Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 47 (2026), No. 3, pp. 977–988
DOI : 10.47974/JIOS-1779

On the covering radius of DNA repetition codes in *R*

P. Chella Pandian
Department of Mathematics
Vivekananda College (Autonomous)
Tiruvedakam West
Madurai 625234
Tamil Nadu
India

Abstract

In this correspondent, obtain to bounds of covering radius of DNA codes in $R = \mathbb{Z}_2 + v\mathbb{Z}_2$, $v^2 = v$ by using different distance. Construct of various Repetition DNA codes and its covering radius of codes, Simplex Repetition DNA code for Unit Type and Zero divisor Type and finally MacDonald Repetition DNA codes of both type over *R* are determined.

Subject Classification (2020): *11T71, 11T30, 68P30, 94B05.*

Keywords: *Finite ring, DNA code, Different distance, Covering radius, Simplex codes.*

1. Introduction

Deoxyribonucleic acid (DNA) is a double helical structure of two strands that twist around each other in an antiparallel orientation to form a helix. Each DNA strand is composed of nucleotides that form a chain. Hydrogen bonds hold the strands together between nitrogenous bases: Adenine (Ae), Thymine (Te), Cytosine (Ce), and Guanine (Ge). A double strand of DNA is formed when the respective bases are complementary to each other according to Watson - Crick base pairing: Adenine (Ae) pairing by Thymine (Te), and Cytosine (Ce) pairing by Guanine (Ge). In base pairing, Adenine (Ae) forms two hydrogen bonds with Thymine (Te), while Cytosine (C) forms three hydrogen bonds with Guanine (Ge). The

ORCID-ID: `0000-0002-4112-868X`

E-mail: `chellapandianpc@gmail.com`

Ae - Te and Ce - Ge pairs occur in such a way that, on the 5′ to 3′ strand, Adenine (Ae) pairing by Thymine (Te), and Cytosine (Ce) pairing by Guanine (Ge) on the opposite strand, which runs in the 3′ to 5′ direction [1, 25].

The most challenging aspect of constructing DNA codes is creating sets of codeword for the constant length n in the nucleotide Alphabets (Ae, Te, Ce, Ge) that satisfy specific joined constraints. This has applications for reliably storing with restores tidings in synthetic DNA strands, particularly DNA chips and other molecular bar-codes.

The researchers working as on the codes over $\mathbb{Z}_4$ in [2, 4, 5, 6, 16, 17, 19, 22, 23, 24] and also on code over $\mathbb{Z}_2^3$ [9]. The binary linear codes with covering radius are studied [13, 14]. In [2], the authors given some results of bounds from the code in $\mathbb{Z}_4$ by using various distances. By using this ring $\mathbb{Z}_2 + v\mathbb{Z}_2,\ v^2 = v$ and $\mathbb{Z}_2 + u\mathbb{Z}_2,\ u^2 = 0,$ found the covering radii of some repetition code and also a DNA codes in the above ring [11, 12].

In $\mathbb{Z}_2 + v\mathbb{Z}_2, v^2 = v$ and $\mathbb{Z}_2 + u\mathbb{Z}_2,\ u^2 = 0$ for covering radius of some block repetition code in $\mathbb{Z}_4$ had been determined in [10, 6, 7, 8] and also are obtained [11, 12].

2. Preliminary

In Genetics and Bioengineering is the one of main application of Algebraic Coding Theory. Design of the problem with DNA codes has applications for reliably storing and retrieving information in synthetic DNA strands in Alphabet $R = \{Ae, Ce, Ge, Te\}$.

Let (y_i) with $y_i \in R^n,\quad i = 1, 2, \cdots, n$ be the set of codewords, that is represent the four nucleotides in DNA, here n is a length. In [25], a nucleotide is denoted to Ae match by Te and Ce match by Ge. The DNA codewords is the fixed length n in the alphabet R and it is well known that the elements $0, 1, v, 1+v$ of the finite ring $\mathbb{Z}_2 + v\mathbb{Z}_2$ with $v^2 = v$ are in one to one correspondence with the nucleotide DNA basis *Ae Ce Te Ge* respectively such that the map $Ae \to 0,\ Ce \to 1,\ Te \to v$ and $Ge \to 1 + v.$

So, the DNA codes problem is compatible to problematic in R-vector codes. The deportation do not assail the *Ge Ce*-weight of the word. The work of the correspondent is the alphabet R with above map and by using with different weights(such as, Lee(Le), Euclidean(En), Chinese Euclidean(CE) and Bachoc(Bc)).

