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Multi-objective optimization revisited

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Abstract

In this paper, the multi-objective (multiple criteria, vector) optimization is considered. Some examples of such optimization problems are presented, Pareto optimality is introduced, and arbitrage schemes are described. The special case of multi-objective linear optimization is studied, and methods for solving such problems are presented. Pareto optimality for the nonlinear optimization problem is also considered.

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1. Introduction

In some situations, an object (a system, a process) is described by a model which has several, sometimes possibly contradictory, objectives (criteria, goals). In such cases, usually it turns out difficult or even impossible to find a single solution which is optimal for the contradictory objectives. Instead, we can try to find a compromise solution, which is based on the relative importance or significance of each objective (each criterion).

Multi-objective programming (*multiple criteria optimization, vector optimization*) is part of Mathematical optimization, which studies mathematical models for optimal decision making based on several criteria simultaneously, that is, choice of optimality principles and methods for optimization problems with several criteria (several objective functions). These criteria can describe evaluations of various qualities of

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the object or process which is subject to decision making, or evaluation of one characteristic of this object or process but from different viewpoints.

The main idea of multi-criteria optimization is to transform the original multi-criteria problem into an optimization problem with a single objective function. As a result, the obtained problem (model) leads to an efficient solution, which may not be optimal regarding all contradictory objectives (criteria) of the initial problem.

A multi-criteria optimization problem is described by a set X of *feasible solutions* (*alternatives*) and a set of numerical functions $f_1, \ldots, f_l$ (*objective functions*, *criteria*) which are defined in the set X. The essence of the multi-criteria optimization consists in finding an optimal solution, that is, in finding $x^* \in X$, which maximizes or minimizes, in some sense, values of all the functions f_i, $i = 1, \ldots, l$. Since in the general case, several functions do not attain their extrema at the same point, then the existence of an optimal solution, which exactly maximizes or minimizes all functions $f_1, \ldots, f_l$, is an exception. That is why, the above statement of the considered problem is not quite correct, and therefore the concept of an *optimal solution* should be reconsidered. Therefore, in multi-criteria optimization the concept of *optimality* receives various, and at the same time not trivial, interpretations. In multi-criteria optimization, an *optimal solution* is natural to be understood as a subset X^* of the set X (that is, $X^* \subset X$), such that the values of numerical functions $f_1, \ldots, f_l$ in X^* correspond to intuitive notions of "best" values of these functions when they are maximized or minimized on the set X. These intuitive notions are formalized into various optimality principles. In such a sense, the essence of multi-objective optimization consists in the development of such optimality principles, in the proof that these principles are implementable, and in finding such implementations.

Multi-objective optimization problems and related to them are considered in [1]-[12], etc. In [1], a multi-objective model predictive control scheme is proposed. Paper [2] is devoted to multi-objective selection based on dominated hypervolume. In book [3], some aspects of multi-objective optimization are studied, and in book [9], goal programming is considered. Some examples are inspired by the latter book. In [4], a new method for generating the Pareto surface in nonlinear multi-criteria optimization problems is proposed. In paper [5], a global formulation for interactive multi-objective optimization is presented. In paper [6], a multi-objective evolutionary algorithm for SSSC-based controller design is suggested. Paper [7] is devoted to the multi-objective branch and bound, and paper [10] to the multiple objective branch and bound for mixed 0-1 linear programming. In [8], improving the computational efficiency in a global formulation (GLIDE) for interactive multi-objective optimization problems is proposed. Paper [11] is devoted to the mathematical basis for satisficing decision making. The topic of multi-objective optimization via visualization is studied in [12].

Rest of the paper is organized as follows. In Section 2, some basic methods for solving multi-objective problems are presented. In Section 3, the special case of multi-objective linear optimization is considered. In Section 4, the Pareto optimality for nonlinear multi-objective optimization is discussed. In Section 5, content of the paper is summarized and an open problem for future research is formulated.

