Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 47 (2026), No. 3, pp. 941–950
DOI : 10.47974/JIOS-1601

Multivariate fractional optimal stopping problems

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Abstract

This paper present a development of new multivariate optimal stopping problems which associated the discrete time or continuous time stochastic processes with a fractional reward criterion. By applying the established reasoning approach in mathematical programming theory, the existence of an optimal stopping time for these problems will be established by assuming certain regularity conditions on the stochastic processes.

Subject Classification: *60G40, 93E20.*

Keywords: *Fractional reward, Optimal stopping, Parametric method, Stopping time.*

1. Introduction

Let $\mathbb{N} = \{1,2,\cdots\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Suppose (Ω, F, P) is a probability space and $\{F_t, t \in \mathbb{N}\}$ is a filtration of F and $F_\infty = \sigma(\cup_{t\in\mathbb{N}} F_t)$. $\sigma: \Omega \to \overline{\mathbb{N}}$ is called a stopping time if for each $t \in \overline{\mathbb{N}}$, $\{\sigma = t\} \in F_t$, and we use Σ to denote the collection of all stopping times. $\xi: \Omega \times [0,1] \to \overline{\mathbb{N}}$ is called a randomized stopping time if for each $t \in \overline{\mathbb{N}}$, $\{\xi = t\} \in F_t \times \mathcal{X}([0,1])$, where $([0,1], \mathcal{X}([0,1]), L)$ denotes the Lebesgue measure space. For each randomized stopping time ξ, a unique randomized stopping time $\tilde{\xi}: \Omega \times [0,1] \to \overline{\mathbb{N}}$ exists for which

$$\mathcal{L}(\{a | \xi(\omega, a) = t\}) = \mathcal{L}(\{a | \tilde{\xi}(\omega, a) = t\})$$

for every $\omega \in \Omega$ and $t \in \overline{\mathbb{N}}$, and for which for every $\omega \in \Omega$ $\tilde{\xi}(\omega, \cdot)$ is an increasing and left-continuous function. We use Ξ to denote the collection of all

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randomized stopping times which are increasing and left-continuous in the variable v.

The topology on Ξ known as the Baxter-Chacon topology is defined.

Definition 1.1: ([2], [3]). The weakest topology on Ξ, for which the following condition holds; for $t \in \mathbb{N}, \xi \in \Xi$,

$$\xi \mapsto \int_{\Omega} h(\omega)\mathcal{L}(\{a \mid \xi(\omega, a) = t\})dP \qquad t \in \mathbb{N}, h \in L^1(\Omega, F, P)$$

is continuous, is called the Baxter-Chacon topology on Ξ.

Theorem 1.1: ([2], [3]). *Ξ is compact relative to the Baxter-Chacon topology.*

We take $(\mathcal{X}, ||\cdot||)$ as a Banach space as well as $\{V(t), t \in \overline{\mathbb{N}}\}$ as an $\{F_t, t \in \overline{\mathbb{N}}\}$-adapted $\mathcal{X}$-valued integrable (in the sense of Bochner) stochastic process with

$$||V(t) - V(\infty)|| \to 0 \text{ a.e. as } t \to \infty.$$

Moreover we take $\Phi: \mathcal{X} \to \mathbb{R}$ as a convex and continuous function, as well as $\Psi: \mathcal{X} \to \mathbb{R}$ as a concave and continuous function such that $\Psi > 0$.

Proposition 1.1: If $E[\sup_{t\in\overline{\mathbb{N}}}||V(t)||] < \infty$, then the set $\{E[V(\xi)] | \xi \in \Xi\}$ is compact.

Proof: Since $V(t) \to V(\infty)$ strongly a.e. and $E[\sup_{t\in\overline{\mathbb{N}}}||V(t)||] < \infty$, we have

$$E[V(\xi_k)] \to E[V(\xi)] \ (k \to \infty)$$

for each $\xi_k, \xi \in \Xi$ such that $\xi_k \to \xi(\text{BC})$.

Hence the map $G: \xi \to \mathcal{X}$ defined by $G(\xi) = E[V(\xi)]$ is continuous. Therefore $G(\xi) = \{E[V(\xi)] | \xi \in \Xi\}$ is compact.