In $w_k(l) = |\{j \mid z_j = k\}|$, here $l \in R^n, n$ be a length of code and the element k from {*Ae, Ce, Ge, Te*}, where $w_k(l)$ is the weight of l at k.

Let C be a DNA linear code in R is a additive subgroup of R^n, where n length. An the member of C is said to be a DNA *codeword* of C. The codewords C is generator matrix if form of a matrix whose rows generate C, it is denoted by G.

The weight are define to Lee(Le), Euclidean(En), Chinese Euclidean(CE) and Bachoc(Bc) is given

R	Le	En	CE	Bc
0	0	0	0	0
1	1	1	2	1
v	2	4	4	2
$1+v$	1	1	2	2

in [21, 3].

Let $\varphi : R \to \mathbb{Z}_2^2$ be a linear gray map is defined by $\varphi(a+2b) = (b, a+b)$, $\forall a + 2b \in R$. In ref. [23], $\varphi[C(\mathbb{Z}_2^2)]$ be a image of linear code C underneath R of a length n. From Gray map with $\mathbb{Z}_2$ code of length $2n$ with same cardinality.

Let $G = \begin{bmatrix} I_{k_0} & X & Y \\ 0 & 2I_{k_1} & 2Z \end{bmatrix}$, where X, Y *and* Z are matrices in R. be a generator matrix of DNA linear code $C \in R$.

Let $[c_0, c_1]$ be a codewords of C and so, C contain all DNA codewords $[c_0, c_1]G$, here c_0 is a linear of length k_1 over R and c_1 is a linear of length k_2 over $\mathbb{Z}_2$. Therefore, the cardinality of codewords of C is $4^{k_1}2^{k_2}$. Then, the parameters of $C : [n, 4^{k_1}2^{k_2}, d]$.

So, the DNA code contained plenary DNA words of $[c_0, c_1]G$, where c_0, c_1 are the linear of length k_1, k_2 atop R, $\mathbb{Z}_2$. Hence, the code C contains a aggregate of $4^{k_1}2^{k_2}$ codewords. So the code C are given $[n, 4^{k_1}2^{k_2}, d]$.

Thus, a DNA linear code $C : [n, k, d]$ atop R, here n, k and d are the length, dimension and minimum distance(such as, Le, En, CE, Bc) of code.

3. DNA Repetition Code with Covering Radius in R

A *Covering Radius* of code C is

$$cr_d(C) = \max_{u \in R^n}\{\min_{c \in C}\{d(c, u)\}\},$$

here d is a minimum distance of code and n is the length of code in R.

Let $\mathbb{F}_q = \{u_0 = 0,\ u_1 = 1,\ u_2,\ u_3,\ \cdots,\ u_{q-1}\}$ be a finite field with a q element and let $C = \{\overline{u} \mid u \in \mathbb{F}_q\}$, where $\overline{u} = (u, u, \cdots, u)$ be a q-ary repetition code. In [20], the covering radii of C is $\left\lceil \frac{n(q-1)}{q} \right\rceil$, here C is a $[n, 1, n]$ code.

Let $G = [\overbrace{11\cdots1}^{n}\ \overbrace{u_2u_2\cdots u_2}^{n}\ \overbrace{u_3u_3\cdots u_3}^{n}\ \cdots\ \overbrace{u_{q-1}u_{q-1}\cdots u_{q-1}}^{n}]$ be a generator matrix for the block of size n repetition code $[n(q-1),\ 1,\ n(q-1)]$ and a covering radius is $\left\lceil \frac{n(q-1)^2}{q} \right\rceil$, since it will be equivalent to a repetition code of length $(q-1)n$ by using the above it.

In DNA repetition code over R. So, two type are

1. A code $C_u \in R$ is an $[n, 1, n, n, n, n]$ code is generated by G_u = $[\overbrace{Ce\,Ce\cdots Ce}^{n}]$ is said to be an Cytosine repetition code.