2. Basic methods for solving multi-objective problems

I. A natural approach for solving multi-objective problems with commensurable criteria is their transformation to the traditional (that is, single-criteria) mathematical optimization problem by replacing the objective functions $f_1, \dots, f_l$ by a *single* "combined" ("general", "aggregate") objective function $F(f_1, \dots, f_l)$. In such a way, the original problem is transformed to maximization (minimization) of the combined function $F(f_1, \dots, f_l)$, which is a function of the given criteria $f_1, \dots, f_l$ and it is a monotone function with respect to each of these criteria (that is, monotone with respect to each argument of the combined function). The various methods differ in the way of constructing this combined objective function.

The most common special case of this principle of union of the objectives is the so-called *weighted sum* of the functions which represents the objectives (criteria) in the given multi-objective problem.

Let the multiple criteria problem have l objectives (criteria) and let the i – th criterion be given as follows:

$$\max(\min)\{f_i, i = 1, \dots, l\}.$$

Then the combined objective function, using the *weighted sum method*, is

$$\max(\min)\left\{F(f_1, \dots, f_l) = \sum_{i=1}^{l} w_i f_i\right\},$$

where $w_i \geq 0,\ i = 1, \dots, l,$ are the *weights*, which reflect the decision makers preference(s) with respect to the relative significance of each objective (criterion). If $w_i = 1,\ i = 1,\dots,l,$ this means that all criteria (goals) have equal weights. Determining the values of the weights $w_i,\ i = 1,\dots,l,$ is subjective. There are analytic methods for determining the weights, however, these methods are also subjective.

Another approach to construct the combined objective function is by the so-called *weighted maxima* (*weighted minima*, respectively):

$$\max_i w_i f_i \quad (\min_i w_i f_i, \text{respectively}), \quad i = 1,\dots,l.$$

There are also other ways to combine the given objective functions (criteria). However, the described approaches are most convenient from both conceptual and technical viewpoint. Their main drawback is the difficult-to-fulfill requirement for meaningful comparability (commensurability) of the values of the objective functions (criteria), as well as the uncertainty (and sometimes arbitrariness) of the choice of the combined objective function F and, in particular, the choice of the weights $w_i,\ i = 1,\dots,l.$ For this choice, it is recommended to use expert evaluations.

When incommensurable criteria are involved in the statement of the multi-objective problem, the corresponding optimality principle can be chosen axiomatically.

II. Except for the approaches, described in **I.**, another possible approach is as follows. The objectives (criteria) are ordered by priority according to their importance. After that, a *main* (deciding) criterion f_{i_0}, $1 \le i_0 \le l,$ is chosen, that is, all weights $w_i,\ i = 1,\dots,l,$ except for some weight $w_{i_0},\ 1 \le i_0 \le l,$ are set equal to zero, and instead of the given multi-objective problem, the following problem

$$\max_{x \in X} f_{i_0}(x) \quad (\min_{x \in X} f_{i_0}(x), \text{respectively})$$

is considered, where this is a single-criteria mathematical optimization problem, and the totality of its optimal solutions can be considered as a totality of feasible solutions of a new multi-objective problem with objective functions $f_i,\ i = 1,\dots,l,\ i \ne i_0$, that is, if the set

$$X_{i_0} \stackrel{\text{def}}{=} \arg\max_{x \in X} f_{i_0}(x) \quad (X_{i_0} \stackrel{\text{def}}{=} \arg\min_{x \in X} f_{i_0}(x), \text{respectively})$$

consists of more than one point, then the choice of feasible solutions (alternatives) is made by solving the multi-objective problem with objectives (criteria) $f_i,\ i = 1,\dots,l,\ i \ne i_0$ in the feasible set X_{i_0}.

More specifically, let the objectives (criteria) be ordered as follows

$$\min\{f_1 = \rho_1\} \quad \text{(Highest priority)}$$

$$\vdots$$

$$\min\{f_l = \rho_l\} \quad \text{(Lowest priority).}$$

In multi-objective linear optimization (Section 3 below), ρ_i is component of the slack variable s_i^+ or s_i^-, describing the objective *i*.

Thus, the optimal value of the criterion with higher priority is *not degraded* to a lower-priority criterion. This means that for all indices $i \geq 1$, if $F(f_i)$ is an optimal value of the objective function for criterion (goal) f_i, then optimization of lower-priority goals f_j $(j > i)$ cannot give a solution, which degrades the value of $F(f_i)$. At each stage, a single-criteria problem is solved, starting with the highest-priority goal f_1 and finishing with the lowest-priority goal f_l. The method described in **II.** is called the *preemptive method*.

