l dx\right|\leq c_1\sum_{l=1}^{N}\left\|y_l(t-\tau)\right\|_{[L^2(\Omega)]^N}\left\|\xi_l\right\|^2_{[L^2(\Omega)]^N},c_1$$

Lipschitz constant.

$$\left|\int_{\Omega}\sum_{l=1}^{N}f_l\xi_l dx\right|\leq\sum_{l=1}^{N}\left\|f_l\right\|_{[L^2(\Omega)]^N}\left\|\xi_l\right\|^2_{[L^2(\Omega)]^N}$$

$$\&\ \left|\int_{\partial\Omega}\sum_{l=1}^{N}\frac{g_l}{\beta}\xi_l\,ds\right|\leq\sum_{l=1}^{N}\left\|\frac{g_l}{\beta}\right\|_{[L^2(\partial\Omega)]^N}\left\|\xi_l\right\|^2_{[L^2(\Omega)]^N},\ t\in[-\tau,0)$$

Using Poincare inequality $|L(\xi_l)| \le C_b \| \xi_l \|_{[H^1(\Omega)]^N}$, C, C_b are constants. $L(\xi)$ is continuous on $(H^1(\Omega))^2$ then by Lax-Milgram lemma there exist a unique solution $y = (y_1, y_2, y_3, ..., y_N) \in (H^1(\Omega))^N$, such that $b(y, \xi) = L(\xi) \ \ \forall \xi \in (H^1(\Omega))^N$.

2.2 *Formulation of the distributed control problem*

Let $(L^2(\Omega))^N$ being the space of controls.

The energy function $\phi(\xi) = b(y, \xi) - 2L(\xi)$ (3) for a control $u = (u_l)_{l=1}^N \in \underbrace{(L^2(\Omega) \times ... \times L^2(\Omega))}_{N-times}$ the state of the system $y(u) = \{y_l(u)\}_{l=1}^N \in \underbrace{(L^2(\Omega) \times ... \times L^2(\Omega))}_{N-times}$ is given by the solution of the following system:-

$$\begin{cases} -\sum_{l=1}^{N} [(\Delta + k_l^2) y_l] + \sum_{l,m=1}^{N} a_{lm} y_m = \sum_{l,m=1}^{N} [f_l + \lambda_l y_l(t-\tau)] + u_l \ in \ \Omega \\ \sum_{l=1}^{N} \left[\beta_l \frac{\partial y_l}{\partial n} + \alpha_l y_l \right] = \sum_{l=1}^{N} g_l, \qquad l \neq m \qquad \text{on } \partial\Omega \end{cases}, \tag{4}$$

the same conjugation condition (*). It is given that the observation equation is $z_m(u) = \{z_m(u)\}_{m=1}^N \equiv y_m(u)$. Minimize the cost function as

$$C(y, u) = (\chi u, u)_{(L^2(\Omega))^N} + \left\| y_i(u) - O_{id} \right\|_H^2, H \subset (L^2(\Omega))^N \tag{5}$$

$$C(u) = (\chi u, u)_{(L^2(\Omega))^N} + \sum_{j=1}^{N} \int_{\Omega} (y(u) - O_{jd})^2 dx$$

Where $O_d = \{O_{1d}, O_{2d}, O_{3d}, .., O_{Nd}\} \in (L^2(\Omega))^N$.

On, χ can be defined as a hermitian positive-definite operator. on $\underbrace{(L^2(\Omega) \times ... \times L^2(\Omega))}_{N-times}$ such that:

$$(\chi u, u) \ge \xi \| u \|^2_{(L^2(\Omega))^N}, \ \zeta > 0 \tag{6}$$

The control problem summarizes is to find:

$$\begin{cases} u \in U_{ad} \ S.t \\ C(y, u) = \inf C(y, v) \ \forall v \in U_{ad} \end{cases},$$

