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Multi-objective fuzzy optimization under uncertainty for investment portfolio selection in emerging financial markets

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Abstract

In the problem of selecting an investment portfolio, especially in emerging financial markets, the investor usually intends to make the best possible choice for investment while using several different objectives, such as returns, liquidity, and risk. For this purpose, a new method is proposed using goal values and the maximum possible deviations for each goal. Fuzzy goals, fuzzy interval values for each asset, and their combination with satisfaction functions are employed to develop a fuzzy goal programming model. It is also suggested that the goals be weighted based on the comments of decision-making experts. This approach can determine an optimal solution based on the uncertainties of the financial markets (fuzzy interval values of the goals) and choose the portfolio based on the best possible level of the goals. The objective is to model a multi-objective investment portfolio selection problem, taking into account the fuzzy interval values, in which the distributions and fuzzy intervals are expressed using the data available in the decision-making environment. In addition, the opinions of market experts have been used directly to weigh goals. Finally, the proposed approach for choosing the investment portfolio in the crypto-currency market is

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implemented, and the 100% satisfaction level of the investment portfolio in all the objectives of the problem, i.e., return, liquidity, and risk, is obtained as the final output.

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1. Introduction

Investment portfolio selection (IPS) addresses the problem of determining the most suitable stocks based on the characteristics of each stock. This problem can be described as a two-step process, i.e., evaluating the stocks to select those that best match the investor's preferences and estimating the amount of capital that should be invested in each of the selected stocks [1]. In 1952, Harry Markowitz introduced the mean-variance (M-V) model for the investment portfolio selection problem, in which the two objectives of return and risk are optimized [2]. In the last five decades, Markowitz's model of financial theory has undergone many changes and developments. However, the original form of this model has not been employed to implement a large-scale portfolio because of significant problems. These problems are A. the computational complexity associated with solving quadratic planning problems with large-scale covariance matrices [3] and B. the specific type of data required for the investment portfolio selection problem. It is not easy to obtain an accurate prediction of the input data required for the model, especially in the variance–covariance matrix among stocks [4]. Markowitz's model, due to the lack of satisfactory performance and the presence of unstable stock weights, which resulted in much dispersion in the resulting portfolio, has been criticized. Several researchers have proposed various approximation methods to reduce these difficulties. Absolute deviation portfolio optimization models [5] and semi-absolute deviation optimization models [6] are among these schemes.

Several finance models have been developed to aid investors in making decisions. In addition to the Markowitz model, other index-based models have been proposed by [3] and [7], which simplify calculations by incorporating factors that affect stock prices. The capital asset pricing model (CAPM), developed by [8] and [9], and the multifactor arbitrage pricing theory (APT), also known as the equilibrium model, introduced by [10], assume that the return of each stock is a linear combination of a small number of factors, in addition to a stock-specific random variable. These models have been further improved by incorporating additional constraints or decision criteria, such as negative risk criteria, cardinality, and rebalancing, as proposed by [11], [12], [13], [14], and [15].

The literature review shows that in choosing an investment portfolio, the investor seeks to combine non-comparable, conflicting objectives and structural preferences simultaneously. In addition to return and risk, the problem of choosing an investment portfolio can be developed to include other factors such as liquidity, portfolio size and social responsibility, or the size of dividends. [16] Solving the multi-objective investment portfolio selection problem requires comprehensive methods such as goal programming (GP). This approach has been widely used in the selection of investment portfolios when investors can set goals for the levels of achievement for each investment objective. [17] presented the first goal programming model for the investment portfolio selection problem. The researchers used a goal programming model considering the return, risk, and dividend objectives. Later, several works, such as ([18]; [19]; [20]; [21]; [22]; [23]; [24] and [25]), well demonstrated the applications of classical goal programming for the multi-objective investment portfolio selection problem.