Using another approach, the maximization (minimization) of the main objective is performed not on the whole feasible set *X* but on its subset, which is defined by constraints of the form $f_i(\mathrm{x}) \geq \alpha_i$, $i = 1,\ldots,l$, $i \neq i_0$, on the values of the remaining criteria, where α_i, $i = 1,\ldots,l$, are some minimum levels. A multi-objective problem, in which the "desirable levels" β_i, $i = 1,\ldots,l$, are given for the criteria f_i, and which requires minimization of the weighted sum of the "deviations" of f_i from β_i, $i = 1,\ldots,l$, is called the *goal programming problem*.

The methods, considered in **I.** and **II.**, are different in the sense that in the general case, they do not give the same optimal solution of a certain multi-objective problem. However, for none of the methods we cannot state that it is better than another method, because each approach is constructed (and therefore it is intended) to meet certain preferences in the decision making process.

One of the most natural optimality principles is the so-called *Pareto optimality principle* (*Pareto efficiency principle*). The point $\mathrm{x}^* \in X$ is called *Pareto effective* (*Pareto optimal*) for a maximization problem with objectives $f_1,\ldots,f_l$, if there does not exist a feasible point $\mathrm{y} \in X$, such that $f_i(\mathrm{y}) \geq f_i(\mathrm{x}^*)$ for each $i = 1,\ldots,l$, and this inequality is strict for at least one index *i*; in the case of a minimization problem, the inequalities are as follows $f_i(\mathrm{y}) \leq f_i(\mathrm{x}^*)$ for each $i = 1,\ldots,l$. In this sense, the Pareto optimal solutions cannot be improved by any other criterion. A disadvantage of this approach is the "multiplicity" of Pareto optimal solutions: in most cases, the set of Pareto effective points is quite wide, which makes it difficult to choose a concrete optimal solution and this requires the introduction of some "secondary" optimality principles. In practical problems, these secondary optimality principles often are performed as iterative procedures, which are based on information about the preferences of the person, who makes the decision, in the set X of alternatives or in the set of values of the objective functions.

This disadvantage can be overcome by the *method of arbitration solutions*, proposed by John Nash. This method substantially reduces the number of the chosen solutions among the optimal solutions in the sense of Pareto. On the basis of meaningful considerations, this method finds minimal feasible values f_i^0, and after that finds a feasible point x^* which maximizes $\prod_{i=1}^{l}(f_i(x) - f_i^0)$.

An important class of multi-objective problems, which are related to the Game theory, are the so-called *arbitration schemes*, which are cooperative games without additional payments, such that the permissible coalitions are only the singleton coalition and coalition, consisting of the whole set of players. The multi-objective problem can be considered as a game and can be solved by using various methods for solving games. For example, if the feasible solution x is chosen to maximize one of the objective functions $f_1, \ldots, f_l$, but which one is not known to the person, who makes the decision, then the weighted sum of these functions can be used, where the components of the mixed strategy of the considered game can be used as the weights of this weighted sum. The multi-objective problem can also be interpreted as a non-coalitional game, including as a cooperative game, and the related optimality principles can be applied. Multi-objective problems with non-numerical criteria, defined by ordered sets of alternatives, are studied in the theory of group decision making. In many cases, multi-objective problems are interpreted as decision making problems under uncertainty, and as a result of the choice of alternatives $x \in X$, the set of possible inputs $f_1(x), \ldots, f_l(x)$ becomes known, but it is not known which output will be realized in practice.

3. Multi-objective linear optimization

The following example shows how a real problem can be formulated mathematically as a multi-objective linear optimization problem.

Example 1: The council of a city is in the process of development of a new taxation system. The annual taxation base for the public real estate property is 50000000 Euros. The annual taxation bases for food and school facilities are 4000000 and 4500000 Euros, respectively. The annual public transport budget is 750000 Euros. The new taxation system should be based on the following 4 objectives (goals, criteria):

1. The tax income must be at least 15000000 Euros in order to ensure the financial endeavors in the city.
2. The food and the school facility taxes should not exceed 10% of all taxes collected.
3. Annual sales taxes should not exceed 20% of all taxes collected.
4. Public transport prices should not exceed 2 Euros per ticket.

