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# Some new results on total irregularity of graphs

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## Abstract

The overall variation in vertex connections within a graph, known as the total irregularity, is quantified by summing half the absolute differences between the degrees of every vertex pair. This study investigates how this measure of irregularity behaves when a graph undergoes two specific transformations: the creation of a k-subdivision and the $K_r$-gluing of two graphs.

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## 1. Introduction

Consider a connected, non-looping graph $G$ with $n$ vertices and $m$ edges. The highest and lowest number of connections a vertex possesses are represented by $\Delta(G)$ and $\delta(G)$, respectively. The degree of a specific vertex $u$, denoted as $d_G(u)$, indicates the count of its directly linked neighbors. A graph is considered $t$-regular when all vertices share the same degree $t$; otherwise, it's deemed irregular. Quantifying the degree of irregularity in a graph is crucial in various applications. Researchers have developed numerous metrics to assess this irregularity.The oldest irregularity measure was proposed by Collatz and Sinogowitz [1] in 1957. Collatz-Sinogowitz index [2] and Sigma index [3, 4] are some of the defined irregularity measures. Albertson's work [5] introduced the concept of edge imbalance, defining it as the absolute difference in degrees between the two vertices connected by an edge. He then extended this idea to define the irregularity of the entire graph as the sum of the imbalances of all its edges, i.e.,

$$irr(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|.$$

Albertson's work revealed that the total edge imbalance of a graph with $n$ vertices, denoted as $irr(G)$, is always less than $\frac{4n^3}{27}$, and graphs can be constructed to get arbitrarily close to this upper bound. He further analyzed irregularity constraints for various graph types, including general graphs, those divisible into two independent vertex sets (bipartite graphs), and graphs lacking triangular connections. Additionally, he determined the maximum irregularity achievable by tree-like structures. Albertson identified that the most irregular graphs, for a given vertex count, possess a distinct composition: a complete subgraph paired with unconnected vertices. For a deeper understanding of graph irregularity and alternative measurement techniques, refer to the works cited in [6, 7, 8, 9, 10, 11]. Also irregularity for Sudoku graph and Sierpin′ski graph studied in [12, 13].

Abdo, Brandt, and Dimitrov [14] introduced a metric for measuring a graph's irregularity, termed the total irregularity, calculated as half the sum of the absolute degree differences between all vertex pairs. This measure, denoted as $irr_t(G)$, is defined as:

$$irr_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)|.$$

It's evident that $irr_t(G)$ is always greater than or equal to the standard irregularity measure, $irr(G)$. Equality occurs precisely when all non-adjacent vertices share the same degree. Furthermore, it has been shown that ([15]) for a connected graph with $n$ vertices, $irr_t(G)$ is bounded by $\frac{n^2}{4}irr(G)$. In the specific case of a tree with $n$ vertices, $irr_t(G)$ is bounded by $(n-2)irr(G)$, with equality holding only for path graphs.

The $k$-subdivision of a graph $G$, represented as $G^{\frac{1}{k}}$, involves replacing each edge in $G$ with a path of length $k$, referred to as a superedge. Each newly inserted vertex on a superedge is uniquely identified by its position along the path.

Specifically, a vertex $x_l^{\{v_i,v_j\}}$ lies on the superedge $P^{\{v_i,v_j\}}$ at a distance $l$ from vertex $v_i$, where $l$ ranges from $1$ to $k-1$.

When $k$ equals 1, the $k$-subdivision simply reproduces the original graph $G$. If G has $n$ vertices and $m$ edges, its $k$-subdivision will contain $n+(k-1)m$ vertices and $km$ edges. For instance, the star graph $S_n$ and its 3-subdivision $S^{\frac{1}{3}}$ illustrate this concept and are shown in Figure 1. More comprehensive details on graph subdivisions can be found in [16, 17, 18, 19, 20, 21, 22].