U_{ad} represents a closed convex subset from $\frac{(L^2(\Omega)\times ...\times L^2(\Omega))}{N-times}$, since it is possible to describe the cost function (5) as

$$C(y,v)=(\chi v,v)_{(L^2(\Omega))^N}+\sum_{m=1}^{N}\int_\Omega (y_m(v)-y_m(0),y_m(0)-O_{md})^2\,dx$$
$$=b(v,v)-2L(v)+\sum_{m=1}^{N}\left\|O_{md}-y_m(0)\right\|^2, m=1,2,..,N.$$

Where

$$b(v,v)=(\chi v,v)_{(L^2(\Omega))^N}+\sum_{m=1}^{N}(y_m(u)-y_m(0),y_m(v)-y_m(0))_{(L^2(\Omega))^N},$$

$$L(v)=\sum_{m=1}^{N}(O_{md}-y_m(0),y_m(v)-y_m(0))_{(L^2(\Omega))^N}, m=1,2,..,N.$$

Assuming that (2) and (6) are valid, the cost function provided (5) is necessary and sufficient for u to be an optimal control.

$$\begin{cases} -\sum_{l=1}^{N}[(\Delta+k_l^2)s_l]+\sum_{l,m=1}^{N}a_{ml}s_m-\sum_{l=1}^{N}\lambda_l s_l(t+\tau)=\sum_{m=1}^{N}y_m(u)-O_{md} & \text{in } \Omega \\ \sum_{l=1}^{N}\left[\beta_l\frac{\partial s_l}{\partial n}+\alpha_l s_l\right]=0, \quad l\neq m & \text{on } \partial\Omega, \\ \left\{\sum_{l=1}^{N}\frac{\partial s_l}{\partial V_{A^*}}\right\}^{\pm}=\frac{\left[\sum_{l=1}^{N}s_l\right]}{R_1+R_2}, \left[\sum_{l=1}^{N}\frac{\partial s_l}{\partial V_{A^*}}\right]=0 & \text{on } \gamma \end{cases} \qquad (7)$$

$s(u)=\{s_l(u)\}_{l=1}^{N}$ is the state that is adjacent.

Overview of proof

Since $C(v)$ can be differentiable and U_{ad} can be bound, optimal control u is defined by $\sum_{m=1}^{N}C'(v)(v_m-u_m)\geq 0 \ \ \forall v\in U_{ad}$ which is equivalent to

$$\sum_{m=1}^{N}(\chi_m u_m,v_m-u_m)_{(L^2(\Omega))^N}+(y_m(u)-O_{md},y_m(v)-y_m(u))\geq 0 \qquad (8)$$

Since $(A^*s,y)=(s,Ay)$ where A is defined by:

$$A\varphi=A\{y_1(u),y_2(u),y_3(u),..,y_N(u)\}=-\sum_{l=1}^{N}[(\Delta+k_l^2)y_l]+\sum_{l,m=1}^{N}a_{lm}y_m,$$

$$\text{then } (s,Ay)=\sum_{m=1}^{N}\left(s_m,-\sum_{l=1}^{N}[(\Delta+k_l^2)y_l]+\sum_{l,m=1}^{N}a_{lm}y_m\right)_{(L^2(\Omega))^N}$$

the sense of distribution, Green's formula or derivative can be used.

$$(A^*s,y)=\left(-\sum_{l=1}^{N}[(\Delta+k_l^2)y_l]+\sum_{l,m=1}^{N}a_{lm}y_m,y_m\right)$$

$A\,s(u)+N^Ts(u)=0$, N^T is transpose of N, $s(u)$ adjoint state

$A=-\sum_{l=1}^{N}[(\Delta+k_l^2)]+\sum_{l,m=1}^{N}a_{lm}, N=a_{lm},\ l,m=1,2,...,N,$ (8) is the same as

$$\sum_{m=1}^{N}(\chi_m u_m,v_m-u_m)_{(L^2(\Omega))^N}+\left(-\sum_{l=1}^{N}[(\Delta+k_l^2)s_l]+\sum_{l,m=1}^{N}a_{lm}s_m,y_m(v)-y_m(u)\right)\geq 0.$$