Let the variables x_1, x_2, and x_3 denote the tax rates, presented as proportions, of the public real estate, food and school facilities, and annual sales taxes, respectively, and let the variable x_4 denote the public transport prices in Euros per ticket.

Then the objectives of the city council, after scaling the values accordingly, can be presented as follows:

$500x_1 + 40x_2 + 45x_3 + 7.5x_4 \geq 15$ (Tax income)

$40x_2 \leq 0,1(500x_1 + 40x_2 + 45x_3 + 7.5x_4)$ (Food and school facility taxes)

$45x_3 \leq 0,2(500x_1 + 40x_2 + 45x_3 + 7.5x_4)$ (Annual sales taxes)

$x_4 \leq 2$ (Public transport tax)

$x_1, x_2, x_3, x_4 \geq 0$ (Economic interpretation of the variables).

After simplifying we get

$$\begin{aligned}
&500x_1 + 40x_2 + 45x_3 + 7.5x_4 \geq 15 \\
&50x_1 - 36x_2 + 4.5x_3 + 0.75x_4 \geq 0 \\
&100x_1 + 8x_2 - 36x_3 + 1.5x_4 \geq 0 \\
&x_4 \leq 2 \\
&x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

Each of the first four inequalities above represents a criterion (goal, objective) which the city council is trying to achieve. However, these goals can be conflicting (contradictory), and the best that can be done in this situation, is to obtain a compromise solution. This can be achieved as follows. Each inequality is converted into a "flexible" goal, such that, if necessary, the corresponding constraint may be violated. Concerning Example 1, applying this approach, we obtain

$$\begin{aligned}
&500x_1 + 40x_2 + 45x_3 + 7.5x_4 + d_1^- - d_1^+ = 15 \\
&50x_1 - 36x_2 + 4.5x_3 + 0.75x_4 + d_2^- - d_2^+ = 0 \\
&100x_1 + 8x_2 - 36x_3 + 1.5x_4 + d_3^- - d_3^+ = 0 \\
&x_4 + d_4^- - d_4^+ = 2 \\
&x_1, x_2, x_3, x_4 \geq 0, \quad d_i^+, d_i^- \geq 0, i = 1,2,3,4,
\end{aligned}$$

where the nonnegative variables d_i^+ and d_i^-, $i = 1,2,3,4$, are the *slack variables*, and they represent the "deviation above" and "deviation below" the right-hand side of constraint No. i, $i = 1,2,3,4$, respectively.

The slack variables d_i^+ and d_i^-, $i = 1,2,3,4$, are dependent by definition, and therefore they cannot simultaneously be basic variables for the obtained linear optimization problem, due to the requirements of the linear optimization theory. This means that in each simplex table, at each simplex iteration, at most one of the two slack variables d_i^+ and d_i^- can be a basic variable, that is, to have a positive value.

If the initial inequality No. i is of the type "less than or equal to" $(\leq)$ and the corresponding $d_i^- \geq 0$, then the goal No. i will be satisfied, otherwise, the goal No. i will not be satisfied. The definitions of slack variables d_i^+ and d_i^-, $i = 1,2,3,4$, allow to fulfill or to violate goal No. i. This is the "flexibility", which is sought by the multi-objective problem, when it tries to find a compromise optimal solution. Naturally, a good compromise solution tries to minimize the values, with which each of the goals is violated.

In Example 1, since the first 3 constraints have the form "$\geq$", and the 4th constraint has the form "$\leq$", then the slack variables d_1^-, d_2^-, d_3^-, and d_4^+ represent the amounts, by which the corresponding goal can be violated, respectively. That is why, the compromise solution should be sought so that the following 4 goals be satisfied as much as possible:

$$\min\{G_1 = d_1^-\}$$

$$\min\{G_2 = d_2^-\}$$

$$\min\{G_3 = d_3^-\}$$

$$\min\{G_4 = d_4^+\}.$$

The functions (goals) G_1, G_2, G_3, and G_4 must be minimized subject to the equality constraints of Example 1.