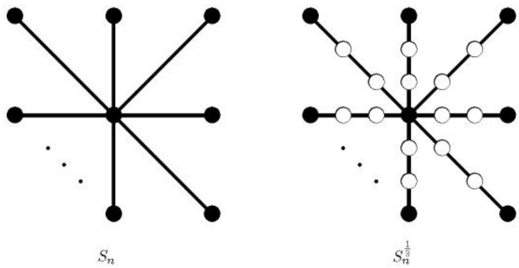


**Figure 1**

**Graphs $S_n$ and $S_n^{\frac{1}{3}}$**

Given two graphs, $G_1$ and $G_2$, and a non-negative integer $r$ that does not exceed the size of the largest complete subgraphs (cliques) in either graph, we can perform an $r$-gluing (or $K_r$-gluing) operation. This involves selecting a clique of size $r$ from each graph and then merging these two cliques together in any way. The resulting graph is the $r$-gluing of $G_1$ and $G_2$ and is denoted as $G_1 \cup_{K_r} G_2$ (see [23]).

If $r$ is zero, the $r$-gluing simply combines the two graphs without any overlap. Special cases of $r$-gluing include vertex gluing $(r=1)$ and edge gluing $(r=2)$. It is important to note that multiple distinct $r$-gluings can be created for the same pair of graphs depending on the specific cliques chosen and how they are merged (see Figure 2). We refer the reader for some results on the $r$-gluing of two graphs to [24, 25].

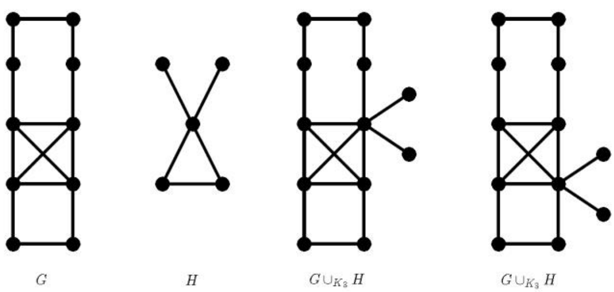


**Figure 2**

**Graph $G$, graph $H$, and all non-isomorphic graphs $G \cup_{K_3} H$.**

The following sections delve into the analysis of total irregularity. Specifically, the impact of $k$-subdivision on total irregularity is examined in the subsequent section, while Section 3 investigates the changes in total irregularity resulting from $K_r$-gluing of two graphs.

## 2. Subdivision

We now turn our attention to analyzing the total irregularity of graphs that have undergone $k$-subdivision. Our investigation begins with the following theorem.

**Theorem 2.1:** *Consider a graph denoted by $G$ and $V_1 = \{x_1^{\{v_i,v_j\}} | v_iv_j \in E(G)\}$. Then*

$$irr_t(G^{\frac{1}{2}}) = irr_t(G) + \frac{1}{2}\sum_{u\in V(G),v\in V_1} |d_G(u) - 2|.$$

***Proof:*** It is clear that $V(G^{\frac{1}{2}}) = V(G) \cup V_1$. According to the established definition of total graph irregularity, it follows that:

$$\begin{aligned}
irr_t(G^{\frac{1}{2}}) &= \frac{1}{2}\sum_{u,v\in V(G^{\frac{1}{2}})} \left|d_{G^{\frac{1}{2}}}(u) - d_{G^{\frac{1}{2}}}(v)\right| \\
&= \frac{1}{2}\Big(\sum_{u,v\in V(G)} \left|d_{G^{\frac{1}{2}}}(u) - d_{G^{\frac{1}{2}}}(v)\right| + \sum_{u,v\in V_1} \left|d_{G^{\frac{1}{2}}}(u) - d_{G^{\frac{1}{2}}}(v)\right| \\
&\quad + \sum_{u\in V(G),v\in V_1} \left|d_{G^{\frac{1}{2}}}(u) - d_{G^{\frac{1}{2}}}(v)\right|\Big) \\
&= \frac{1}{2}\left(\sum_{u,v\in V(G)} |d_G(u) - d_G(v)| + \sum_{u,v\in V_1} |2-2| + \sum_{u\in V(G),v\in V_1} |d_G(u) - 2|\right) \\
&= irr_t(G) + \frac{1}{2}\sum_{u\in V(G),v\in V_1} |d_G(u) - 2|.
\end{aligned}$$