From (7), we obtain

$$\sum_{m=1}^{N}(\chi_m u_m,v_m-u_m)_{(L^2(\Omega))^N}+\int_{\Omega}\sum_{m=1}^{N}s_m(v_m-u_m)dx\geq 0.$$

If constraints are absent i.e. when $\mu_\Omega=\mu$ the equality

$$s_m(u)+\chi_m u_m=0$$

i.e., $\frac{-s_m(u)}{\chi_m}=u_m$ subsequently, the differential problem of determining the vector-function $(y_m,s_m)^T$

$$\begin{cases}
-\sum_{l=1}^{N}[(\Delta+k_l^2)s_l]+\sum_{l,m=1}^{N}a_{ml}s_m-\sum_{l=1}^{N}\lambda_l s_l(t+\tau)=\sum_{m=1}^{N}y_m(u)-O_{md} & \text{in } \Omega\\
-\sum_{l=1}^{N}[(\Delta+k_l^2)y_l]+\sum_{l,m=1}^{N}a_{ml}y_m=\sum_{l=1}^{N}[f_l+\lambda_l y_l(t-\tau)]+\frac{s_l}{\chi_l} & \text{in } \Omega\\
\sum_{l=1}^{N}\left[\beta_l\frac{\partial y_l}{\partial n}+\alpha_l y_l\right]=\sum_{l=1}^{N}g_l,\sum_{l=1}^{N}\left[\beta_l\frac{\partial s_l}{\partial n}+\alpha_l s_l\right]=0 \qquad l\neq m & \text{on } \partial\Omega\\
\left\{\sum_{l=1}^{N}\frac{\partial s_l}{\partial V_{A^*}}\right\}^{\pm}=\frac{\left[\sum_{l=1}^{N}s_l\right]}{R_1+R_2},\left[\sum_{l=1}^{N}\frac{\partial s_l}{\partial V_{A^*}}\right]=0 & \text{on } \gamma\\
R_1\left\{\sum_{l=1}^{N}\frac{\partial y_l}{\partial V_A}\right\}^{+}+R_2\left\{\sum_{l=1}^{N}\frac{\partial y_l}{\partial V_A}\right\}^{-}=\left[\sum_{l=1}^{N}y_l\right]+\varepsilon,\left[\sum_{l=1}^{N}\frac{\partial y_l}{\partial V_A}\right]=\varpi_l & \text{on } \gamma
\end{cases}$$

Remark 1: Let N=2, the optimality system is given by

$$(\chi_1 u_1, v_1 - u_1)_{(L^2(\Omega))^N} + (\chi_2 u_2, v_2 - u_2)_{(L^2(\Omega))^N} \cdot + \int_\Omega s_1(v_1 - u_1)dx + \int_\Omega s_2(v_2 - u_2)dx \geq 0$$

$$\begin{cases} -(\Delta + k_1^2)s_1 + a_{21}s_2 - \lambda_1 s_1(t+\tau) = y_1(u) - O_{1d} & \text{in } \Omega \\ -(\Delta + k_2^2)s_2 + a_{12}s_1 - \lambda_2 s_2(t+\tau) = y_2(u) - O_{2d} & \text{in } \Omega \\ -(\Delta + k_1^2)y_1 + a_{12}y_2 = [f_1 + \lambda_1 y_1(t-\tau)] + u_1 & \text{in } \Omega \\ -(\Delta + k_2^2)y_1 + a_{21}y_1 = [f_2 + \lambda_2 y_2(t-\tau)] + u_2 & \text{in } \Omega \\ \left[\beta_1 \dfrac{\partial y_1}{\partial n} + \alpha_1 y_1\right] = g_1, \beta_1 \dfrac{\partial s_1}{\partial n} + \alpha_1 s_1 = 0 & \text{on } \partial\Omega \\ \left[\beta_2 \dfrac{\partial y_2}{\partial n} + \alpha_2 y_2\right] = g_2, \beta_2 \dfrac{\partial s_2}{\partial n} + \alpha_2 s_2 = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} R_1\left\{\dfrac{\partial y_1}{\partial V_A}\right\}^+ + R_2\left\{\dfrac{\partial y_1}{\partial V_A}\right\}^- = [y_1] + \varepsilon, \left[\dfrac{\partial y_1}{\partial V_A}\right] = \varpi_1 & \text{on } \gamma \\ R_1\left\{\dfrac{\partial y_2}{\partial V_A}\right\}^+ + R_2\left\{\dfrac{\partial y_2}{\partial V_A}\right\}^- = [y_2] + \varepsilon, \left[\dfrac{\partial y_2}{\partial V_A}\right] = \varpi_2 & \text{on } \gamma \\ \left\{\dfrac{\partial s_1}{\partial V_{A^*}}\right\}^\pm = \dfrac{[s_1]}{R_1 + R_2}, \left[\dfrac{\partial y_1}{\partial V_A}\right] = 0 & \text{on } \gamma \\ \left\{\dfrac{\partial s_2}{\partial V_{A^*}}\right\}^\pm = \dfrac{[s_2]}{R_1 + R_2}, \left[\dfrac{\partial y_2}{\partial V_A}\right] = 0 & \text{on } \gamma \end{cases}$$