Now we consider the following problems:

$$(P_\xi) \qquad \sup_{\xi\in\Xi} \frac{\Phi(E[V(\xi)])}{\Psi(E[V(\xi)])},$$

$$(P_\Sigma) \qquad \sup_{\sigma\in\Sigma} \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])},$$

$$(P_\theta) \qquad d(\theta) = \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta\Psi(E[V(\sigma)])\}, \theta \in \mathbb{R}.$$

Many authors have studied mathematical optimization problems under fractional costs/rewards. For example, Aggarwal et al. [1], Iwamoto and Fujita [4], and Ren and Krogh [6] studied Markov decision processes under fractional criteira. Morimoto [5], Tanaka [9, 10] studied one-parameter or multiparameter optimal stopping problems under fractional criteira. Sawasaki, Kimura and Tanaka [7] studied a two-player zero-sum game under fractional criteira in the subject of game theory. Stancu-Minasian [8] treated fractional programming in the subject of mathematical programming.

This paper present a development of new multivariate optimal stopping problems which combined the discrete tme or continuous time stochastic processes with a fractional reward criterion. By applying the established reasoning approach in mathematical programming theory, especially, fractional programming, and the Baxter-Chacon topology on Ξ, we shall show the existence of optimal stopping times for these problems by assuming certain regularity conditions on the stochastic processes.

2. Parametric problem (P_θ)

In this section we study properties of the optimal value function $d(\theta)$.

Proposition 2.1: The function $\theta \mapsto d(\theta)$ is convex.

Proof: For all $\theta_1 < \theta_2$ and $\mathcal{L} \in [0,1]$, we have

$$\begin{aligned} d(\mathcal{L}\theta_1 + (1-\mathcal{L})\theta_2) &= \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - (\mathcal{L}\theta_1 + (1-\mathcal{L})\theta_2)\Psi(E[V(\sigma)])\} \\ &= \sup_{\sigma\in\Sigma}\{\mathcal{L}(\Phi(E[V(\sigma)]) - \theta_1\Psi(E[V(\sigma)])) \\ &\quad +(1-\mathcal{L})(\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)]))\} \\ &\leqq \mathcal{L}\sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_1\Psi(E[V(\sigma)])\} \\ &\quad +(1-\mathcal{L})\sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)])\} \\ &= \mathcal{L}d(\theta_1) + (1-\mathcal{L})d(\theta_2). \end{aligned}$$

In order to prove the next result, we prepare the theorem.

Theorem 2.1: *If* $E[\sup_{t\in\overline{\mathbb{N}}}||V(t)||] < \infty$ *and* $\eta: \mathcal{X} \to \mathbb{R}$ *is convex and continuous, it follows that there is* $\tau \in \Sigma$ *for which*

$$\eta(E[V(\tau)]) = \sup_{\sigma\in\Sigma}\eta(E[V(\sigma)]) = \sup_{\xi\in\Xi}\eta(E[V(\xi)]) < \infty$$

Proposition 2.2: The function $\theta \mapsto d(\theta)$ is strictly decreasing.

Proof: Because Ψ is concave and by applying $-\Psi$ to Theorem 2.1, we can find $\tau \in \Sigma$ for whcih

$$\Psi(E[V(\tau)]) = \inf_{\sigma\in\Sigma}\Psi(E[V(\sigma)]) = \inf_{\xi\in\Xi}\Psi(E[V(\xi)]) > -\infty.$$

By $E[V(\tau)] \in \mathcal{X}$, $\Psi(E[V(\tau)]) > 0$. Then we obtain $\inf_{\sigma\in\Sigma}\Psi(E[V(\sigma)]) > 0$.

For all $\theta_1 < \theta_2$, we have

$$\begin{aligned} d(\theta_1) &= \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_1\Psi(E[V(\sigma)])\} \\ &= \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)]) + (\theta_2-\theta_1)\Psi(E[V(\sigma)])\} \\ &\geqq \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)]) + (\theta_2-\theta_1)\inf_{\sigma\in\Sigma}\Psi(E[V(\sigma)])\} \\ &= \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)])\} + (\theta_2-\theta_1)\inf_{\sigma\in\Sigma}\Psi(E[V(\sigma)]) \end{aligned}$$

$$\begin{aligned} &> \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)])\} \\ &= d(\theta_2). \end{aligned}$$

Proposition 2.3: $\lim_{\theta\to\infty} d(\theta) = -\infty$ and $\lim_{\theta\to-\infty} d(\theta) = \infty$.