The high volatility of market environments makes it difficult to use historical financial data to predict the future state of the stock market. Furthermore, investors may have difficulty defining their objectives for the stocks in their portfolio, leading to decision variables that can be considered either fuzzy or random parameters. To address this challenge, various extensions of the goal programming model have been proposed that employ fuzzy methodology to address the uncertainty inherent in the investment portfolio selection problem.

Bellman and Zadeh's theory of fuzzy sets, introduced in 1970, has proven to be a valuable tool for managing inaccurate data and human uncertainty in decision-making [26]. A novel optimization method utilizing fuzzy values was presented in [27]. Several researchers have employed fuzzy decision theory to address the investment portfolio selection problem. For example, in 2000, Tanaga, et al. developed two types of IPS models based on fuzzy possibility distributions, with the objectives of variance minimization and increasing stock portfolio returns [28]. Watada formulated an IPS model in 2001 by considering the expected return and risk as fuzzy objectives [29]. Based on the Markowitz model, researchers described the levels of achievement for the considered objectives using logistic membership functions. Arenas-Parra et al. formulated a fuzzy-adaptive programming model in 2006 that applies the concept of a fuzzy goal solution. In this approach, the difference between the ideal solution and target values is evaluated using fuzzy parameters and expected intervals and the difference between the fuzzy numbers [30]. Additionally, researchers have studied sensitivity analysis to analyze the difference between expert opinion and investor preferences. Gladish et al. developed a three-stage model in 2007 based on a multi-objective model involving several markets with random scenarios. This model takes into consideration the uncertainty of market scenarios and data inaccuracies [31]. [32] proposed a nondeterministic goal programming model using the concept of satisfaction functions to explicitly consider investor portfolio preferences. However, in this approach, different objectives are expressed through time intervals, and other parameters are considered constant with specified values. [33] proposed weighted additive models for investment decisions using fuzzy goal programming. Based on Zimmerman's approach in 1978, Gupta and Bhattacharjee proposed a minimum-sum weighted fuzzy-goal programming technique ([34]; [35]). In this approach, by assigning the highest degree as the ideal level, only the variables with small deviations are considered in each membership function. In 2011, Liu proposed a fuzzy approach to the portfolio optimization problem in which stock returns were fuzzy numbers [36]. To solve the proposed model, the absolute mean deviation risk function model and the Zadeh expansion principle are used. However, in Liu's approach, the membership functions of a specific objective value should be applied without using mathematical methods. [37] suggested an FGP model in which investors' preferences are integrated with social responsibility. [38] proposed a fuzzy interval goal programming approach to select a portfolio by considering different objectives, in which fuzzy interval values are used in the final model.

In recent years, various approaches have been proposed to address the investment portfolio selection problem using fuzzy decision theory. [39] proposed a fuzzy IPS model that considers risk preferences according to market dynamics trends and a risk-return trade-off, allowing decision-makers to prioritize their objectives. However, this model requires expert knowledge and interpretation to form fuzzy membership functions based on investor behavior and market dynamics trends. [40] proposed a fuzzy hierarchical GP that expresses investor preferences at two levels: first, by comparing the same criteria, and second, by comparing two superior-level criteria (financial goals and sustainable and responsible goals). [41] developed a fuzzy multi-objective optimization model with portfolio selection, dividend return, and risk control, using a multiplicative factor to construct the objective function. However, the results of their research showed that this operator may not be suitable for modeling human decision-making [42]. [43] introduced a goal-fuzzy programming model with limited chance, including expected return, risk, and profit rate, and considered two types of stochastic and fuzzy uncertainty. [44] proposed an imprecise GP model for portfolio selection by using the concept of satisfaction functions and considering the direct opinion of the investor based on the defuzzified values of returns in the original model.