3. Control under conjugation conditions for N×N elliptic system that involves a Helmholtz operator

In this section, we consider the following N×N elliptic system (1) with Conjugation conditions

$$\begin{cases} R_1\left\{\sum_{l=1}^N \dfrac{\partial y_l}{\partial V_A}\right\}^+ + R_2\left\{\sum_{l=1}^N \dfrac{\partial y_l}{\partial V_A}\right\}^- = \left[\sum_{l=1}^N y_l\right] + \varepsilon & \text{on } \gamma \\ \left[\sum_{l=1}^N \dfrac{\partial y_l}{\partial V_A}\right] = \varpi_l + u_l, & \text{on } \gamma \end{cases} \tag{9}$$

are two chains listed below:-

$$(W^1(\gamma))^N \subseteq (L^2(\gamma))^N \subseteq (W^{-1}(\gamma))^N,$$
$$(W_0^1(\gamma))^N \subseteq (L^2(\gamma))^N \subseteq (W_0^{-1}(\gamma))^N$$

3.2 *The control problem's formulation*

The space for controls can be referred to as $(L^2(\gamma))^N$.
The energy function

$$\phi(v) = b(y,v) - 2L(\eta) \tag{10}$$

The solution of the following system determines the state of system

$$y(u) = \{y_l(u)\}_{l=1}^N \in \underbrace{(L^2(\Omega)\times\ldots\times L^2(\Omega))}_{N-times}$$

for a control $u = (u_l)_{l=1}^N \in (L^2(\gamma))^N$.

$$\begin{cases} -\sum_{l=1}^N [(\Delta + k_l^2)y_l] + \sum_{l,m=1}^N a_{lm}y_m = \sum_{l=1}^N [f_l + \lambda_l y_l(t-\tau)] \text{ in } \Omega \subset \mathbb{R}^N \\ \sum_{l=1}^N \left[\beta_l \frac{\partial y_l}{\partial n} + \alpha_l y_l\right] = \sum_{l=1}^N g_l, \qquad l \neq m \qquad \text{on } \partial\Omega \\ R_1 \left\{\sum_{l=1}^N \frac{\partial y_l}{\partial V_A}\right\}^+ + R_2 \left\{\sum_{l=1}^N \frac{\partial y_l}{\partial V_A}\right\}^- = \left[\sum_{l=1}^N y_l\right] + \varepsilon \qquad \text{on } \gamma \\ \left[\sum_{l=1}^N \frac{\partial y_l}{\partial V_A}\right] = \varpi_l + u_l, \qquad \text{on } \gamma \end{cases} \tag{11}$$