The following corollaries are a direct consequence of Theorem 2.1:

**Corollary 2.2:** *Let $G$ be a graph without isolated vertices with order $n$, size $m$, and $V_1 = \{x_1^{\{v_i,v_j\}} | v_iv_j \in E(G)\}$. Then*

$$irr_t(G) + m^2 - mn \le irr_t(G^{\frac{1}{2}}) \le irr_t(G) + m^2.$$

***Proof:*** By Theorem 2.1 and since $G$ does not have isolated vertices, we have

$$\begin{aligned}
irr_t(G^{\frac{1}{2}}) &= irr_t(G) + \frac{1}{2}\sum_{u\in V(G),v\in V_1} |d_G(u) - 2| \\
&\le irr_t(G) + \frac{1}{2}\Big(\sum_{u\in V(G),v\in V_1} d_G(u)\Big) \\
&= irr_t(G) + \frac{m}{2}\Big(\sum_{u\in V(G)} d_G(u)\Big) \\
&= irr_t(G) + m^2.
\end{aligned}$$

Now, we find the lower bound. By using Theorem 2.1, we have

$$\begin{aligned}
irr_t(G^{\frac{1}{2}}) &= irr_t(G) + \tfrac{1}{2}\textstyle\sum_{u\in V(G),v\in V_1} |d_G(u) - 2| \\
&\geq irr_t(G) + \tfrac{1}{2}\Big(\textstyle\sum_{u\in V(G),v\in V_1} (|d_G(u)| - 2)\Big) \\
&= irr_t(G) + \tfrac{1}{2}\big(\textstyle\sum_{u\in V(G),v\in V_1} |d_G(u)| - \sum_{u\in V(G),v\in V_1} 2\big) \\
&= irr_t(G) + \tfrac{m}{2}\big(\textstyle\sum_{u\in V(G)} |d_G(u)| - \sum_{u\in V(G)} 2\big) \\
&= irr_t(G) + m^2 - mn.
\end{aligned}$$

Consequently, the result is established.

**Corollary 2.3:** *Given a $k$-regular graph $G$ with $n$ vertices, where $k$ is at least $2$, we have*

$$irr_t(G^{\frac{1}{2}}) = \frac{n^2k(k-2)}{4}.$$

***Proof:*** Since $G$ is a $k$-regular graph, then $irr_t(G) = 0$. So by using Theorem 2.1 and the fact that the size of the graph is $\frac{nk}{2}$, we have the result.

In the following, we give an infinite class of graphs that shows the lower bound in Corollary 2.2 is sharp. Also, it is sharp on the upper bound.

**Remark 2.4:** The lower bound presented in Corollary 2.2 is tight. To demonstrate this, it is sufficient to examine the case where $G$ is a $k$-regular graph, with $k$ being at least $2$, and use Corollary 2.3. To show that the upper bound is tight, for this purpose, taking $G$ to be the path graph $P_2$, then $P_2^{\frac{1}{2}} = P_3$. Since $irr_t(P_2) = 0$ and $irr_t(P_3) = 1$, we are done. Moreover one can easily check that for the disjoint union of $n$ graphs $P_2$, say $H$, the total irregularity of the 2-subdivision of $H$ is $n^2$, and equality on the upper bound is hold. So we have an infinite class of graphs that shows the upper bound is sharp.

