Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 47 (2026), No. 3, pp. 1085–1097
DOI : 10.47974/JIOS-1972

Optimality for parabolic systems operators with an infinite number of variables beneath conjugation conditions

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Abstract

Rationale for this manuscript is to investigate the optimality for N×N non cooperative parabolic systems involving operators with an infinite number of variables under conjugation conditions. By employing Green's formula and, we drive the existence and uniqueness for the stats of these systems by Lax-Milgram lemma. Moreover, we examine the essential conditions for the solutions to be considered optimal.

Subject Classification: *Primary 35K40, Secondary 49J99.*

Keywords: *Parabolic systems, Optimal control.*

1. Context and Motivation

The sufficient and necessary criteria for the optimality of systems regulated by various forms of partial differential operators, specified within finite-dimensional variable spaces, are explored in references [2–7]. Gali et al. initiated and formalized the study of optimal control problems for such systems in [7, 8], focusing on systems characterized by a single equation. Optimal control approaches for both deterministic and stochastic systems have since been developed, as outlined in references [1–15]. In Section 2, we investigate the optimal boundary control of an N×N non-cooperative elliptic system subject to Dirichlet boundary conditions and conjugation constraints. Initially, we demonstrate the existence of a solution to system (1); subsequently, we derive the necessary and sufficient

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conditions for optimality. Section 3 extends the analysis to the case involving Neumann boundary conditions.

2. Boundary control of an N×N parabolic system governed by operators in infinite-dimensional spaces, subject to Dirichlet and conjugation conditions

This section is devoted to the study of the following N×N parabolic system

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)y_m+\sum_{m=1}^{N}\frac{\partial y_m}{\partial t}+\sum_{n=1}^{N}a_{mn}y_m=h_m & \text{in } \mathbb{R}^\infty \\ y_m=0 \qquad \forall 1\le m\le N & \text{on } B \end{cases}, \tag{1}$$

and conjugation conditions

$$\left[\frac{\partial y_m}{\partial V_A}\right]=[Y], \left\{\frac{\partial y_m}{\partial V_A}\right\}^{\pm}=R[y_m],\ D_k^2=\frac{\partial^2 y}{\partial x_k^2},$$

Our model is defined on $(W_0^1(\mathbb{R}^\infty))^N$, $a_{mn}=\begin{cases}1 & m\ge n\\ -1 & m<n\end{cases}$, γ represents x-axis or y-axis. Now we define a continuous bilinear form on $(W_0^1(\mathbb{R}^\infty))^N$ by

$$b(y,\chi)=\sum_{m=1}^{N}\sum_{k=1}^{\infty}\int_{\mathbb{R}^\infty}(-D_k^2 y_m)\chi_m\,d\rho + \sum_{m,n=1}^{N}\int_{\mathbb{R}^\infty}a_{mn}\ y_m\chi_m\,d\rho+\sum_{m,n=1}^{N}\int_{\gamma}R[y_m][\chi_n]d\gamma$$

1.1. *Well-Posedness Solution*

The two chains under consideration are as follows

$$(W^1(\mathbb{R}^\infty))^N\subseteq(L^2(\mathbb{R}^\infty))^N\subseteq(W^{-1}(\mathbb{R}^\infty))^N,$$
$$(W_0^1(\mathbb{R}^\infty))^N\subseteq(L^2(\mathbb{R}^\infty))^N\subseteq(W_0^{-1}(\mathbb{R}^\infty))^N,$$

we define the following bilinear form on $(W_0^1(\mathbb{R}^\infty))^N$

$$b(y,\chi)=\sum_{m=1}^{N}\sum_{k=1}^{\infty}\int_{\mathbb{R}^\infty}(-D_k^2 y_m)\chi_m\,d\rho + \sum_{m,n=1}^{N}\int_{\mathbb{R}^\infty}a_{mn}\ y_m\chi_m\,d\rho+\sum_{m,n=1}^{N}\int_{\gamma}R[y_m][\chi_n]d\gamma$$

$$L(\chi) = \sum_{m=1}^{N} \int_{\mathbb{R}^\infty} h_m \ \chi_j \, d\rho, \ d\rho = dx_1 \times dx_2 \times dx_3 \times \ldots$$

For $y = (y_1, y_2, y_3, .., y_N) \in (W_0^1(\mathbb{R}^\infty))^N$, with the help of the Lax-Milgram lemma, it can be proved that.