Proof: Note that $\inf_{\sigma\in\Sigma}\Psi(E[V(\sigma)]) > 0$ from the proof of the previous proposition. We assume that $\theta > 0$.

$$\begin{aligned} d(\theta) &= \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta\Psi(E[V(\sigma)])\} \\ &\leqq \sup_{\sigma\in\Sigma}\Phi(E[V(\sigma)]) + \sup_{\sigma\in\Sigma}\{-\theta\Psi(E[V(\sigma)])\} \\ &= \sup_{\sigma\in\Sigma}\Phi(E[V(\sigma)]) - \theta\inf_{\sigma\in\Sigma}\Psi(E[V(\sigma)]) \\ &\to -\infty\ (\theta\to\infty). \end{aligned}$$

Next we assume that $\theta < 0$.

$$\begin{aligned} d(\theta) &= \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta\Psi(E[V(\sigma)])\} \\ &\geqq \inf_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta\Psi(E[V(\sigma)])\} \\ &\geqq \inf_{\sigma\in\Sigma}\Phi(E[V(\sigma)]) + \inf_{\sigma\in\Sigma}\{-\theta\Psi(E[V(\sigma)])\} \\ &= \inf_{\sigma\in\Sigma}\Phi(E[V(\sigma)]) - \theta\inf_{\sigma\in\Sigma}\{\Psi(E[V(\sigma)])\} \\ &\to \infty\ (\theta\to-\infty). \end{aligned}$$

Proposition 2.4: There exists only one $\theta^* \in \mathbb{R}$ for whcih $d(\theta^*) = 0$.

Proposition 2.5: The function $\theta \mapsto d(\theta)$ is Lipschitz continuous.

Proof: For any θ_1, θ_2, we have

$$\begin{aligned} |d(\theta_1) - d(\theta_2)| &= |\sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_1\Psi(E[V(\sigma)])\} \\ &\quad -\sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)])\}| \\ \leq\ & \sup_{\sigma\in\Sigma}|\{\Phi(E[V(\sigma)]) - \theta_1\Psi(E[V(\sigma)])\} - \{\Phi(E[V(\sigma)]) - \theta_2\Psi(E[V(\sigma)])\}| \\ =\ & \sup_{\sigma\in\Sigma}|\theta_1 - \theta_2|\Psi(E[V(\sigma)]) \\ \leq\ & |\theta_1 - \theta_2|\sup_{\sigma\in\Sigma}\Psi(E[V(\sigma)]). \end{aligned}$$

By Proposition 1.1, $\{E[V(\xi)] | \xi \in \Xi\}$ is compact. Because Ψ is continuous, we obtain

$$\sup_{\xi\in\Xi}\Psi(E[V(\xi)]) < \infty,$$

and

$$\sup_{\sigma\in\Sigma}\Psi(E[V(\sigma)])\sup_{\xi\in\Xi}\Psi(E[V(\xi)]) < \infty.$$

Proposition 2.6: We set $\theta = \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])}$ for any $\sigma \in \Sigma$. Then $d(\theta) \geq 0$.

Proof: For any $\sigma \in \Sigma$, we get

$$d(\theta) \geq \Phi(E[V(\sigma)]) - \theta\Psi(E[V(\sigma)]) = 0.$$

Theorem 2.2: *$\sigma^* \in \Sigma$ is optimal for (P_Σ) and $\theta^* = \frac{\Phi(E[Z(\sigma^*)])}{\Psi(E[Z(\sigma^*)])}$ if and only if $\sigma^* \in \Sigma$ is optimal for (P_{θ^*}) and $d(\theta^*) = 0$.*

Proof: Let $\sigma^* \in \Sigma$ be optimal for (P_Σ). Then for any $\sigma \in \Sigma$ we obtain

$$\theta^* = \frac{\Phi(E[Z(\sigma^*)])}{\Psi(E[Z(\sigma^*)])} \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])},$$

and hence

$$\Phi(E[Z(\sigma^*)]) - \theta^*\Psi(E[Z(\sigma^*)]) = 0,$$
$$\Phi(E[V(\sigma)]) - \theta^*\Psi(E[V(\sigma)])0 \text{ for } \sigma \in \Sigma.$$

Therefore we have $d(\theta^*) = \Phi(E[Z(\sigma^*)]) - \theta^*\Psi(E[Z(\sigma^*)]) = 0$.