2. A code $C_{zd} \in R$ is an $(n, 2, d_{Le,En,Bc(CE)}$ = $2n(4n))$ code is generated by G_{zd} = $[\overbrace{Te\,Te\cdots Te}^{n}]$ is said to be an Thymine repetition code.

Theorem 3.1:

1. $\left\lfloor \frac{n}{2} \right\rfloor \le cr_{Le}(C_{zd}) \le 2n$ *and* $n \le cr_{Le}(C_u) \le \frac{5n}{3}$,
2. $cr_{En}(C_{zd}) = 2n$ *and* $cr_{En}(C_u) \le \frac{3n}{2}$,
3. $cr_{CE}(C_{zd}) = 2n$ *and* $cr_{CE}(C_u) = 2n$,
4. $2\left\lfloor \frac{n}{2} \right\rfloor \le cr_{Bc}(C_u) \le 2n$ *and* $n \le cr_{Bc}(C_u) \le 2n$,

where C_u *and* C_{zd} *are the DNA repetition code for unit type and zero divisors type in generator matrices* G_u *and* G_{zd}.

Proof: Let $C = \{Ae\ Ae\ \cdots\ Te\ Te\ \cdots\ Te\}$ be a code in R and if y = $\overbrace{Te\,Te\cdots Te}^{\lfloor \frac{n}{2} \rfloor}$ $\overbrace{Ae\,Ae\cdots Ae}^{\lceil \frac{n}{2} \rceil} \in R^n$. So, $d_{Le}(y,\ AeAe\ \cdots\ Ae)$ = $w_{Le}(y$ − $AeAe\ \cdots\ Ae)$ = $\left\lceil \frac{n}{2} \right\rceil$ and $d_{Le}(y,\ TeTe\ \cdots\ Te)$ = $wt_{Le}(y$ − $TeTe\ \cdots\ Te)$ = $\left\lfloor \frac{n}{2} \right\rfloor$. So, $d_{Le}(y, C_{zd})$ = $\min\left\{\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. From the definition: $cr_{Le}(C_u) \ge \left\lfloor \frac{n}{2} \right\rfloor$

If y be any codeword in R^n and take it of y has u_0 coordinated as 0's, u_1 coordinated as 1's, u_2 coordinated as 2's, u_3 coordinated as 3's, then $u_0 + u_1 + u_2 + u_3 = n$. Since $C_u = \{AeAe \cdots Ae, TeTe \cdots Te\}$ and using by lee weight of R. Therefore, $d_{Le}(y, 00\cdots0) = n - u_0 + u_1 + u_2 + u_3$ and $d_{Le}(y, TeTe \cdots Te) = n - u_2 + u_1 + u_0 + u_3$ and so, $d_{Le}(y, C_{zd}) = \min\{n - u_0 + u_1 + u_2 + u_3, n - u_2 + u_1 + u_{0+}u_3\}$ and hence, $d_{Le}(y, C_{zd}) \leq n + n = 2n$. Thus, $\left\lfloor \frac{n}{2} \right\rfloor \leq cr_{Le}(C_u) \leq 2n$.

For any codeword $y \in R$, then

$$d_{Le}(y, AeAe\cdots Ae) = n - u_0 + u_1 + u_2 + u_{3,}$$

$$d_{Le}(y, CeCe\cdots Ce) = n - u_1 + u_0 + u_2 + u_{3,}$$

$$d_{Le}(y, TeTe\cdots Te) = n - u_2 + u_1 + u_3 + u_0$$

$$d_{Le}(y, GeGe\cdots Ge) = n - u_3 + u_1 + u_0 + u_2.$$

This implies $d_{Le}(y, C_u) = \min\{n - u_0 + u_1 + u_2 + u_3, n - u_1 + u_0 + u_2 + u_3, n - u_2 + u_1 + u_3 + u_0, n - u_3 + u_1 + u_0 + u_2\} \leq \frac{5n}{3}$ and hence $cr(C_u) \leq \frac{5n}{3}$.