It is given that the observation equation is

$$z_m(u) = \{z_m(u)\}_{m=1}^N \equiv y_m(u).$$

Minimize the cost function as

$$C(y,u) = (\sigma u, u)_{(L^2(\Omega))^N} + \left\|y(u) - O_d\right\|_H^2, H \subset L^2(\gamma) \tag{12}$$

$$C(y,u) = (\sigma u, u)_{(L^2(\Omega))^N} + \sum_{m=1}^N \int_\gamma (y(u) - O_{md})^2 d\gamma$$

Where $O_d = \{O_{1d}, O_{2d}, O_{3d}, .., O_{Nd}\} \in (L^2(\gamma))^N$

On, ϖ can be defined as a hermitian positive-definite operator.

on $\underbrace{(L^2(\Omega)\times\ldots\times L^2(\Omega))}_{N-times}$ such that:

$$(\sigma u, u) \geq \varsigma \left\| u \right\|^2_{(L^2(\Omega))^N}, \varsigma > 0 \tag{13}$$

The challenge lies in identifying:

$$\begin{cases} u \in U_{ad} \ S.t \\ C(u) = \inf C(v) \ \forall v \in U_{ad} \end{cases}, \ U_{ad} \text{ is a closed convex subset from } (L^2(\gamma))^N,$$

as the cost function (12) can be interpreted as

$$\begin{aligned} C(y, v) &= (\sigma u, u)_{(L^2(\Omega))^N} + \sum_{m=1}^{N} \int_{\gamma} (y_m(v) - y_m(0), y_m(0) - O_{md})^2 d\gamma \\ &= b(v, v) - 2L(v) + \left\| O_{md} - y_m(0) \right\|^2, m = 1, 2, ..., N. \end{aligned}$$

Where

$$b(v, v) = (\sigma u, u)_{(L^2(\Omega))^N} + \sum_{m=1}^{N} (y_m(u) - y_m(0), y_m(v) - y_m(0))_{(L^2(\Omega))},$$

$$L(v) = \sum_{m=1}^{N} (O_{md} - y_m(0), y_m(v) - y_m(0))_{(L^2(\Omega))}, m = 1, 2, .., N.$$

Assuming that (2) and (13) are valid, the cost function provided (12) is necessary and sufficient for u to be an optimal control.

$$\begin{cases} -\sum_{l=1}^{N} [(\Delta + k_l^2) s_l] + \sum_{l,m=1}^{N} a_{ml} s_m = \sum_{l=1}^{N} \lambda_l s_l (t + \tau) \text{ in } \Omega \subset \mathbb{R}^N \\ \sum_{l=1}^{N} \left[\beta_l \dfrac{\partial s_l}{\partial n} + \alpha_l s_l \right] = 0, \qquad l \neq m \qquad \text{on } \partial\Omega \\ R_1 \left\{ \sum_{l=1}^{N} \dfrac{\partial s_l}{\partial V_{A^*}} \right\}^{+} + R_2 \left\{ \sum_{l=1}^{N} \dfrac{\partial s_l}{\partial V_{A^*}} \right\}^{-} = \left[\sum_{l=1}^{N} s_l \right] \qquad \text{on } \gamma \\ \left[\sum_{l=1}^{N} \dfrac{\partial y_l}{\partial V_A} \right] = \varpi_l + \sum_{m=1}^{N} [y_m(u)] - O_{md}, \qquad \text{on } \gamma \end{cases} \tag{14}$$

$s(u) = \{s_l(u)\}_{l=1}^{N}$ is the state that is adjacent,

Overview of proof

Since $C(v)$ can be differentiable and U_{ad} can be bound, optimal control u is defined by $\sum_{m=1}^{N} C'(v)(v_m - u_m) \geq 0 \ \forall v \in U_{ad}$ which is equivalent to

$$(\varpi_m u_m, v_m - u_m)_{(L^2(\Omega))^N} + (y_m(u) - O_{md}, y_m(v) - y_m(u)) \geq 0 \tag{15}$$

Since $(A^*s, y) = (s, Ay)$ where A is defined by:

$$A\varphi = A\{y_1(u), y_2(u), y_3(u), .., y_N(u)\} = -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{l=1}^{N} a_{ml}y_m,$$

then $(s, Ay) = \sum_{m=1}^{N}\left(s_m, -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{l=1}^{N} a_{ml}y_m\right)_{(L^2(\Omega))^N}$