Next let $\sigma^* \in \Sigma$ be optimal for (P_{θ^*}) and $d(\theta^*) = 0$. Then for any $\sigma \in \Sigma$ we obtain

$$\Phi(E[V(\sigma)]) - \theta^*\Psi(E[V(\sigma)]) \leq \Phi(E[Z(\sigma^*)]) - \theta^*\Psi(E[Z(\sigma^*)]) = 0,$$

and hence

$$\theta^* = \frac{\Phi(E[Z(\sigma^*)])}{\Psi(E[Z(\sigma^*)])},$$
$$\theta^* \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])} \text{ for } \sigma \in \Sigma.$$

Therefore we have the conclusions.

3. Fractional problems (P_ξ) and (P_Σ)

In this section we assume that

$$\theta \geqq 0,$$
$$\Phi \geqq 0.$$

Remark 3.1:

(1) If $\theta \geqq 0$, then $\Phi - \theta\Psi$ is convex.

(2) If $\Phi \geqq 0$, θ^* in Proposition 2.4 is nonnegative.

Before stating the main results, we prepare the useful result.

Theorem 3.1: ([3]). *If $E[\sup_{t\in\overline{\mathbb{N}}}||V(t)||] < \infty$ and $\eta: \mathcal{X} \to \mathbb{R}$ is continuous, then there is $\delta \in \Xi$ for whcih*

$$\eta(E[V(\delta)]) = \sup_{\xi\in\Xi} \eta(E[V(\xi)]).$$

Theorem 3.2: *If $E[\sup_{t\in\overline{\mathbb{N}}}||V(t)||] < \infty$, then*

$$\sup_{\sigma\in\Sigma}\frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])}=\sup_{\xi\in\Xi}\frac{\Phi(E[V(\xi)])}{\Psi(E[V(\xi)])}.$$

Proof: We set $\overline{d}(\theta)=\sup_{\xi\in\Xi}\{\Phi(E[V(\xi)])-\theta\Psi(E[V(\xi)])\},\theta\geqq 0$. Since $\Phi-\theta\Psi$ is convex and Theorem 2.1, we have $d(\theta)=\overline{d}(\theta),\theta\geqq 0$. By Proposition 2.4, there exists only one $\theta^*\geqq 0$ for whcih $d(\theta^*)=\overline{d}(\theta^*)=0$. Moreover by Theorem 2.1, we can find $\sigma^*\in\Sigma$ such that

$$0=d(\theta^*)=\Phi(E[V(\sigma^*)])-\theta^*\Psi(E[V(\sigma^*)]).$$

Hence we obtain

$$\theta^*=\frac{\Phi(E[V(\sigma^*)])}{\Psi(E[V(\sigma^*)])}.$$

And also

$$0=d(\theta^*)\geqq\Phi(E[V(\sigma)])-\theta^*\Psi(E[V(\sigma)]),\sigma\in\Sigma$$

and then

$$\theta^*\geqq\frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])}.$$

Therefore we obtain

$$\theta^*=\frac{\Phi(E[V(\sigma^*)])}{\Psi(E[V(\sigma^*)])}=\sup_{\sigma\in\Sigma}\frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])}.$$

Next, since $\Phi-\theta^*\Psi$ is continuous and Theorem 3.1, we can obtain $\xi^*\in\Xi$ such that $0=\overline{d}(\theta^*)=\Phi(E[V(\xi^*)])-\theta^*\Psi(E[V(\xi^*)])$. Hence we get

$$\theta^*=\frac{\Phi(E[V(\xi^*)])}{\Psi(E[V(\xi^*)])}.$$

And also

$$0=d(\theta^*)\geqq\Phi(E[V(\xi)])-\theta^*\Psi(E[V(\xi)]),\xi\in\Xi$$

and then

$$\theta^*\geqq\frac{\Phi(E[V(\xi)])}{\Psi(E[Z(\xi)])}.$$

Therefore we obtain

$$\theta^*=\frac{\Phi(E[Z(\xi^*)])}{\Psi(E[Z(\xi^*)])}=\sup_{\xi\in\Xi}\frac{\Phi(E[V(\xi)])}{\Psi(E[V(\xi)])}.$$