The following examples illustrate applications of the *weighted method* and the *preemptive method* for solving some particular multi-objective problems.

Example 2: A company with 4 employers in its PR office wants to advertise its new product. Product advertisement can be by radio, television (TV), and Internet. The next Table 1 presents the exposition (information about the number of people, reached by any of the three methods of advertisement per day), the cost, and the number of assigned PR officers.

Table 1

Data for Example 2

Data	**Advertising data per minute**		
	Radio	**TV**	**Internet**
Exposition (in millions of persons per minute)	3	6	2
Cost (in thousand of Euros per minute)	6	20	3
Number of PR officers per minute	1	2	1

The time for radio advertisement is at most 5 minutes. Furthermore, advertising in any of the three ways must reach at least 10 million people. The company has a budget of 30000 Euros for advertisement.

The company has to determine how much time (in minutes) should be used for advertising by radio, TV, and Internet.

Let x_1, x_2, and x_3 be the number of minutes for advertisement via radio, TV, and Internet, respectively. The given situation can be described as a multi-objective problem as follows:

$$\min\{G_1 = d_1^-\} \quad \text{(Exposition requirement be satisfied)} \tag{1}$$

$$\min\{G_2 = d_2^+\} \quad \text{(Budget requirement be satisfied)} \tag{2}$$

$$\begin{gathered} 3x_1 + 6x_2 + 2x_3 + d_1^- - d_1^+ = 10 \quad \text{(Exposition objective)} \\ 6x_1 + 20x_2 + 3x_3 + d_2^- - d_2^+ = 30 \quad \text{(Budget objective)} \\ x_1 + 2x_2 + x_3 \le 4 \text{ (PR officers limit)} \\ x_1 \le 5 \quad \text{(Limit on radio advertisement)} \\ x_1, x_2, x_3, d_1^+, d_1^-, d_2^+, d_2^- \ge 0 \text{ (Economic interpretation of the variables).} \end{gathered} \tag{3}$$

It is estimated that the exposition objective is twice as expensive as the budget objective. Then the combined objective (goal) function is as follows

$$\min\{g = 2G_1 + G_2 = 2d_1^- + d_2^+\}. \tag{4}$$

As it was pointed out in Section 1, the multi-criterion optimization gives only an *effective solution* of the problem, which only satisfies the criteria of the model, and this solution is not necessarily optimal for all criteria.

In multi-objective linear optimization, a version of the simplex method is used, which guarantees nondegradation of the higher-priority solutions. This modification of the simplex method uses the so-called *column-dropping rule*, in which, a nonbasic variable x_j is eliminated, with $\Delta_j \equiv c_j - \sum_i \alpha_{ij} x_j \neq 0$ in the last (optimal) simplex table regarding the goal (criterion) G_k, before the optimization problem be solved regarding the goal G_{k+1}. The rule assumes that if the values of such nonbasic variables are increased to values, greater than 0 while the next criteria are optimized, the quality of a higher-priority criterion can only degrade, not to be improved. The preemptive method requires modification of the simplex table, such that the objective functions of all goals (criteria) of the considered model be included in this table.

Unfortunately, this modification of the simplex method makes the multi-objective optimization more complicated, than it really is.

We will show that the same results can be obtained in a simpler way by using the following steps.

Step 0. Identify the criteria (goals) of the model and order them by priority as follows: $G_1 = \rho_1 \succ G_2 = \rho_2 \succ \cdots \succ G_l = \rho_l$. Set $i := 1$.

Step i. Solve the linear optimization problem LP_i of minimizing G_i. Let $\rho_i = \rho_i^*$ be the corresponding optimal value of the slack variable ρ_i. If $i = l$, stop; problem LP_l solves the $l-$criterial problem. Otherwise, add a new constraint $\rho_i = \rho_i^*$ to the system of constraints of the problem for criterion G_i in order to ensure nondegradation of the value of ρ_i in the next problems. Set $i := i + 1$, and then repeat Step No. *i*.