Proposition 1: For $h = (h_1, h_2, h_3, .., h_N) \in (L^2(\mathbb{R}^\infty))^N$, there exists a unique state $y \in (W_0^1(\mathbb{R}^\infty))^N$ of the system (1).

Verification

It is possible to represent the bilinear form as:

$$b(y,\chi) = \sum_{m=1}^{N} \sum_{k=1}^{\infty} \int_{\mathbb{R}^\infty} D_k y_m \, D_k \chi_m \ d\rho + \sum_{m,n=1}^{N} \int_{\mathbb{R}^\infty} a_{mn} \ y_m \, \chi_m \ d\rho$$
$$+ \sum_{m,n=1}^{N} \int_{\gamma} R[y_m][\chi_m] d\gamma$$

$$b(y,\chi) = \sum_{m=1}^{N} \sum_{k=1}^{\infty} \int_{\mathbb{R}^\infty} D_k y_m \, D_k \chi_m \, d\rho$$
$$+ \sum_{m,n=1}^{N} \int_{\mathbb{R}^\infty} y_m \ \chi_m \, d\rho + \sum_{m>n}^{N} \int_{\mathbb{R}^\infty} y_m \, \chi_m \, d\rho$$
$$+ \sum_{m,n=1}^{N} \int_{\gamma} R[y_m][\chi_m] d\gamma - \sum_{m<n}^{N} \int_{\mathbb{R}^\infty} y_m \, \chi_m \, d\rho$$

$$b(\zeta,\zeta) = \sum_{m=1}^{N} \sum_{k=1}^{\infty} \int_{\mathbb{R}^\infty} |D_k \zeta_m|^2 \ d\rho + \sum_{m=1}^{N} \int_{\mathbb{R}^\infty} |\zeta_m|^2 \ d\rho$$
$$+ \sum_{m=1}^{N} \int_{\gamma} (R[\zeta_m])^2 \, d\gamma$$

$$b(\zeta,\zeta) \geq \sum_{m=1}^{N} \sum_{k=1}^{\infty} \| D_k \zeta_m \|^2_{L^2(\mathbb{R}^\infty)} + \sum_{m=1}^{N} \| \zeta_m \|^2_{L^2(\mathbb{R}^\infty)}$$

$$b(\zeta,\zeta) \geq \sum_{m=1}^{N} \| \zeta_m \|^2_{W_0^1(\mathbb{R}^\infty)} + \sum_{m=1}^{N} \| \zeta_m \|^2_{L^2(\mathbb{R}^\infty)}$$

$$b(\zeta,\zeta) \geq C \sum_{m=1}^{N} \| \zeta_m \|^2_{W_0^1(\mathbb{R}^\infty)} \ \text{[Coerciveness]} \ \forall \zeta \in [W_0^1(\mathbb{R}^\infty)]^N \tag{2}$$

Since $(R[z_m])^2 \geq 0$, $c = const$, $b(\zeta,\eta) \leq C_1 \ \| \zeta \|_{W_0^1(\mathbb{R}^\infty)} \ \| \eta \|_{W_0^1(\mathbb{R}^\infty)} \ \forall \zeta,$ $\eta \in W_0^1 \ (\mathbb{R}^\infty)$ and $L(\eta)$ is continuous on $(W_0^1(\mathbb{R}^\infty))^N$ then by Lax-Milgram lemma there exist a unique solution $\zeta=(\zeta_1, \zeta_2, \zeta_3, ..., \zeta_N) \in (W_0^1(\mathbb{R}^\infty))^N$, such that $b(\zeta,\eta) = L(\eta)$ for all $\eta \in (W_0^1(\mathbb{R}^\infty))^N$.