In the sense of distribution, Green's formula or derivative can be used.

$$(A^*s, y) = \left(-\sum_{k=1}^{N}(D_k^2)s_m + \sum_{i=1}^{N} a_{lm}s_m, y_m\right),\ A\ s(u) + N^T s(u) = 0,\ N^T$$

is transpose of N, $s(u)$ adjoint state

$$A = -\sum_{k=1}^{N}(D_k^2) + \sum_{l=1}^{N} a_{lm}, N = a_{lm}, l, m = 1, 2, ..., N,$$

(15) is equivalent to

$$(\varpi_m u_m, v_m - u_m)_{(L^2(\Omega))^N} + \left(-\sum_{k=1}^{N}(D_k^2)s_m + \sum_{l=1}^{N} a_{lm}s_m, y_m(v) - y_m(u)\right) \geq 0$$

Utilizing Green's formula, we have

$$\left(-\mu\left\{\sum_{l=1}^{N}\frac{\partial s_l}{\partial V_{A^*}}\right\}^{+} - \nu\left\{\sum_{l=1}^{N}\frac{\partial s_l}{\partial V_{A^*}}\right\}^{-} + \sum_{l=1}^{N}\varpi_l u_l, v_m - u_m\right)_{(L^2(\Omega))^N} \geq 0.$$

$$\mu = \frac{R_1}{R_1 + R_1}, \nu = \frac{R_2}{R_1 + R_1}$$

Remark 2: Let N=2, the optimality system is given by

$$\left(\left\{-\mu\left\{\frac{\partial s_1}{\partial V_{A^*}}\right\}^{+} - \nu\left\{\frac{\partial s_1}{\partial V_{A^*}}\right\}^{-} + \{\varpi_1 u_1\}\right\}, \{v_1 - u_1\}\right)_{(L^2(\gamma))^N} \geq 0,$$

$$\left(\left\{-\mu\left\{\frac{\partial s_2}{\partial V_{A^*}}\right\}^{+} - \nu\left\{\frac{\partial s_2}{\partial V_{A^*}}\right\}^{-} + \{\varpi_2 u_2\}\right\}, \{v_2 - u_2\}\right)_{(L^2(\gamma))^N} \geq 0.$$

$$\begin{cases} (\Delta + k_1^2)s_1 + a_{21}s_2 = \lambda_1 s_1(t+\tau) & \text{in } \Omega \subset \mathbb{R}^N \\ (\Delta + k_1^2)s_2 + a_{12}s_1 = \lambda_2 s_2(t+\tau) & \text{in } \Omega \subset \mathbb{R}^N \\ \left[\beta_1 \dfrac{\partial s_1}{\partial n} + \alpha_1 s_1\right] = 0 & \text{on } \partial\Omega \\ \left[\beta_2 \dfrac{\partial s_2}{\partial n} + \alpha_2 s_2\right] = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} R_1\left\{\dfrac{\partial s_1}{\partial V_{A^*}}\right\}^+ + R_2\left\{\dfrac{\partial s_1}{\partial V_{A^*}}\right\}^- = [s_1] & \text{on } \gamma \\ R_1\left\{\dfrac{\partial s_2}{\partial V_{A^*}}\right\}^+ + R_2\left\{\dfrac{\partial s_2}{\partial V_{A^*}}\right\}^- = [s_2] & \text{on } \gamma \\ \left[\dfrac{\partial y_1}{\partial V_A}\right] = \varpi_1 + [y_1(u)] - O_{1d}\ , & \text{on } \gamma \\ \left[\dfrac{\partial y_2}{\partial V_A}\right] = \varpi_2 + [y_2(u)] - O_{2d}\ , & \text{on } \gamma \end{cases}$$

Conclusion

Finding the existence and uniqueness for solution (state of the system) and necessary and sufficient condition for optimality of the system in N×N and special case 2×2 in the case of distributed and boundary control on conditions.

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