Let $y = \overbrace{AeAe\cdots Ae}^{p}\overbrace{CeCe\cdots Ce}^{p}\overbrace{TeTe\cdots Te}^{p}\overbrace{GeGe\cdots Ge}^{n-3p}$, here $p = \left\lfloor \frac{n}{4} \right\rfloor$.

Thus, $d_{Le}(y, AeAe \cdots Ae) = n$, $d_{Le}(y, CeCe \cdots Ce) = 2n - 4t$, $d_{Le}(y, TeTe \cdots Te) = n$ and $d_{Le}(y, GeGe \cdots Ge) = 4p$. Therefore $cr_{Le}(C_u) \geq \min\{2n, 2n - 4p, 4p\} \geq n$. The remaining part of proof is follows from above.

Block DNA Repetition Code

Let $B = \{c_0, c_1, c_2, c_3\}$ be a Block DNA Repetition Code in R with length $3n$ is generated by $G = [\overbrace{Ce\,Ce\cdots Ce}^{n}\ \overbrace{Te\,Te\cdots Te}^{n}\ \overbrace{Ge\,Ge\cdots Ge}^{n}]$, where $c_0 = Ae\cdots Ae\ Ae\cdots Ae\ Ae\cdots Ae$, $c_1 = Ce\cdots Ce\ Te\cdots Te\ Ge\cdots Ge$, $c_2 = Te\cdots Te\ Ae\cdots Ae\ Te\cdots Te$, $c_3 = Ge\cdots Ge\ Te\cdots Te\ Ce\cdots Ce$. The Parameter of this code is given:

The length of all the weight are $3n$ with one dimension and distance $d_{Le,Bc} = 4n$ but $d_{En} = 6n$ and $d_{CE} = 8n$.

Theorem 3.2

1. $3n \le cr_{Le}(B^{3n}) \le \frac{7n}{2}$
2. $cr_{En}(B^{3n}) = 5n.$
3. $cr_{CE}(B^{3n}) = 6n,$
4. $cr_{Bc}(B^{3n}) = 4n.$

Proof: Take $y = Ae \cdots Ae \in R^{3n}$. Then, $d_{Le}(y, B^{3n}) = 3n$ and so, $cr_{LE}(B^{3n}) \ge 3n$.

If $y = (a|b|c) \in R^{3n}$ with a, b and c have compositions $i_0, i_1, i_2, i_3, j_0, j_1, j_2, j_3$ and k_0, k_1, k_2, k_3 respectively such that $\sum_{x=0}^{3} i_x = n$, $\sum_{x=0}^{3} j_x = n$ and $\sum_{x=0}^{3} k_x = n$, then

$$d_{Le}(y, c_0) = 3n - i_0 + i_1 + i_2 + i_3 - j_0 + j_1 + j_2 + j_3 - k_0 + k_1 + k_2 + k_3,$$

$$d_{Le}(y, c_1) = 3n - i_1 + i_0 + i_2 + i_3 - j_2 + j_0 + j_1 + j_0 - k_3 + k_0 + k_1 + k_2,$$

$$d_{Le}(y, c_2) = 3n - i_2 + i_1 + i_0 + i_3 - j_0 + j_1 + s_j + j_3 - k_2 + k_0 + k_1 + k_3$$

and

$$d_{Le}(y, c_3) = 3n - i_3 + i_1 + i_0 + i_2 - j_2 + j_0 + j_1 + j_3 - k_1 + k_3 + k_0 + k_2.$$

So, $d_{Le}(y, B^{3n}) = \min\{d_{Le}(y, c_0),\ d_{Le}(y, c_1),\ d_{Le}(y, c_2),\ d_{Le}(y, c_3)\} \le \frac{7n}{2}$ and hence, $cr_{Le}(B^{3n}) \le \frac{7n}{2}$. The proof of other part is follows from above way.

Two Block DNA Repetition Code:

Let $G = [\overbrace{CeCe \cdots Ce}^{n}\ \overbrace{TeTe \cdots Te}^{n}]$ be a generator matrix is given the following parameters:

The length of all the weight are 2n with one dimension and distance $d_{Le,En,Bc} = 2n$ but $d_{CE} = 4n$.