2.2 *Presentation of the control problem*

Consider $(L^2(B))^N$ as the space of controls.
The energy function

$$\phi(v)=b(y,v)-2L(\eta) \tag{3}$$

for a control $w=(w_1, w_2, w_3, ..., w_N) \in (L^2(\mathrm{B}))^N$ the state of the system $y=\{y_1, y_2, y_3, ..., y_N\} \in (L^2(B))^N$ is obtained by solving the following system:-

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)y_m+\sum_{m=1}^{N}\frac{\partial y_m}{\partial t}+\sum_{n=1}^{N}a_{mn}y_m=h_m \ \text{ in } \mathbb{R}^\infty \\ y_m=w_m \qquad \forall 1\le m\le N \qquad \text{on}\, B \end{cases}, \tag{4}$$

Conjugation condition (*). The observation equation takes the form

$$z_m=\{z_1,z_2,z_3,..,z_N\}\equiv y_m.$$

The cost functional is given by:

$$C(w)=(\varpi w,w)_{(L^2(B))^N}+\|y(w)-O_d\|_H^2, H\subset L^2(B) \tag{5}$$

$$C(w)=(\varpi w,w)_{(L^2(B))^N}+\sum_{m=1}^{N}\int(y(w)-O_{md})^2 d\rho$$

Where $\qquad O_d=\{O_{1d},O_{2d},O_{3d},..,O_{Nd}\}\in(L^2(B))^N$,

ϖ is a positive definite self-adjoint operator on $(L^2(B))^N$ for which:

$$(\varpi w,w)\ge\gamma\|w\|^2_{(L^2(B))^N},\gamma>0 \tag{6}$$

The control problem reduces to finding:

$$\begin{cases} w\in W_{ad}\ S.t \\ C(w)=\inf C(v)\ \forall v\in W_{ad} \end{cases},$$

where W_{ad} is a convex, closed set within $(L^2(B))^N$, given the form of the cost function (5)

$$C(v) = (\varpi v, v)_{\underset{N-times}{\underline{(L^2(B)\times\ldots L^2(B))}}} + \sum_{m=1}^{N} \int_B (y_m(v) - y_m(0), y_m(0) - O_{md})^2 d\rho$$
$$= b(v,v) - 2L(v) + \| O_{md} - y_m(0) \|^2, m = 1,2.$$

Where

$$b(v,v) = (\varpi v, v)_{\underset{N-times}{\underline{(L^2(B)\times\ldots L^2(B))}}} + (y_m(u) - y_m(0), y_m(0) - y_m(0))_{\underset{N-times}{\underline{(L^2(B)\times\ldots L^2(B))}}},$$

$$L(v) = (O_{md} - y_m(0), y_m(v) - y_m(0))_{\underset{N-times}{\underline{(L^2(B)\times\ldots L^2(B))}}}, m = 1,2,..,N.$$

Under the conditions specified in (2) and (6), and with the cost function from (5), w is optimal if the following relations are met

$$\begin{cases} -\sum_{k=1}^{\infty} (D_k^2) s_m + \sum_{m=1}^{N} \frac{\partial s_m}{\partial t} + \sum_{n=1}^{N} a_{mn} s_m = 0 \text{ in } \mathbb{R}^\infty \\ s_m = y_m(w) - O_{md} \quad \forall 1 \le m \le N \qquad \text{on } B \\ \left[\frac{\partial s_m}{\partial V_{A^*}} \right] = [s_m] = 0 \qquad , x \in \gamma, \\ \left\{ \frac{\partial s_m}{\partial V_{A^*}} \right\}^{\pm} = R[s_m] \qquad , x \in \gamma \end{cases} \tag{7}$$

$s = \{s_1, s_2, s_3, .., s_N\}$ is referred to as the adjoint state.