Before we give the formula for the total irregularity of $G^{\frac{1}{k}}$ in general case, we consider $G^{\frac{1}{3}}$:

**Theorem 2.5:** *Let $G$ be a graph and $V_2 = \{x_1^{\{v_i,v_j\}}, x_2^{\{v_i,v_j\}} | v_iv_j \in E(G)\}$. Then*

$$irr_t(G^{\frac{1}{3}}) = irr_t(G) + \tfrac{1}{2}\textstyle\sum_{u\in V(G),v\in V_2} |d_G(u) - 2|.$$

***Proof:*** We have $V(G^{\frac{1}{3}}) = V(G) \cup V_2$. According to the definition of total irregularity, we can establish:

$$irr_t(G^{\frac{1}{3}}) = \tfrac{1}{2}\textstyle\sum_{u,v\in V(G^{\frac{1}{3}})} \left|d_{G^{\frac{1}{3}}}(u) - d_{G^{\frac{1}{2}}}(v)\right|$$

$$= \tfrac{1}{2}\Big(\textstyle\sum_{u,v\in V(G)} \left|d_{G^{\frac{1}{3}}}(u) - d_{G^{\frac{1}{3}}}(v)\right| + \sum_{u,v\in V_2} \left|d_{G^{\frac{1}{3}}}(u) - d_{G^{\frac{1}{3}}}(v)\right|$$

$$+\sum_{u\in V(G),v\in V_2}\left|d_{G^{\frac{1}{3}}}(u)-d_{G^{\frac{1}{3}}}(v)\right|\Big)$$
$$=\frac{1}{2}\big(\sum_{u,v\in V(G)}|d_G(u)-d_G(v)|+\sum_{u,v\in V_2}|2-2|+\sum_{u\in V(G),v\in V_2}|d_G(u)-2|\big)$$
$$=irr_t(G)+\frac{1}{2}\sum_{u\in V(G),v\in V_2}|d_G(u)-2|.$$

By a similar argument as Theorems 2.1 and 2.5 in its general form, the result is:

**Theorem 2.6:** *Assume we have a graph $G$, and*

$$V_{k-1}=\{x_1^{\{v_i,v_j\}},x_2^{\{v_i,v_j\}},\dots,x_{k-1}^{\{v_i,v_j\}}|v_iv_j\in E(G)\}.$$

*Then*

$$irr_t(G^{\frac{1}{k}})=irr_t(G)+\frac{1}{2}\sum_{u\in V(G),v\in V_{k-1}}|d_G(u)-2|.$$

By a similar argument as Corollary 2.2, as a consequence of Theorem 2.6 we have:

**Corollary 2.7:** *Let $G$ be a graph without isolated vertices with order $n$ and size $m$. Then*

$$irr_t(G)+(k-1)m^2-(k-1)mn\le irr_t(G^{\frac{1}{k}})\le irr_t(G)+(k-1)m^2.$$

As we have shown that in Remark 2.4, the bounds in Corollary 2.7 are sharp. By a similar argument as Corollary 2.3, as a result of Theorem 2.6, we also have the following Corollary:

**Corollary 2.8:** *If $G$ is a $k$-regular graph of order $n$ $(k\ge 2)$, then*

$$irr_t(G^{\frac{1}{t}})=\frac{n^2k(k-2)(t-1)}{4}.$$

## 3. $K_r$-Gluing

This section examines the total irregularity resulting from the $K_r$-gluing operation applied to two graphs. We begin by analyzing the specific case of $K_0$-gluing, which corresponds to the separate combination of graphs $G_1$ and $G_2$.

**Theorem 3.1:** *Consider two graphs, $G_1$ and $G_2$, where $G_1$ has vertex set $V_1$ and edge set $E_1$, and $G_2$ has vertex set $V_2$ and edge set $E_2$. Assume $G_1$ has $n_1$ vertices and $m_1$ edges, and $G_2$ has $n_2$ vertices and $m_2$ edges. Also, let $\gamma=max\{n_2m_1-n_1m_2,n_1m_2-n_2m_1\}$ and $\theta=n_2m_1+n_1m_2$. Then*

$$irr_t(G_1)+irr_t(G_2)+\gamma\le irr_t(G_1\cup_{K_0}G_2)\le irr_t(G_1)+irr_t(G_2)+\theta.$$