Corollary 3.1: *If* $E[\sup_{t\in\overline{\mathbb{N}}}||V(t)||]<\infty$, *it follows that there is* $\sigma^*\in\Sigma$ *for which*

$$\theta^*=\frac{\Phi(E[V(\sigma^*)])}{\Psi(E[V(\sigma^*)])}=\sup_{\sigma\in\Sigma}\frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])}.$$

Corollary 3.2: *If* $E[\sup_{t\in\overline{\mathbb{N}}}||V(t)||]<\infty$, *it follows that there is* $\xi^*\in\Xi$ *for which*

$$\theta^*=\frac{\Phi(E[V(\xi^*)])}{\Psi(E[V(\xi^*)])}=\sup_{\xi\in\Xi}\frac{\Phi(E[V(\xi)])}{\Psi(E[V(\xi)])}.$$

4. Dinkelbach algorithm

We consider the following algorithm. Let $\delta > 0$ be fixed.

(1) Set $\theta_1 = 0$.

(2) For $\sigma_1 \in \Sigma$, set $\theta_2 = \frac{\Phi(E[V(\sigma_1)])}{\Psi(E[V(\sigma_1)])}$.

(3) Find $\sigma_2 \in \Sigma$ so that $d(\theta_2) = \Phi(E[V(\sigma_2)]) - \theta_2\Psi(E[V(\sigma_2)])$.

(i) If $d(\theta_2) < \delta$, then stop. In particular, σ_2 is exactly optimal if $d(\theta_2) = 0$.

(ii) If $d(\theta_2) \geqq \delta$, then set $\theta_3 = \frac{\Phi(E[V(\sigma_2)])}{\Psi(E[V(\sigma_2)])}$ and go to (4).

(4) Find $\sigma_3 \in \Sigma$ so that $d(\theta_3) = \Phi(E[V(\sigma_3)]) - \theta_3\Psi(E[V(\sigma_3)])$.

(i) If $d(\theta_3) < \delta$, then stop. In particular, σ_3 is exactly optimal if $d(\theta_3) = 0$.

(ii) If $d(\theta_3) \geq \delta$, then set $\theta_4 = \frac{\Phi(E[V(\sigma_3)])}{\Psi(E[V(\sigma_3)])}$, $k := 4$ and go to (k).

(k) Find $\sigma_k \in \Sigma$ so that $d(\theta_k) = \Phi(E[V(\sigma_k)]) - \theta_k\Psi(E[V(\sigma_k)])$.

(i) If $d(\theta_k) < \delta$, then stop. In particular, σ_k is exactly optimal if $d(\theta_k) = 0$.

(ii) If $d(\theta_k) \geq \delta$, then set $\theta_{k+1} = \frac{\Phi(E[V(\sigma_k)])}{\Psi(E[V(\sigma_k)])}$, $k := k + 1$ and go to (k).

Proposition 4.1: For any k such that $d(\theta_k) \geq \delta$, we have $\theta_{k+1} > \theta_k$.

Proof:

$$\begin{aligned} 0 < d(\theta_k) &= \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta_k\Psi(E[V(\sigma)])\} \\ &= \Phi(E[V(\sigma_k)]) - \theta_k\Psi(E[V(\sigma_k)]) \\ &= \theta_{k+1}\Psi(E[V(\sigma_k)]) - \theta_k\Psi(E[V(\sigma_k)]) \\ &= (\theta_{k+1} - \theta_k)\Psi(E[V(\sigma_k)]). \end{aligned}$$

From $\Psi(E[V(\sigma_k)]) > 0$, we obtain $\theta_{k+1} > \theta_k$.

Proposition 4.2: $lim_{k\to\infty}\theta_k = \theta^*$.