Overview of the proof

Given the differentiability of $C(v)$ and the boundedness of W_{ad}, the optimal control w is defined as follows $\sum_{m=1}^{N} C^{/}(v)(v_m - w_m) \ge 0 \;\; \forall v \in W_{ad}$ which is equivalent to

$$(\varpi_m w_m, v_m - w_m)_{(L^2(B))^N} + (y_m(w) - O_{md}, y_m(v) - y_m(w)) \ge 0 \tag{8}$$

Since $(A^* s, y) = (s, Ay)$ where A is defined by:

$$A\Upsilon = A\{y_1, y_2, y_3, .., y_N\} = -\sum_{k=1}^{\infty} (D_k^2) y_m + \sum_{n=1}^{N} a_{mn} y_m,$$

then $$(s, Ay) = \sum_{m=1}^{N} \left(s_m, -\sum_{k=1}^{\infty} (D_k^2) y_m + \sum_{m=1}^{N} a_{mn} y_m \right)_{(L^2(B))^N}$$

Employing Green's formula or interpreting derivatives in the weak sense

$$(A^{*}s, y) = \left(-\sum_{k=1}^{\infty} (D_k^2)s_m + \sum_{n=1}^{N} a_{nm} s_m, y_m \right)$$

$A\, s(u) + N^T s(u) = 0$, N^T is transpose of N, $s(u)$ adjoint state $A = -\sum_{k=1}^{\infty}(D_k^2) + \sum_{n=1}^{N} a_{nm}$, $N = a_{nm}$, $n, m = 1, 2, \ldots, N$, (8) is equivalent to

$$(\varpi_m w_m, v_m - w_m)_{(L^2(B))^N} + \left(-\sum_{k=1}^{\infty} (D_k^2)s_m + \sum_{n=1}^{N} a_{nm} s_m, y_m(v) - y_m(w) \right) \geq 0$$

From (7), we obtain

$$(\varpi_m w_m, v_m - w_m)_{(L^2(B))^N} + \int_B \sum_{m=1}^{N} s_m (v_m - w_m) d\rho \geq 0.$$

In the absence of constraints, that is, when $\mu_B = \mu$ the identity $s_m(w) + \varpi_m w_m = 0$ i.e., $\frac{-s_m(w)}{\varpi_m} = w_m$ accordingly the differential problem of solving for the vector-function $(y_m, s_m)^T$

$$\begin{cases} -\sum_{k=1}^{\infty} (D_k^2)s_m + \sum_{m=1}^{N} \frac{\partial s_m}{\partial t} + \sum_{n=1}^{N} a_{mn} s_m = 0 \text{ in } \mathbb{R}^{\infty} \\ -\sum_{k=1}^{\infty} (D_k^2)y_m + \sum_{m=1}^{N} a_{mn} y_m = h_m \text{ in } \mathbb{R}^{\infty} \\ y_m + \frac{s_m}{\varpi_j} = 0, s_m = y_m(w) - O_{md} \quad \forall 1 \leq m \leq N \qquad \text{on } B \end{cases}$$

$$\begin{cases} \left[\frac{\partial s_m}{\partial V_{A^*}} \right] = [s_m] = 0 & , x \in \gamma \\ \left\{ \frac{\partial s_m}{\partial V_{A^*}} \right\}^{\pm} = R[s_m] & , x \in \gamma \end{cases}$$

Insight 1: The optimality system, in case of N=2, is formulated as

$$(\varpi_1 w_1, v_1 - w_1)_{(L^2(B))^N} + \left(\varpi_2 w_2, v_2 - w_2\right)_{(L^2(B))^N} \cdot \\ + \int_B s_1(v_1 - w_1) d\rho + \int_B s_2(v_2 - w_2) d\rho \geq 0$$