***Proof:*** Let $G=G_1\cup_{K_r}G_2$. By the definition of total irregularity of graphs, we have

$$irr_t(G_1 \cup_{K_0} G_2) = \frac{1}{2}\sum_{u,v\in V(G_1\cup_{K_0}G_2)} |d_G(u) - d_G(v)|$$
$$= \frac{1}{2}\Big(\sum_{u,v\in V(G_1)} |d_G(u) - d_G(v)| + \sum_{u,v\in V(G_2)} |d_G(u) - d_G(v)|$$
$$+ \sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|\Big)$$
$$= irr_t(G_1) + irr_t(G_2) + \frac{1}{2}\Big(\sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|\Big).$$

Here, we find and upper bound for $\frac{1}{2}\left(\sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|\right)$.

$$\frac{1}{2}\Big(\sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|\Big) \le \frac{1}{2}\Big(\sum_{u\in V_1,v\in V_2} |d_G(u)| + \sum_{u\in V_1,v\in V_2} |d_G(v)|\Big)$$
$$= \frac{1}{2}\Big(n_2\sum_{u\in V_1} |d_G(u)| + n_1\sum_{,v\in V_2} |d_G(v)|\Big)$$
$$= n_2m_1 + n_1m_2.$$

Therefore we have the upper bound. Now, we find the lower bound.

$$\frac{1}{2}\Big(\sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|\Big) \ge \frac{1}{2}\Big(\sum_{u\in V_1,v\in V_2} |d_G(u)| - \sum_{u\in V_1,v\in V_2} |d_G(v)|\Big)$$
$$= \frac{1}{2}\Big(n_2\sum_{u\in V_1} |d_G(u)| - n_1\sum_{,v\in V_2} |d_G(v)|\Big)$$
$$= n_2m_1 - n_1m_2.$$

Similarly, we have $\frac{1}{2}\left(\sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|\right) \ge n_1m_2 - n_2m_1$. Therefore we have the lower bound, and we are done.

The following theorem gives an upper bound for $irr_t(G_1 \cup_{K_r} G_2)$ $(r \in \mathbb{N})$:

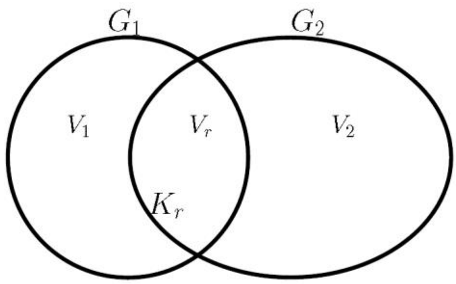


**Figure 3**

**Graph $G_1 \cup_{K_r} G_2$ with vertex set $V = V_1 \cup V_r \cup V_2$**

**Theorem 3.2:** *Consider two graphs, $G_1$ and $G_2$, where $G_1$ has $n_1$ vertices and $m_1$ edges, and $G_2$ has $n_2$ vertices and $m_2$ edge and $r \in \mathbb{N}$. Also, let*

$$\alpha = \min\Big\{\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)|, \sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|\Big\},$$

*and $V = V(G_1 \cup_{K_r} G_2) = V_1 \cup V_r \cup V_2$, as we see in Figure 3, $V(G_1) = V_1 \cup V_r$ and $V(G_2) = V_2 \cup V_r$. Then*

$$irr_t(G_1 \cup_{K_r} G_2) \le irr_t(G_1) + irr_t(G_2) + \alpha + \frac{r(r-1)(n_1+n_2-r)}{2} +$$
$$m_1(n_2 - r) + m_2(n_1 - r).$$