Multi-objective decision-making methods (MODMs) are decision-making approaches that involve the simultaneous optimization of several objectives and are used to solve ongoing problems with unlimited options. The main objective of MODM is to determine a set of potential solutions to identify a single optimal solution that balances different objectives. On the other hand, the multiple attribute decision-making method (MADM) is a decision-making approach that involves evaluating and ranking alternatives based on multiple attributes or criteria. MADM techniques are used to discretely represent a problem with multiple conflicting criteria and with multiple possible options that each with multiple attributes should be evaluated. There are two main goals for solving practical problems with MCDM methods: calculating the optimal criterion weights and determining the ranking of alternatives. Researchers have presented a new approach to design methods for improving the quality of decision-making in recent decades, and they have mentioned several techniques for analyzing the weights of criteria and methods for ranking options ([45] and [46]). The fuzzy criterion-based method (F-BCM) is one of the latest MCDM methods for calculating criteria weights [47]. In this method, among all the determined criteria for decision-making, one criterion is selected as an index and the main criterion. Then, the relative importance of the other criteria compared to the primary criterion is determined. For an application of this method in portfolio selection, we refer to the research of [48] and [49].

Although the studies mentioned in the literature review have tried to solve some of the issues of investment portfolio selection, there are few studies on determining investment portfolios in emerging financial markets because the historical data of these markets are not completely analyzed, and investors have been looking for their attractiveness to obtain a suitable portfolio in that market. In addition, investors who operate in these markets often do not have enough experience with their performance, and their ability to make decisions on elements such as return, liquidity, and risk is reduced. Therefore, the concept of satisfaction functions (for example, [50] and [51]) can be used to select and achieve the desired levels of goals in determining an investment portfolio. On the other hand, the lack of historical data about the assets of this market causes more uncertainty than for other financial markets. As one of the appropriate solutions for addressing this uncertainty, the assumption of fuzzy interval values is suggested. The use of fuzzy interval values in the original model instead of their defuzzification causes the uncertainty considered in the model to be at the maximum level. On the other hand, each of the goals considered in decision-making has different weights compared to each other, and to determine the weights of these goals, the opinions of the experts in that market can be considered directly. Therefore, this article aims to determine an investment portfolio in emerging financial markets using the concepts of satisfaction functions, fuzzy interval values, and the direct opinions of experts in that market.

In the current research, a multi-objective investment portfolio selection approach is proposed, including the return, liquidity, and risk fuzzy interval objectives, in which trapezoidal fuzzy numbers are obtained from the data provided in the decision-making environment. Finally, these numbers are used to construct the fuzzy intervals for each objective in the final model. In addition, to use the experience of the financial market experts, we will consider their opinions directly by combining them with the concepts of satisfaction functions and multi-objective decision-making in the final model. The proposed approach allows all investors (professionals and beginners) to choose a suitable investment portfolio by considering several conflicting and uncertain objectives in emerging financial markets. In other words, the innovations of this research are as follows:

I. Construction and use of fuzzy distribution functions and calculation of goal values and expected intervals for all desired objectives according to the market conditions under investigation using simple mathematical modeling.

II. We use the experience of financial market experts in the planning model to choose suitable investment portfolios in emerging financial markets.
III. Using fuzzy theory, we present an approach for calculating portfolio risk under the condition of insufficient data.
IV. Development of a fuzzy-based criterion method to weight the goals investigated in the problem, taking into account the experience of financial market experts.
V. Simple modeling considers fuzzy interval values in the model that are applicable for all investors with different levels of investment knowledge.

The rest of the article is organized as follows. In section 2, a fuzzy method is introduced to determine the fuzzy distributions of objectives and their expected intervals, i.e., return, liquidity, and fuzzy risk. Section 3 proposes a fuzzy programming method that directly incorporates the opinions of financial market experts and considers the fuzzy interval of values in modeling the investment portfolio selection problem. In section 4, the development of the fuzzy-based criterion method to determine the weights of the objectives is introduced. In section 5, the final model of the problem for choosing the investment portfolio is presented. In section 6, the implementation of the proposed approach to choosing an investment portfolio in the crypto-currency market is presented. Section 7 presents the final summary and directions for future research, and finally, section 8 provides practical conclusions and management perspectives.