The column-dropping rule successively reduces the dimension (that is, number of variables) of the problem by removing variables, whereas the approach, suggested above, enlarges the problem by adding new constraints. However, taking into account the nature of the additional constraints $\rho_i = \rho_i^*$, we can modify the simplex algorithm such that the new additional constraints can be implemented implicitly by replacing $\rho_i = \rho_i^*$ by $\rho_i \le \rho_i^*$. This replacement affects only the constraints, containing ρ_i, and in practice, the number of variables is reduced when moving from one goal (criterion) to another.

Example 3: Solve the problem of Example 2 by the *preemptive method*.

The problem is solved algorithmically.

Step 0. Order the goals by priority as follows $G_1 \succ G_2$.

$G_1 : \min d_1^-$ (Exposition requirement be satisfied)

$G_2 : \min d_2^+$ (Budget requirement be satisfied).

Step 1. Solve the linear problem LP_1:

$$\min\{G_1 = d_1^-\}$$

subject to

$3x_1 + 6x_2 + 2x_3 + d_1^- - d_1^+ = 10$ (Exposition objective)

$6x_1 + 20x_2 + 3x_3 + d_2^- - d_2^+ = 30$ (Budget objective)

$x_1 + 2x_2 + x_3 \le 4$ (PR officers limit)

$x_1 \le 5$ (Limit on radio advertisement)

$x_1, x_2, x_3, d_1^+, d_1^-, d_2^+, d_2^- \ge 0$ (Economic interpretation of variables).

In problem LP_1 we have $\rho_1 = d_1^-$. That is why the additional constraint, which will be used in the problem concerning the goal G_2, is $d_1^- \le \rho_1^*$, where ρ_1^* is defined above.

Step 2. Solve the linear problem LP_2 with objective function

$$\min\{G_2 = d_2^+\}$$

subject to the constraints of problem LP_1 (Step 1) plus the additional constraint $d_1^- \le \rho_1^*$.

Since $d_2^+ = 0$ in the optimal solution of problem LP_1 and since LP_2 is a minimization problem, then the optimal solution of problem LP_1 is an optimal solution of problem LP_2 as well; this conclusion can also be drawn by solving problem LP_2 directly by the simplex method. The value $d_2^+ = 0$ in the optimal solution of problem LP_2 shows that the goal G_2 is achieved.

The additional constraint $d_1^- \le \rho_1^*$ can be obtained by replacing d_1^- in the 1st constraint. As a result, the right-hand side of the 1st constraint of problem LP_1 will be reduced by ρ_1^*, and problem LP_2 is formulated as follows

$$\min\{G_2 = d_2^+\}$$

subject to

$3x_1 + 6x_2 + 2x_3 - d_1^+ = 10 - \rho_1^*$ (Exposition objective)

$6x_1 + 20x_2 + 3x_3 + d_2^- - d_2^+ = 30$ (Budget objective)

$x_1 + 2x_2 + x_3 \le 4$ (PR officers limit)

$x_1 \le 5$ (Limit on radio advertisement)

$x_1, x_2, x_3, d_1^+, d_2^+, d_2^- \ge 0.$

Number of variables in this formulation of problem LP_2 is 6 whereas the number of variables in problem LP_1 is 7. As pointed out above, the column-dropping rule is based on the idea of successive reduction of the number of variables.

The next Example 4 shows an application of the *preemptive method* to the problem, considered in Example 2 (and Example 3) for *optimizing the objective functions instead of satisfying the objectives (criteria)*. Example 4 also illustrates the column-dropping rule for solving multi-objective linear problems.

Example 4: The goals (criteria) of Example 2 can be reformulated as follows:

Priority No. 1: Maximization of exposition (G_1)

Priority No. 2: Minimization of the cost (G_2).

These two goals (criteria) mathematically can be formulated as follows:

$\max\{G_1(\mathrm{x}) = 3x_1 + 6x_2 + 2x_3\}$ (Exposition objective)

$\min\{G_2(\mathrm{x}) = 6x_1 + 20x_2 + 3x_3\}$ (Cost objective).