Theorem 3.3:

1. $2n \le cr_{Le}(B^{2n}) \le \frac{11n}{3}$

2. $cr_{En}(B^{2n}) = \frac{7n}{2}$.
3. $4\left\lfloor \frac{n}{2} \right\rfloor + 2n \le cr_{CE}(B^{2n}) \le 4n$,
4. $2\left\lfloor \frac{n}{2} \right\rfloor + n \le cr_{Bc}(B^{2n}) \le 4n$.

Proof: Use the above theorem of proof with two block of same length.

Let $G = [\overbrace{Ce\,Ce\cdots Ce}^{m}\ \overbrace{Te\,Te\cdots Te}^{m_1}]$ be the generator matrix for m and m_1 block DNA repetition code of length are m and m_1 respectively. The following table are denoted by the parameters of m and m_1 block repetition code (B^{m+m_1}):

R	Le	En	CE	Bc
l	$m + m_1$	$m + m_1$	$m + m_1$	$m + m_1$
d	1	1	1	1
d(w)	w^*	w^*	$min\{4m, 3m + 3m_1\}$	w^*

where $w^* = min\{2m, m + m_1\}$

Theorem 3.4:

1. $m + m_1 \le cr_{Le}(B^{m+m_1}) \le 2m + \frac{5m_1}{3}$,
2. $cr_{En}(B^{m+m_1}) \le \frac{3m}{2} + 2m_1$
3. $2m + 4\left\lfloor \frac{m_1}{2} \right\rfloor \le cr_{CE}(B^{m+m_1}) \le 2m + 2m_1$,
4. $m + 2\left\lfloor \frac{m_1}{2} \right\rfloor + \le cr_{Bc}(B^{2n}) \le 2m + 2m_1$.

Proof: By Theorem 3.3 with m, m_1 block length.

4. DNA Repetition code of Simplex Code for Unit Type and Zero divisor Type in *R*

In this section, the studies of Quaternary simplex codes of type *u* and type *zd*, ref. [4], but i will consider for unit type and zero divisor type in this paper.

Unit Type Simplex code S_k^u :
Let

$$G_k^u = \left[\begin{array}{c|c|c|c} Ae\cdots Ae & Ce\cdots Ce & Te\cdots Te & Ge\cdots Ge \\ \hline G_{k-1}^u & G_{k-1}^u & G_{k-1}^u & G_{k-1}^u \end{array}\right] \tag{4.1}$$

be a generator matrix for unit type simplex code of DNA repetition code in *R*. The parameter of this code $[4^k, k]$. If $k=2$, then $M_1^u = [Ae\, Ce\, Te\, Ge]$.

Zero Divisor Type Simplex code S_k^{zd} :

Let S_k^{zd} be a zero divisor Type Simplex code is a punctures version of S_k^u. Then the parameter $[2^{k-1}, (2^k-1), k]$.

The generator matrix of Zero Divisor Type Simplex code with $k=2$ is

$$G_2^{zd} = \left[\begin{array}{cccc|c|c} Ce & Ce & Ce & Ce & Ae & Te \\ \hline Ae & Ce & Te & Ge & Ce & Ce \end{array}\right] \tag{4.2}$$

and $k>2$ is

$$G_k^{zd} = \left[\begin{array}{l|l|l} CeCe\cdots Ce & AeAe\cdots Ae & TeTe\cdots Te \\ \hline G_{k-1}^u & G_{k-1}^{zd} & G_{k-1}^{zd} \end{array}\right] \tag{4.3}$$

, where G_{k-1}^u is the generator matrix of S_{k-1}^u. Further is referred to [4].

Theorem 4.1

1. $cr_{Le}(S_k^u) \le 2^{2k}+1$,
2. $cr_{En}(S_k^u) \le \frac{5\cdot 4^k+5}{3}$,
3. $cr_{CE}(S_k^u) \le 2^{2k+1}-3$,
4. $cr_{Bc}(S_k^u) \le \frac{2^{2(k+1)}-1}{3}$.