Proof: By Proposition 4.1, there exists $\overline{\theta} := \lim_{k\to\infty}\theta_k$. Suppose that $\theta^* > \overline{\theta}$. From the proof of Proposition 4.1, we have

$$0 < d(\theta_k) = (\theta_{k+1} - \theta_k)\Psi(E[V(\sigma_k)]) \leq (\theta_{k+1} - \theta_k)\sup_{\xi\in\Xi}\Psi(V[V(\xi)]),$$

and $\sup_{\xi\in\Xi}\Psi(E[V(\xi)]) < \infty$ from Proposition 1.1. Hence we have $\lim_{k\to\infty}d(\theta_k) = 0$. Since d is continuous, we obtain $d(\theta) = 0$. On the other hand, from $\theta^* > \overline{\theta}$ and Proposition 2.2, we have $d(\theta^*) < d(\overline{\theta})$. This is a contradiction. By the same arguments, we can conclude $\theta = \overline{\theta}$.

5. Continuous time case

Suppose (Ω, F, P) is a probability space equipped with a filtration $\{F_t, t \in [0,\infty]\}$ of F, which is assumed to satisfy the usual conditions. $\sigma: \Omega \to [0,\infty]$ is called a stopping time if for each $t \in [0,\infty]$, $\{\sigma = t\} \in F_t$ for each $t \in [0,\infty]$, and we use Σ to denote the collection of all stopping times. $\xi: \Omega \times [0,1] \to [0,\infty]$ is called a randomized stopping time if for each $t \in [0,\infty]$, $\{\xi \leq t\} \in F_t \times \mathcal{X}([0,1])$, and for each $\omega \in \Omega$, $\xi(\omega, \cdot)$ is an increasing and left-continuous function. We use Ξ to denote the collection of all randomized stopping times.

Definition 5.1: ([2], [3]). The weakest topology for which the following condition holds ; for each $f \in C([0,\infty])$ and $h \in L^1(\Omega, F, P)$

$$\xi \mapsto \int_{\Omega\times[0,1]} h(\omega) f(\xi(\omega, a)) P(d\omega) L(da)$$

is continuous, is called the Baxter-Chacon topology on Ξ.

Theorem 5.1: ([3]). Ξ is compact relative to the Baxter-Chacon topology.

We take $(\mathcal{X}, ||\cdot||)$ as a Banach space as well as $\{V(t), t \in [0,\infty]\}$ as an $\{F_t, t \in [0,\infty]\}$-adapted $\mathcal{X}$-valued integrable (in the sense of Bochner) stochastic process with

$$||V(t) - V(\infty)|| \to 0 \text{ a.e. as } t \to \infty.$$

Moreover we take $\Phi: \mathcal{X} \to \mathbb{R}$ as a convex and continuous function, as well as $\Psi: \mathcal{X} \to \mathbb{R}$ as a concave and continuous function such that $\Psi > 0$.

We consider the following problems:

$$(P_\xi) \qquad \sup_{\xi\in\Xi} \frac{\Phi(E[V(\xi)])}{\Psi(E[V(\xi)])},$$

$$(P_\Sigma) \qquad \sup_{\sigma\in\Sigma} \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])},$$

$$(P_\theta) \qquad d(\theta) = \sup_{\sigma\in\Sigma}\{\Phi(E[V(\sigma)]) - \theta\Psi(E[V(\sigma)])\}, \theta \in \mathbb{R}.$$

Theorem 5.2: ([3]). *If $E[\sup_{t\in[0,\infty]}||V(t)||] < \infty$, the map $t \in [0,\infty] \mapsto V(t) \in \mathcal{X}$ is right-continuous a.e., and $\eta: \mathcal{X} \to \mathbb{R}$ is convex and continuous, it follows that*

$$\sup_{\sigma\in\Sigma} \eta(E[V(\sigma)]) = \sup_{\xi\in\Xi} \eta(E[V(\xi)]).$$