***Proof:*** Let $G = G_1 \cup_{K_r} G_2$. By the definition of total irregularity of graphs, we have

$$irr_t(G_1 \cup_{K_r} G_2) = \frac{1}{2}\sum_{u,v\in V} |d_G(u) - d_G(v)|$$
$$= \frac{1}{2}\Big(\sum_{u,v\in V(G_1)} |d_G(u) - d_G(v)| + \sum_{u,v\in V(G_2)} |d_G(u) - d_G(v)|$$
$$- \sum_{u,v\in V_r} |d_G(u) - d_G(v)| + \sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|\Big).$$

First, we find $\frac{1}{2}\sum_{u,v\in V(G_1)} |d_G(u) - d_G(v)|$.

$$\frac{1}{2}\sum_{u,v\in V(G_1)} |d_G(u) - d_G(v)| = \frac{1}{2}\Big(\sum_{u,v\in V_1} |d_G(u) - d_G(v)|$$
$$+ \sum_{u,v\in V_r} |d_G(u) - d_G(v)| + \sum_{u\in V_1,v\in V_r} |d_G(u) - d_G(v)|\Big)$$
$$= \frac{1}{2}\Big(\sum_{u,v\in V_1} \left|d_{G_1}(u) - d_{G_1}(v)\right|$$
$$+ \sum_{u,v\in V_r} \left|d_{G_1}(u) + d_{G_2}(u) - r + 1 - (d_{G_1}(v) + d_{G_2}(v) - r + 1)\right|$$
$$+ \sum_{u\in V_1,v\in V_r} \left|d_{G_1}(u) - (d_{G_1}(v) + d_{G_2}(v) - r + 1)\right|\Big)$$
$$= \frac{1}{2}\Big(\sum_{u,v\in V_1} \left|d_{G_1}(u) - d_{G_1}(v)\right|$$
$$+ \sum_{u,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v) + d_{G_2}(u) - d_{G_2}(v)\right|$$
$$+ \sum_{u\in V_1,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v) - (d_{G_2}(v) - r + 1)\right|\Big)$$
$$\le \frac{1}{2}\Big(\sum_{u,v\in V_1} \left|d_{G_1}(u) - d_{G_1}(v)\right|$$
$$+ \sum_{u,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v)\right| + \sum_{u,v\in V_r} \left|d_{G_2}(u) - d_{G_2}(v)\right|$$
$$+ \sum_{u\in V_1,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v)\right| + \sum_{u\in V_1,v\in V_r} \left|d_{G_2}(v) - r + 1\right|\Big).$$

So we have

$$\frac{1}{2}\sum_{u,v\in V(G_1)} |d_G(u) - d_G(v)| \quad \le irr_t(G_1) + \frac{1}{2}\Big(\sum_{u,v\in V_r} \left|d_{G_2}(u) - d_{G_2}(v)\right|$$
$$+ \sum_{u\in V_1,v\in V_r} \left|d_{G_2}(v) - r + 1\right|\Big)$$
$$\le irr_t(G_1) + \frac{1}{2}\Big(\sum_{u,v\in V_r} \left|d_{G_2}(u) - d_{G_2}(v)\right|$$
$$+ \sum_{u\in V_1,v\in V_r} d_{G_2}(v) + \sum_{u\in V_2,v\in V_r} r - 1\Big)$$
$$\le irr_t(G_1) + \frac{1}{2}\Big(\sum_{u,v\in V_r} \left|d_{G_2}(u) - d_{G_2}(v)\right|$$
$$+|V_1| \sum_{v\in V_r} d_{G_2}(v) + |V_2||V_r|(r - 1)\Big).$$

By a similar argument, we have

$$\frac{1}{2}\sum_{u,v\in V(G_2)} |d_G(u) - d_G(v)| \quad \le irr_t(G_2) + \frac{1}{2}\Big(\sum_{u,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v)\right|$$
$$+|V_2| \sum_{v\in V_r} d_{G_1}(v) + |V_1||V_r|(r - 1)\Big).$$