Proof: Let $y = CeCe\cdots Ce \in R^n$. By equation(4.1), ref. [13], Theorem 3.3.

$$cr_{Le}(S_k^u) \le cr_{Le}(S_{k-1}^u) + cr_{Le}(<\overbrace{**}^{p}\overbrace{**}^{p}\overbrace{***}^{p}>), \text{ here } * = CeCe\cdots Ce, ** = TeTe\cdots Te,$$

$$*** = GeGe\cdots Ge \text{ and } p = 4^{(k-1)}$$

$$\le cr_{Le}(S_{k-1}^4) + 3.4^{(k-1)} \le 3[4^{(k-1)} + \ldots + 4^1] + cr_{Le}(S_1^u)$$

$$cr_{Le}(S_k^u) \le 4^k + 1 \text{ (since, } cr_{Le}(\mathrm{S}_1^u) = 5)$$

The other part is alike way from 1.

Theorem 4.2:

1. $cr_{Le}(S_k^{zd}) \le 2^k(2^k-1)-1$,
2. $cr_{En}(S_k^{zd}) \le \frac{5\cdot 4^k - 6\cdot 2^k - 8}{6}$,
3. $cr_{CE}(S_k^{zd}) \le (2^{2k}-2^k)-7$,
4. $cr_{Bc}(S_k^{zd}) \le \frac{(2^{2k}+3\cdot 2^{2k-3}-9\cdot 2^{k-3}-10)}{3}$

Proof: From eqn. (4.3), ref. [13], Theorem 3.4, so

$$\begin{aligned}
cr_{Le}(S_k^{zd}) &\le cr_{Le}(S_{k-1}^{zd}) + cr_{Le}(<\overbrace{*}^{2^{2(k-1)}}\ \overbrace{**}^{2^{(2k-3)}-2^{(k-2)}}>), \text{ where } * = Ce\cdots Ce, ** = Te\cdots Te \\
&= cr_{Le}(S_{k-1}^{zd}) + 2^{(2k-2)} + 2^{(2k-3)} - 2^{(k-2)} \\
&\le 2(4^{(k-2)} + 4^{(k-2)} + \ldots + 4^2) + (4^{(2k-3)} + 4^{(2k-5)} + \ldots + 4^3) - \\
&\quad (4^{(k-2)} + 4^{(k-3)} + \ldots + 4) + cr_{Le}(S_2^{zd}) \\
cr_{Le}(S_k^{zd}) &\le 2^{k-1}(2^k-1)-1 \text{ (since } cr_{Le}(S_2^{zd}) = 5).
\end{aligned}$$

The other part is the same arguments to 1.

5. DNA Repetition code of MacDonald Code for Unit Type and Zero divisor Type in *R*

Let F_q be a finite field, and let $M_{k,t}(q)$ be a q-ary MacDonald code in F_q. So a unique parameter of this code is a $\left[\frac{q^k-q^t}{q-1}, k, q^{k-1}-q^{t-1}\right]$ code. In particular, the $M_{k,t}(q)$ weight code is either q^{k-1} or $q^{k-1}-q^{t-1}$ from [18].

In [20] the authors were working to cover the radius of the code in F_q and also gave one or more exact values for the small dimension of the code. Define a MacDonald code in the ring with generator matrices of simplex codes [15].

$M_{k,t}(q)$ Unit Type:

$$G_{k,t}^u = \left[G_k^u \backslash \frac{0}{G_t^u} \right] \tag{5.1}$$

here 0 is a $(k-t)\times 2^{2t}$, for $2 \le t \le k-1$.

$M_{k,t}(q)$ Zero Divisor Type:

$$G_{k,t}^{zd} = \left[G_k^{zd} \backslash \frac{0}{G_t^{zd}} \right] \tag{5.2}$$

here 0 is a $(k-t) \times 2^{t-1}(2^t -)$, for $2 \le t \le k-1$. From the above, the parameter of $M^u_{k,t}$ code is $[2^{2k} - 2^{2t}, k]$ code and $M^{zd}_{k,t}$ code is $[(2^{k-1} - 2^{t-1})\ (2^k + 2^t - 1),$ $k]$ code in R. The above these code from S^u_k and S^{zd}_k code.