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)y_1 + y_1(w) - y_2(w) = h_1 \text{ in } \mathbb{R}^{\infty} \\ -\sum_{k=1}^{\infty}(D_k^2)y_2 + y_1(w) + y_2(w) = h_2 \text{ in } \mathbb{R}^{\infty} \\ -\sum_{k=1}^{\infty}(D_k^2)s_1 + s_1(w) + s_2(w) = 0 \text{ in } \mathbb{R}^{\infty} \\ \sum_{k=1}^{\infty}(D_k^2)s_2 - s_1(w) + s_2(w) = 0 \text{ in } \mathbb{R}^{\infty} \\ y_1 = w_1, s_1 = y_1(w) - O_{1d}, y_2 = w_2, s_2 = y_2(w) - O_{2d} \text{ on } B \\ \left[\dfrac{\partial s_m}{\partial V_{A^*}}\right] = [s_m] = 0, \left\{\dfrac{\partial s_m}{\partial V_{A^*}}\right\}^{\pm} = R[s_m], m = 1,2,\ x \in \gamma \end{cases}$$

3. Boundary control of an N×N parabolic system with operators having an infinite number of variables, subject to Neumann and conjugation conditions

We now turn to the following N×N elliptic system in this section

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{m=1}^{N}\dfrac{\partial y_m}{\partial t} + \sum_{m=1}^{N} a_{mn}y_m = h_m \text{ in } \mathbb{R}^{\infty} \\ \dfrac{\partial y_m}{\partial V_A} = g_m \qquad \forall 1 \le m \le N \qquad \text{on } B \end{cases}, \tag{9}$$

and conjugation conditions (*), $g_j \in H^{\frac{-1}{2}}(B)$, Now we define a continuous bilinear form on $(W_0^1(\mathbb{R}^{\infty}))^N$ by

$$b(y,\xi) = \sum_{m=1}^{N}\sum_{k=1}^{\infty}\int_{\mathbb{R}^{\infty}} D_k y_m D_k \xi_j d\rho + \sum_{m,n=1}^{N}\int_{\mathbb{R}^{\infty}}(a_{mn}\ y_m)\xi_m d\rho + \sum_{m,n=1}^{N}\int_{\gamma} R[y_m][\xi_m]d\gamma$$

3.1 *Well-Posedness Solution*

We have the following two chains

$$(W^1(\mathbb{R}^{\infty}))^N \subseteq (L^2(\mathbb{R}^{\infty}))^N \subseteq (W^{-1}(\mathbb{R}^{\infty}))^N,$$
$$(W_0^1(\mathbb{R}^{\infty}))^N \subseteq (L^2(\mathbb{R}^{\infty}))^N \subseteq (W_0^{-1}(\mathbb{R}^{\infty}))^N,$$

we define the following bilinear form on $(W_0^1(\mathbb{R}^{\infty}))^N$

$$b(\eta,\xi)=\sum_{m=1}^{N}\sum_{k=1}^{\infty}\int_{\mathbb{R}^{\infty}}(-D_k^2\eta_m)\xi_j d\rho$$
$$+\sum_{m,n=1}^{N}\int_{\mathbb{R}^{\infty}}a_{mn}\ \eta_m\ \xi_m\ d\rho+\sum_{m,n=1}^{N}\int_{\gamma}R[\eta_m][\xi_m]d\gamma$$

$$L(\xi)=\sum_{m=1}^{N}\int_{B}g_m\xi_m dB+\sum_{m=1}^{N}\int_{\mathbb{R}^{\infty}}h_m\xi_m\ d\rho,\ d\rho=dx_1\times dx_2\times dx_3\times\ldots$$

For $\zeta=(\zeta_1,\ \zeta_2,\ \zeta_3,\ \ldots,\ \zeta_N)\ \in\ (W_0^1(\mathbb{R}^{\infty}))^N,\ \eta=(\eta_1,\ \eta_2,\ \eta_3,\ \ldots,\ \eta_N)$ $\in\ (W_0^1(\mathbb{R}^{\infty}))^N$ with the help of lax- Milgram lemma, we derive that:

Proposition 2: For $h=(h_1,\ h_2,\ h_3,\ \ldots,\ h_N)\in(L^2(\mathbb{R}^{\infty}))^N$, there exists a unique solution $y=(y_1,\ y_2,\ y_3,\ \ldots,\ y_N)\in(W_0^1(\mathbb{R}^{\infty}))^N$ of system (1). We have

$$b(y,\xi)=\sum_{m=1}^{N}\sum_{k=1}^{\infty}\int_{\mathbb{R}^{\infty}}-(D_k^2)y_m\xi_m d\rho+\sum_{j,i=1}^{N}\int_{\gamma}R[y_m][\xi_m]d\gamma$$
$$+\sum_{m,n=1}^{N}\int_{\mathbb{R}^{\infty}}a_{mn}\ y_m\xi_m\ d\rho=\sum_{m=1}^{N}\int_{\mathbb{R}^{\infty}}\ h_m\xi_m d\rho$$

Through Green's formula, we get

$$b(y,\xi)=\sum_{m=1}^{N}\sum_{k=1}^{\infty}\int_{\mathbb{R}^{\infty}}D_k y_m D_k\xi_m d\rho+\sum_{m,n=1}^{N}\int_{\gamma}R[y_m][\xi_n]d\gamma$$
$$+\sum_{m,n=1}^{N}\int_{\mathbb{R}^{\infty}}a_{mn}\ y_m\xi_m d\rho-\sum_{m=1}^{N}\int_{B}\frac{\partial y_m(x)}{\partial V_A}\xi_m dB$$

$$b(y,\xi)-\sum_{m=1}^{N}\int_{B}\frac{\partial y_m(x)}{\partial V_A}\xi_m\ dB=\sum_{m=1}^{N}\int_{\mathbb{R}^{\infty}}\ h_m\ \xi_m\ d\rho$$

Hence $\sum_{m=1}^{N}\int_{B}\left(\frac{\partial y_m(x)}{\partial V_A}-g_m\right)\eta_m(x)dB=0\rightarrow g_m\Big|_B=\frac{\partial y_m(x)}{\partial V_A}\Bigg|_B\ \ \forall\, m,\ldots,N$

3.2 *Description of the control problem*

Let $(L^2(B))^N$ be defined as the space of controls.

The energy form

$$\phi(v)=b(y,v)-2L(\eta) \tag{10}$$

with control $w=(w_1, w_2, w_3, ..., w_N) \in (L^2(B))^N$ the system′ state $y(w)$ = $\{y_1(w), y_2(w), y_3(w), ..., y_N(w)\} \in (L^2(B))^N$ is given by the solution of the following system:-

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{m=1}^{N}\frac{\partial y_m}{\partial t} + \sum_{n=1}^{N} a_{mn}y_m = h_m \text{ in } \mathbb{R}^{\infty} \\ \frac{\partial y_m}{\partial V_A} = g_m + w_m \qquad \forall 1 \le m \le N \quad \text{on } B \end{cases}, \tag{11}$$

Conjugation condition (*)

The observation equation takes the form

$$z_j(w) = \{z_1(w), z_2(w), z_3(w), .., z_N(w)\} \equiv y_j(w)$$

The cost functional is given by:

$$C(w) = (\varpi w, w)_{(L^2(B))^N} + \| y(w) - O_d \|_H^2, H \subset L^2(B) \tag{12}$$

$$C(w) = (\varpi w, w)_{(L^2(B))^N} + \sum_{m=1}^{N}\int_B (y(w) - O_{md})^2 d\rho$$