Now, we consider $-\frac{1}{2}\sum_{u,v\in V_r} |d_G(u) - d_G(v)|$. Since $|a + b| \ge ||a| - |b||$ and $|a - b| \ge |a| - |b|$, we have

$$-\frac{1}{2}\sum_{u,v\in V_r} |d_G(u) - d_G(v)| = -\frac{1}{2}\sum_{u,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v) + d_{G_2}(u) - d_{G_2}(v)\right|$$
$$\leq -\frac{1}{2}\sum_{u,v\in V_r} \left||d_{G_1}(u) - d_{G_1}(v)| - |d_{G_2}(u) - d_{G_2}(v)|\right|$$
$$\leq -\frac{1}{2}\sum_{u,v\in V_r} (|d_{G_1}(u) - d_{G_1}(v)| - |d_{G_2}(u) - d_{G_2}(v)|)$$
$$= -\frac{1}{2}\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)| + \frac{1}{2}\sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|.$$

Similarly we have

$$-\frac{1}{2}\sum_{u,v\in V_r} |d_G(u) - d_G(v)| \leq \frac{1}{2}\sum_{u,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v)\right| - \frac{1}{2}\sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|.$$

So we have

$$-\frac{1}{2}\sum_{u,v\in V_r} |d_G(u) - d_G(v)| \leq \min\{\frac{1}{2}\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)| - \frac{1}{2}\sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|, -\frac{1}{2}\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)| + \frac{1}{2}\sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|\}.$$

Now, we find $\frac{1}{2}\sum_{u\in V_1, v\in V_2} |d_G(u) - d_G(v)|$.

$$\frac{1}{2}\sum_{u\in V_1, v\in V_2} |d_G(u) - d_G(v)| \leq \frac{1}{2}\sum_{u\in V_1, v\in V_2} d_G(u) + \frac{1}{2}\sum_{u\in V_1, v\in V_2} d_G(v)$$
$$= \frac{1}{2}|V_2|\sum_{u\in V_1} d_{G_1}(u) + |V_1|\frac{1}{2}\sum_{v\in V_2} d_{G_2}(v).$$

Let $\alpha = \min\left\{\sum_{u,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v)\right|, \sum_{u,v\in V_r} \left|d_{G_2}(u) - d_{G_2}(v)\right|\right\}$, then by our argument we have:

$$irr_t(G_1 \cup_{K_r} G_2) \leq \alpha + irr_t(G_1) + \frac{1}{2}(|V_1|\sum_{v\in V_r} d_{G_2}(v) + |V_2||V_r|(r-1))$$
$$+ irr_t(G_2) + \frac{1}{2}(|V_2|\sum_{v\in V_r} d_{G_1}(v) + |V_1||V_r|(r-1))$$
$$+ \frac{1}{2}|V_2|\sum_{u\in V_1} d_{G_1}(u) + |V_1|\frac{1}{2}\sum_{v\in V_2} d_{G_2}(v)$$
$$= \alpha + irr_t(G_1) + irr_t(G_2) + \frac{1}{2}(|V_2||V_r|(r-1) + |V_1||V_r|(r-1))$$
$$+ \frac{1}{2}(|V_2|\sum_{v\in V(G_1)} d_{G_1}(v) + |V_1|\sum_{v\in V(G_2)} d_{G_2}(v))$$
$$= \alpha + irr_t(G_1) + irr_t(G_2) + \frac{1}{2}(|V_1 \cup V_2||V_r|(r-1))$$
$$+ |V_2||E(G_1)| + |V_1||E(G_2)|.$$

Therefore we have the upper bound.