Theorem 5.3: ([3]). Suppose that $E[\sup_{t\in[0,\infty]}||V(t)||] < \infty$, $t \in [0,\infty] \mapsto V(t) \in \mathcal{X}$ is right-continuous a.e., and for each $\sigma_n \in \Sigma, \sigma_n \nearrow \sigma$, $V(\sigma_n) \to V(\sigma)$ a.e.. For all $\xi_n, \xi \in \Xi$ such that ξ_n converges to ξ with repect to the Baxter-Chacon topology, and $A \in F$,

$$E[1_A V(\xi)]) = \lim_{n\to\infty} E[1_A Z(\xi_n)]).$$

Theorem 5.4: *Suppose that* $E[\sup_{t\in[0,\infty]}||V(t)||] < \infty,\ t \in [0,\infty] \mapsto V(t) \in \mathcal{X}$ *is right-continuous a.e., and for each* $\sigma_n \in \Sigma, \sigma_n \nearrow \sigma,\ V(\sigma_n) \to V(\sigma)$ *a.e., and* $\eta: \mathcal{X} \to \mathbb{R}$ *is convex and continuous. Then there is* $\tau \in \Sigma$ *for whcih*

$$\eta(E[V(\tau)]) = \sup_{\sigma\in\Sigma} \eta(E[V(\sigma)]) = \sup_{\xi\in\Xi} \eta(E[V(\xi)]) < \infty$$

By taking into account these theorems, we get the similar results to those described in section 2 and section 3. Therefore we omit the proofs.

Proposition 5.1: If $E[\sup_{t\in[0,\infty]}||V(t)||] < \infty$, the map $t \in [0,\infty] \mapsto V(t) \in \mathcal{X}$ is right-continuous a.e., and for each $\sigma_n \in \Sigma, \sigma_n \nearrow \sigma,\ V(\sigma_n) \to V(\sigma)$ a.e., then the set $\{E[V(\xi)] | \xi \in \Xi\}$ is compact.

Proposition 5.2: The function $\theta \mapsto d(\theta)$ is convex.

Proposition 5.3: The function $\theta \mapsto d(\theta)$ is strictly decreasing.

Proposition 5.4: $\lim_{\theta\to\infty} d(\theta) = -\infty$ and $\lim_{\theta\to-\infty} d(\theta) = \infty$.

Proposition 5.5: There is only one $\theta^* \in \mathbb{R}$ for which $d(\theta^*) = 0$.

Proposition 5.6: The function $\theta \mapsto d(\theta)$ is Lipschitz continuous.

Proposition 5.7: We set $\theta = \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])}$ for all $\sigma \in \Sigma$. Then $d(\theta) \geqq 0$.

Theorem 5.5: $\sigma^* \in \Sigma$ *is optimal for* (P_Σ) *and* $\theta^* = \frac{\Phi(E[V(\sigma^*)])}{\Psi(E[V(\sigma^*)])}$ *if and only if* $\sigma^* \in \Sigma$ *is optimal for* (P_{θ^*}) *and* $d(\theta^*) = 0$.

Throughout the remainder of this section, we make the assumptions that

$$\theta \geqq 0,$$
$$\Phi \geqq 0.$$

Theorem 5.6: *If* $E[\sup_{t\in[0,\infty]}||V(t)||] < \infty,\ t \in [0,\infty] \mapsto V(t) \in \mathcal{X}$ *is right-continuous a.e., and for each* $\sigma_n \in \Sigma, \sigma_n \nearrow \sigma,\ Z(\sigma_n) \to V(\sigma)$ *a.e., it follows that*

$$\sup_{\sigma\in\Sigma} \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])} = \sup_{\xi\in\Xi} \frac{\Phi(E[V(\xi)])}{\Psi(E[V(\xi)])}.$$

Corollary 5.1: *If* $E\left[\sup_{t\in[0,\infty]}||V(t)||\right] < \infty, t \in [0,\infty] \mapsto V(t) \in \mathcal{X}$ *is right-continuous a.e., and for each* $\sigma_n \in \Sigma, \sigma_n \nearrow \sigma,\ V(\sigma_n) \to V(\sigma)$ *a.e., it follows that there is* $\sigma^* \in \Sigma$ *for which*

$$\theta^* = \frac{\Phi(E[V(\sigma^*)])}{\Psi(E[V(\sigma^*)])} = \sup_{\sigma\in\Sigma} \frac{\Phi(E[V(\sigma)])}{\Psi(E[V(\sigma)])}.$$

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Received February, 2023