2.1. *Determining the fuzzy distribution and the expected intervals of the targets*

Investors always make decisions in an economic environment with uncertain conditions, which leads to uncertainty in their choices. One of the reasons for the uncertainty of this environment is the unavailability of sufficient data for the effective parameters in the market. Since emerging financial markets have much less data than other financial markets, this problem increases uncertainty. Therefore, the investor in these markets cannot determine with certainty the goal values and maximum deviations of return R_p, liquidity L_p, and risk β_p in the portfolio, nor can he define the parameters of return r_j, liquidity l_j, and risk b_j (j=1...,m), which are the components of the portfolio objectives. Since the parameters and goal values of the practical objectives in the investment portfolio selection problem are fuzzy, the solution method and the possibility distributions are also fuzzy. [52]. Thus, the goal values of return, liquidity, and portfolio risk criteria, which are represented by $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$, respectively, are the fuzzy parameters of return, liquidity, and risk, and they are, respectively represented by $\tilde{r}_j$, $\tilde{l}_j$, and $\tilde{b}_j$, denoted by a trapezoidal fuzzy number as $\tilde{r} = (r_1, r_2, r_3, r_4)$, $\tilde{l} = (l_1, l_2, l_3, l_4)$, and $\tilde{b} = (b_1, b_2, b_3, b_4)$. These fuzzy numbers are obtained using environmental data when each fuzzy parameter is expressed by its fuzzy distribution. Figure 1 (1-a, 1-b, 1-c) shows the fuzzy distribution function for the problem parameters.

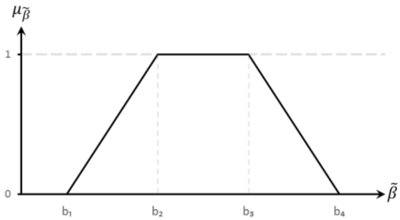

Figure 1-a: Portfolio fuzzy return distribution function

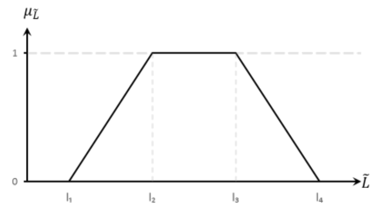

Figure 1-b: Portfolio fuzzy liquidity distribution function

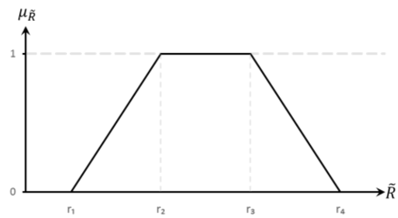

Figure 1-c: Fuzzy portfolio risk distribution function

Figure 1
Distribution functions of portfolio objectives

For example, in Figure 1-a, $\mu_{\tilde{R}}$ with a value of $0 \leq \mu_{\tilde{R}} \leq 1$ shows the possibility of occurrence of $\tilde{R}$. There are four possible values (located at the breakpoints of the figure) for $\tilde{R}: r_1, r_2, r_3, r_4$. The values of the interval $[r_2, r_3]$ are the central values of the trapezoidal fuzzy number with the maximum probability of occurrence, i.e., $\mu_{\tilde{R}} = 1$, and the values of r_1, r_4 are the side values of the trapezoidal fuzzy number with the minimum degree of occurrence, i.e., $\mu_{\tilde{R}} = 0$ (for the other objectives, the results are similar). Since trapezoidal fuzzy numbers are piecewise linear and simplify the calculation process, they are the most commonly used types of numbers in solving mathematical programming problems. To generate the possibility distributions $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$, first, the methods of ideal and anti-ideal solutions for the return, liquidity and risk of the portfolio, which are denoted as R_p^{max}, L_p^{max}, and β_p^{max} and R_p^{min}, L_p^{min}, and β_p^{min}, respectively, are calculated. Then, we determine the possibility distributions of fuzzy solutions $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$, which are represented by $\tilde{R}_p^*$, $\tilde{L}_p^*$, and $\tilde{\beta}_p^*$, and finally, the expected intervals of return, liquidity, and risk fuzzy are calculated.