Theorem 5.1:

1. $cr_{Le}(M^u_{k,t}) \le 2^{2k} - 2^{2r} + cr_{Le}(M^u_{r,t})$,
2. $cr_{En}(M^u_{k,t}) \le \frac{5}{3}(2^{2k} - 2^{2r}) + cr_{En}(M^u_{r,t})$,
3. $cr_{CE}(M^u_{k,t}) \le 2^{2k+1} - 2^{2r+1} + cr_{CEn}(M^u_{r,t})$ and
4. $cr_{Bc}(M^u_{k,t}) \le \frac{4^{k+1} - 4^{r+1}}{3} + cr_{Bc}(M^u_{r,t})$, for $t < r \le k$.

Proof: By using, Eqn.(5.1), ref. [13] and Theorem 3.2, so

$$\begin{aligned} cr_{Le}(M^u_{k,t}) &\le cr_{Le}(<\overbrace{CeCe\cdots Ce}^{2^{2(k-1)}}\overbrace{TeTe\cdots Te}^{2^{2(k-1)}}\overbrace{GeGe\cdots Ge}^{2^{2(k-1)}}>) + cr_{Le}(M^u_{r,t}) \\ &= 3.4^{k-1} + cr_{Le}(M^u_{k-1,t}), \text{for } k \ge r > t. \\ &\le 3^{2(k-1)} + 3^{2(k-2)} + \cdots + 3^{2r} + cr_{Le}(M^u_{r,t}) \text{ for } k \ge r > t \\ cr_{Le}(M^u_{k,t}) &\le 2^{2k} - 2^{2r} + cr_{Le}(M^u_{r,t}), \text{for } k \ge r > t. \end{aligned}$$

The remaining part is suitable arguments to 1.

Theorem 5.2:

1. $cr_{Le}(M^{zd}_{k,t}) \le 2^{(k-1)}(2^k - 1) + 2^{(r-1)}(1 - 2^r) + cr_{Le}(M^{zd}_{r,t})$,
2. $cr_{En}(M^{zd}_{k,t}) \le \frac{(5 \cdot 2^{2k} - 6) + (6 - 5 \cdot 2^{2r})}{6} + cr_{En}(M^{zd}_{r,t})$,
3. $cr_{CE}(M^{zd}_{k,t}) \le (4^k - 2^k) + (2^r - 4^r) + cr_{CEn}(M^{zd}_{r,t})$ and
4. $cr_{Bc}(M^{zd}_{k,t}) \le \frac{2^{2(k+1)} - 2^{2(r+1)} + 3(2^{2(k-1)} - 2^{2(r-1)}) + 9(2^{r-1} - 2^{k-1})}{6} + cr_{Bc}(M^{zd}_{r,t})$, for $t < r \le k$.

Proof: Ref. [13], eqn. (5.2), Theorem 3.4.

$$cr_{Le}(M^{zd}_{k,t}) \le cr_{Le}(<\overbrace{*}^{2^{2(k-1)}}\quad\overbrace{**}^{2^{2(k-1)-1}-2^{(k-1)-1}}>) + cr_{Le}(M^{zd}_{k-1,t}),$$

where $* = CeCe\cdots Ce, ** = TeTe\cdots Te$

$$cr_{Le}(M^{zd}_{k,t}) \le 2^{(2k-1)} - (2^{k-1}) + 2^{(r-1)} - (2^{2r-1}) + cr_{Le}(M^{zd}_{r,t}), \text{ for } t < r \le k.$$

The remaining part is quasi idea to 1.

References

[1] M. L. Adleman, "Molecular computation of solutions to combinatorial problems," *Science*, vol. 266, pp. 1021-1024 (1994).

[2] T. Aoki, P. Gaborit, M. Harada and M. Ozeki P. Sol'e, On the covering radius of Z_4 codes and their lattices, *IEEE Trans. Inform. Theory,* Vol. 45, no. 6, pp. 2162-2168 (1999).

[3] C. Bachoc, Application of coding theory to the construction of modular lattices, *Journal of Combin. Theory Ser.,* A78, 92-119 (1997).

[4] M. C. Bhandari, M. K. Gupta, A. K. Lal, and A. K., "On Z4 Simplex codes and their gray images," Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, AAECC-13, *Lecture Notes in Computer Science,* vol. 1719, pp. 170-180 (1999).