Now, we find a lower bound for $irr_t(G_1 \cup_{K_r} G_2)$ $(r \in \mathbb{N})$:

**Theorem 3.3:** *Consider two graphs, $G_1$ and $G_2$, where $G_1$ has $n_1$ vertices and $m_1$ edges, and $G_2$ has $n_2$ vertices and $m_2$ edge and $r \in \mathbb{N}$. Also, let*

$$\alpha = -\frac{1}{2}n_1(\sum_{v\in V_r} d_{G_2}(v) - \sum_{v\in V_2} d_{G_2}(v)) - n_2 m_1 - \sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)| - \sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|,$$

*and*

$$\beta = -\frac{1}{2}n_2(\sum_{v\in V_r} d_{G_1}(v) - \sum_{v\in V_1} d_{G_1}(v)) - n_1m_2$$
$$-\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)| - \sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|,$$

*and* $V = V(G_1 \cup_{K_r} G_2) = V_1 \cup V_r \cup V_2$, *as we see in Figure 3,* $V(G_1) = V_1 \cup V_r$ *and* $V(G_2) = V_2 \cup V_r$. *Then*

$$irr_t(G_1 \cup_{K_r} G_2) \geq irr_t(G_1) + irr_t(G_2) + max\{\alpha, \beta\} - \frac{r(r-1)(n_1+n_2-r)}{2}.$$

***Proof:*** Let $G = G_1 \cup_{K_r} G_2$. By the definition of total irregularity of graphs, we have

$$irr_t(G_1 \cup_{K_r} G_2) = \frac{1}{2}\sum_{u,v\in V} |d_G(u) - d_G(v)|$$
$$= \frac{1}{2}(\sum_{u,v\in V(G_1)} |d_G(u) - d_G(v)| + \sum_{u,v\in V(G_2)} |d_G(u) - d_G(v)|$$
$$-\sum_{u,v\in V_r} |d_G(u) - d_G(v)| + \sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)|).$$

By a similar argument as proof of Theorem 3.2, we have the following inequalities:

$$\frac{1}{2}\sum_{u,v\in V(G_1)} |d_G(u) - d_G(v)| \geq irr_t(G_1) - \frac{1}{2}(\sum_{u,v\in V_r} \left|d_{G_2}(u) - d_{G_2}(v)\right|$$
$$+|V_1|\sum_{v\in V_r} d_{G_2}(v) + |V_2||V_r|(r-1)),$$
$$\frac{1}{2}\sum_{u,v\in V(G_2)} |d_G(u) - d_G(v)| \geq irr_t(G_2) - \frac{1}{2}(\sum_{u,v\in V_r} \left|d_{G_1}(u) - d_{G_1}(v)\right|$$
$$+|V_2|\sum_{v\in V_r} d_{G_1}(v) + |V_1||V_r|(r-1)),$$
$$-\frac{1}{2}\sum_{u,v\in V_r} |d_G(u) - d_G(v)| \geq -\frac{1}{2}\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)|$$
$$-\frac{1}{2}\sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|,$$

and finally,

$$\frac{1}{2}\sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)| \geq \frac{1}{2}|V_2|\sum_{u\in V_1} d_{G_1}(u) - \frac{1}{2}|V_1|\sum_{v\in V_2} d_{G_2}(v),$$

or

$$\frac{1}{2}\sum_{u\in V_1,v\in V_2} |d_G(u) - d_G(v)| \geq -\frac{1}{2}|V_2|\sum_{u\in V_1} d_{G_1}(u) + \frac{1}{2}|V_1|\sum_{v\in V_2} d_{G_2}(v).$$

Let

$$\alpha = -\frac{1}{2}n_1(\sum_{v\in V_r} d_{G_2}(v) - \sum_{v\in V_2} d_{G_2}(v)) - n_2m_1$$
$$-\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)| - \sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|,$$

and

$$\beta = -\frac{1}{2}n_2(\sum_{v\in V_r} d_{G_1}(v) - \sum_{v\in V_1} d_{G_1}(v)) - n_1m_2$$
$$-\sum_{u,v\in V_r} |d_{G_1}(u) - d_{G_1}(v)| - \sum_{u,v\in V_r} |d_{G_2}(u) - d_{G_2}(v)|.$$

Therefore by a similar argument as final part of proof of Theorem 3.2, we have the result.

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