Where $\qquad O_d = \{O_{1d}, O_{2d}, O_{3d}, ..., O_{Nd}\} \in (L^2(B))^N,$

assuming ϖ is hermitian positive definite operator on $(L^2(B))^N$, we have:

$$(\varpi w, w) \ge \gamma \| w \|^2_{(L^2(B))^N}, \gamma > 0 \tag{13}$$

The control problem is thus to find:

$$\begin{cases} w \in W_{ad} \ S.t \\ C(w) = \inf C(v) \ \forall v \in W_{ad} \end{cases},$$

given that W_{ad} is a closed convex subset of $(L^2(B))^N$, the cost function (12) can be expressed as

$$\begin{aligned} C(v) &= (\varpi v, v)_{(L^2(B))^N} + \sum_{m=1}^{N}\int_B (y_m(v) - y_m(0), y_m(0) - O_{md})^2 d\rho \\ &= b(v,v) - 2L(v) + \| O_{md} - y_m(0) \|^2, m = 1, 2. \end{aligned}$$

Where $b(v,v) = (\varpi v, v)_{(L^2(B))^N} + (y_m(u) - y_m(0), y_m(v) - y_m(0))_{(L^2(B))},$

$$L(v) = (O_{md} - y_m(0), y_m(v) - y_m(0))_{(L^2(B))}, m = 1, 2, .., N.$$

Under the assumptions of (2) and (13) and with the cost function defined in (12), the necessary and sufficient conditions for w to be an optimal control are that the following equations and inequalities hold.

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)s_m + \sum_{m=1}^{N}\frac{\partial s_m}{\partial t} + \sum_{n=1}^{N} a_{mn}s_m = 0 & \text{in } \mathbb{R}^{\infty} \\ \frac{\partial y_m}{\partial V_A} = g_m + w_m\ , \frac{\partial s_m}{\partial V_{A^*}} = y_m(w) - O_{md} \quad \forall 1 \le m \le N & \text{on } B \\ \left[\frac{\partial s_m}{\partial V_{A^*}}\right] = [s_m] = 0 \qquad , x \in \gamma \\ \left\{\frac{\partial s_m}{\partial V_{A^*}}\right\}^{\pm} = R[s_m] \qquad , x \in \gamma \end{cases} , \tag{14}$$

$s = \{s_1, s_2, s_3, .., s_N\}$ is referred to as the adjoint state.

Overview of the proof

Given the differentiability of $C(v)$ and the boundedness of W_{ad}, the optimal control w is defined as follows $\sum_{m=1}^{N} C^{/}(v)\ (v_m - w_m) \ge 0\ \forall v \in W_{ad}$ which is equivalent to

$$(\varpi_m w_m, v_m - w_n)_{(L^2(B))^N} + (y_n(w) - O_{nd}, y_n(v) - y_n(w)) \ge 0 \tag{13}$$

Since $(A^* s, y) = (s, Ay)$ where A is defined by:

$$A\Upsilon = A\{y_1, y_2, y_3, .., y_N\} = -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{n=1}^{N} a_{mn}y_m\ ,$$

then $$(s, Ay) = \sum_{j=1}^{N}\left(s_m, -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{m=1}^{N} a_{mn}y_m\right)_{(L^2(B))^N}$$

Employing Green's formula or interpreting derivatives in the weak sense

$$(A^* s, y) = \left(-\sum_{k=1}^{\infty}(D_k^2)s_m + \sum_{n=1}^{N} a_{nm}s_m, y_m\right)$$

$A\ s(w) + N^T s(w) = 0$, N^T is transpose of N, $s(w)$ adjoint state

$A = -\sum_{k=1}^{\infty}(D_k^2) + \sum_{n=1}^{N} a_{nm}$, $N = a_{nm}$, m, $n = 1, 2, ..., N$, (15) is equivalent to

$$(\varpi_m w_m, v_m - w_m)_{(L^2(B))^N} + \left(-\sum_{k=1}^{\infty}(D_k^2)s_m + \sum_{m=1}^{N} a_{mn}s_m, y_m(v) - y_m(w)\right) \geq 0$$