[5] A. Bonnecaze, P. Sol'e, C. Bachoc and B. Mourrain, Type II codes over Z_4, *IEEE Trans. Inform. Theory,* Vol. 43, pp. 969-976 (1997).

[6] P. C. Pandian, "On covering radius of some codes over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where, $u^2 = 0$, " *International Journal of Research in Applied, Natural and Social Sciences,* vol. 2, no. 1, pp. 61–70, 2014.

[7] P. Chella Pandian, On covering radius of codes over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where, $u^2 = 0$ using chinese euclidean distance, *Journal of Discrete Mathematics, Algorithms and Applications* Vol. 9, no. 2, pp. 1750017(1-8) (2017).

[8] P. Chella Pandian, On Covering Radius of Codes Over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ Using Bachoc Distance, *International Journal of Mathematics And its Applications,* Vol. 5, no.4-C , pp. 277-282 (2017).

[9] P. Chella Pandian, On codes over $\mathbb{Z}_2^3$ and its covering radius for Lee weight and Homogeneous weight, *Journal of Information and Optimization Sciences,* Vol. 39, no.8 , pp. 1705-1715 (2018).

[10] P. Chella Pandian, Bounds on the covering radius of some classes of codes over R, *Open Journal of Discrete Applied Mathematics,* Vol. 2, no. 1, pp. 14-23 (2019).

[11] P. Chella Pandian, On the Covering Radius of DNA Code Over N, *Asian Journal of Mathematical Sciences,* Vol. 7, no. 2, pp.97-104 (2023).

[12] P. Chella Pandian , On the Covering Radius of DNA Code over a Finite Ring, *Journal of Discrete Mathematics and Its Applications,* Vol. 8, No. 2, pp. 75-82 (2023).

[13] G. D. Cohen, M. G. Karpovsky, H. F. Mattson,, J. R. Schatz, Covering radius- Survey and recent results, *IEEE Trans. Inform. Theory,* Vol. 31, no. 3, pp. 328-343 (1985).

[14] C. Cohen, A. Lobstein, N. J. A. Sloane, Further Results on the Covering Radius of codes, *IEEE Trans. Inform. Theory,* Vol. 32, no. 5, pp. 680-694 (1986).

[15] C. J. Colbourn, M. K. Gupta, On quaternary MacDonald codes, *Proc. Information Technology:Coding and Computing* (ITCC), pp. 212-215 (April 2003).

[16] I. Constantinescu, T. Heise, A metric for codes over residue class rings of integers, *Problemy Peredachi Informatsii,* Vol. 33, pp. 22-28 (1997).

[17] J. H. Conway and N. J. A. Sloane, "Self-dual codes over the integers modulo 4," *Journal of Combinatorial Theory Series A*, vol. 62, pp. 30-45 (1993).

[18] S. Dodunekov and J. Simonis, "Codes and projective multisets," *The Electronic Journal of Communications,* vol. 5, R37 (1998).

[19] S. T. Dougherty, M. Harada, and P. Solé, "Shadow codes over Z_4 finite fields and their applications," vol. 7, no. 4, pp. 507-529 (2001).

[20] C. Durairajan, On Covering Codes and Covering Radius of Some Optimal Codes, Ph. D. Thesis, *Department of Mathematics,* IIT Kanpur (1996).

[21] M. K. Gupta, D. G. Glynn, and T. A. Gulliver, "On Senary Simplex Codes," *Lecture Notes in Computer Science*, pp. 112-121 (2001).

[22] M. K. Gupta and C. Durairajan, "On the Covering Radius of some Modular Codes," *Journal of Advances in Mathematics of Computations*, vol. 8, no. 2, pp. 9 (2014).

[23] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Sol'e, The Z_4-linearity of kerdock, preparata, goethals, and related codes. *IEEE Trans. Inform. Theory,* Vol. 40, pp. 301-319 (1994).

[24] M. Harada, New extremal Type II codes over Z_4. *Des. Codes and Cryptogr.* Vol. 13, pp. 271-284 (1998).

[25] D. J. Watson, C. H. F. Crick, *A structure for deoxyribose nucleic acid, Nature,* Vol. 25, pp. 737-738 (1953).

Received January, 2024