Then the new multi-objective problem is formulated as follows

$$\max\{G_1(\mathrm{x}) = 3x_1 + 6x_2 + 2x_3\}$$

$$\min\{G_2(\mathrm{x}) = 6x_1 + 20x_2 + 3x_3\}$$

subject to

$$x_1 + 2x_2 + x_3 \leq 4$$

$$x_1 \leq 5$$

$$x_1, x_2, x_3 \geq 0.$$

The procedure of Example 3 will be used.

Step 1. Solve the single-criteria linear optimization problem LP_1:

$$\max\{G_1(\mathrm{x}) = 3x_1 + 6x_2 + 2x_3\}$$

subject to

$$x_1 + 2x_2 + x_3 \leq 4$$

$$x_1 \leq 5$$

$$x_1, x_2, x_3 \geq 0.$$

After that solve the single-criteria linear optimization problem LP_2:

$$\min\{G_2(\mathrm{x}) = 6x_1 + 20x_2 + 3x_3\}$$

subject to

$$x_1 + 2x_2 + x_3 \leq 4$$

$$x_1 \leq 5$$

$$3x_1 + 6x_2 + 2x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0,$$

where the third constraint is the additional constraint.

The solution process continues in the same manner, if there are more criteria.

4. Pareto optimality for nonlinear optimization problems

In some nonlinear optimization problems, multiple objective functions $f_1,\dots,f_l$ also occur. Let $\mathrm{f}=[f_1,\dots,f_l]$, $\mathrm{g}=[g_1,\dots,g_m]$, $\mathrm{h}=[h_1,\dots,h_p]$ be vector functions.

Consider the problem

(M)

$$\min \mathrm{f}(\mathrm{x})$$

subject to

$$g_i(\mathrm{x})\le 0,\quad i=1,\dots,m,$$

$$h_i(\mathrm{x})=0,\quad i=1,\dots,p.$$

According to the definition of Section 2, a feasible point x (that is, a point satisfying the constraints of problem (M)) is said to be (*weak*) *Pareto optimal* for the considered minimization multi-objective problem (M) if there does not exist a feasible point y such that $f_i(\mathrm{y})<f_i(\mathrm{x})$, $i=1,\dots,l$.

Define the *generalized Lagrangian function* L for problem (M), interpreting λ as a vector in $\mathbb{R}^l$:

$$L(\mathrm{x},\lambda,\mathrm{r},\mathrm{s})=\langle\lambda,\mathrm{f}(\mathrm{x})\rangle+\langle\mathrm{r},\mathrm{g}(\mathrm{x})\rangle+\langle\mathrm{s},\mathrm{h}(\mathrm{x})\rangle\equiv\sum_{i=1}^{l}\lambda_i f_i(\mathrm{x})+\sum_{i=1}^{m}r_i g_i(\mathrm{x})+\sum_{i=1}^{p}s_i h_i(\mathrm{x}),$$

where $\langle .,.\rangle$ denotes the scalar (inner, dot) product.

It turns out that the following theorem, which is analogous to the corresponding theorem for the case of a single-criteria constrained optimization problem, holds true.

Theorem 1: *Let* x *be a Pareto optimal solution of the multi-objective problem* (M). *Then there exist* $\lambda\ge 0$, $\mathrm{r}\ge 0$, *and* s, *which are not equal to* 0 *simultaneously, such that* $\langle\mathrm{r},\mathrm{g}(\mathrm{x})\rangle=0$ (*that is,* $r_i g_i(\mathrm{x})=0$, $i=1,\dots,m$) *and* $0\in\partial_{\mathrm{x}}L(\mathrm{x},\lambda,\mathrm{r},\mathrm{s})$, *where* $\partial_{\mathrm{x}}L$ *denotes the subdifferential of* L *with respect to* x.

The proof of Theorem 1 is omitted.

5. Conclusion

In this paper, the multi-objective (multiple criteria, vector) optimization is considered. Some examples of such optimization problems are presented, Pareto optimality is introduced, and arbitrage schemes are described. The special case of multi-objective linear optimization is studied, and methods for solving such problems are presented. Pareto optimality for the nonlinear optimization problem is also considered.

An open problem for future research is to develop other methods and applications of multi-objective optimization.

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