Using Green's formula

$$(\varpi_m w_m, v_m - w_m)_{(L^2(B))^N} + \sum_{m=1}^{N}\left(s_m, -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{n=1}^{N} a_{mn}y_m\right)$$
$$+ \sum_{m=1}^{N}\left(s_m, \frac{\partial y_m(v) - \partial y_m(w)}{\partial V_A}\right)_{(L^2(B))}$$
$$+ \sum_{m=1}^{N}\left(s_m, \frac{\partial y_m(v) - \partial y_m(w)}{\partial V_A}\right)_{(L^2(B))} + \sum_{m=1}^{N}\left(\frac{\partial s_j}{\partial V_{A^*}}, y_m(v) - y_m(w)\right)_{(L^2(B))} \geq 0$$

We obtain $(\varpi_m w_m, v_m - w_m)_{(L^2(B))^N} + \int_B \sum_{m=1}^{N} s_m(v_m - w_m)\,d\rho \geq 0.$

In the absence of constraints, that is, when $\mu_B = \mu$ the identity $s_m(w)$ + $\varpi_m w_m = 0$ i.e., $\frac{-s_m(w)}{\varpi_m} = w_m$ then the task of solving the differential problem for the vector-function $(y_m, s_m)^T$

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)s_m + \sum_{m=1}^{N}\frac{\partial s_m}{\partial t} + \sum_{m=1}^{N} a_{mn}s_m = 0 \text{ in } \mathbb{R}^{\infty} \\ -\sum_{k=1}^{\infty}(D_k^2)y_m + \sum_{m=1}^{N} a_{mn}y_m = h_m \text{ in } \mathbb{R}^{\infty} \\ \frac{\partial y_m}{\partial V_A} + \frac{s_m}{\varpi_m} = 0, \frac{\partial s_m}{\partial V_{A^*}} = y_j(w) - O_{jd} \quad \forall 1 \leq j \leq N \text{ on } B \end{cases}$$

$$\begin{cases} \left[\frac{\partial s_m}{\partial V_{A^*}}\right] = [s_m] = 0 & , x \in \gamma \\ \left\{\frac{\partial s_m}{\partial V_{A^*}}\right\}^{\pm} = R[s_m] & , x \in \gamma \end{cases}$$

Insight 2: For N=2, the optimality system takes the form

$$(\varpi_1 w_1, v_1 - w_1)_{(L^2(B))^N} + (\varpi_2 w_2, v_2 - w_2)_{(L^2(B))^N}$$
$$+ \int_B s_1(v_1 - w_1)\,d\rho + \int_B s_2(v_2 - w_2)\,d\rho \geq 0.$$

$$\begin{cases} -\sum_{k=1}^{\infty}(D_k^2)y_1 + y_1(w) - y_2(w) = h_1 & \text{in } \mathbb{R}^{\infty} \\ -\sum_{k=1}^{\infty}(D_k^2)y_2 + y_1(w) + y_2(w) = h_2 & \text{in } \mathbb{R}^{\infty} \\ -\sum_{k=1}^{\infty}(D_k^2)s_1 + s_1(w) + s_2(w) = 0 & \text{in } \mathbb{R}^{\infty} \\ \sum_{k=1}^{\infty}(D_k^2)s_2 - s_1(w) + s_2(w) = 0 & \text{in } \mathbb{R}^{\infty} \\ \frac{\partial y_1}{\partial V_A} = g_1 + w_1, \frac{\partial s_1}{\partial V_{A^*}} = y_1(w) - O_{1d}, \frac{\partial y_2}{\partial V_A} = g_2 + w_2, \frac{\partial s_2}{\partial V_{A^*}} = y_2(w) - O_{2d} & \text{on } B \\ \left[\frac{\partial s_m}{\partial V_{A^*}}\right] = [s] = 0, \left\{\frac{\partial s_m}{\partial V_{A^*}}\right\}^{\pm} = R[s_m] & , x \in \gamma \end{cases}$$

Conclusion

The existence and uniqueness of the solution (state of the system) were determined, and the necessary and sufficient conditions for the system's optimality were established in the 2×2 case, with a generalization to the N×N case, for both distributed and boundary controls.

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Received August, 2024