22. *Calculation of the ideal and anti-ideal solutions for the return, liquidity, and risk criteria*

Selecting an investment portfolio is generally a multi-objective linear programming problem that attempts to determine the best portfolio based on several objectives. For example, consider the following fuzzy multi-objective linear programming model.

$$\text{Max}\,\tilde{R}_p = \sum_{j=1}^{m} \tilde{r}_j\, x_j \quad (1)$$

$$\text{Max}\,\tilde{L}_p = \sum_{i=1}^{m} \tilde{l}_j\, x_j \quad (2)$$

$$\text{Opt}\,\tilde{\beta}_p = \sum_{j=1}^{m} \tilde{b}_j\, x_j \quad (3)$$

$$s.t. \qquad \sum_{j=1}^{m} x_j = 1 \quad (4)$$

$$0 \le x_j \le t_1 \quad (5)$$

$$\sum_{v=c_k}^{c'_k} x_v \le w_v.\ \forall v \in 1,\dots,f \quad (6)$$

where $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$ are the return, liquidity, and risk objectives of the investment portfolio selection problem, respectively. t_1 is the maximum amount of selection from each asset, and w_k is the maximum amount of selection from each sector (see section 2-3). Objective functions (1) and (2) are related to the objective of maximizing the return and liquidity of the portfolio, and objective function (3) represents that the portfolio risk should be equal to a predetermined value as a goal solution. Constraints (4) - (6) are related to the assumptions of not allowing short selling, maximum selection from each asset, and selection from different sectors.

To calculate the goal solutions of $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$ for a particular value of $\alpha \in [0,1]$, each of the fuzzy objectives (1) to (3) is decomposed into the following two deterministic problems that are formulated for $\tilde{R}_p$ values as follows:

$$\left(R_p^{max}\right)_\alpha^l = max \sum_{j=1}^{m}\left(r_{j1} + (r_{j2} - r_{j1})\alpha\right) x_j \quad (7)$$

And constraints (4)-(6) (7.1)

$$\left(R_p^{max}\right)_\alpha^u = max \sum_{j=1}^{m}\left(r_{j4} - (r_{j4} - r_{j3})\alpha\right) x_j \quad (8)$$

And constraints (4)-(6) (8.1)

For the objectives of $\tilde{L}_p$ and $\tilde{\beta}_p$, the following results are obtained:

$$\left(L_p^{max}\right)_\alpha^l = max \sum_{j=1}^{m}\left(l_{j1} + (l_{j2} - l_{j1})\alpha\right) x_j \quad (9)$$

And constraints (4)-(6) (9.1)

$$\left(L_p^{max}\right)_\alpha^u = max \sum_{j=1}^{m}\left(l_{j4} - (l_{j4} - l_{j3})\alpha\right) x_j \quad (10)$$

And constraints (4)-(6) (10.1)

$$\left(\beta_p^{max}\right)_\alpha^l = max \sum_{j=1}^{m}\left(b_{j1} + (b_{j2} - b_{j1})\alpha\right) x_j \quad (11)$$

And constraints (4)-(6) (11.1)

$$\left(\beta_p^{max}\right)_\alpha^u = max \sum_{j=1}^{n}\left(b_{j4} - (b_{j4} - b_{j3})\alpha\right) x_j \quad (12)$$

And constraints (4)-(6) (12.1)

The optimizations in (7), (9) and (11) are designed to determine the goal solutions of the lower limit for the maximum value of the objective functions, i.e., $\left\{\left(R_p^{max}\right)_\alpha^l, \left(L_p^{max}\right)_\alpha^l, \left(\beta_p^{max}\right)_\alpha^l \ , \forall\alpha \in [0,1]\right\}$, and models (8), (10), and (12) are designed to determine the goal solutions of the upper limit for the maximum value of the objective functions, i.e., $\left\{\left(R_p^{max}\right)_\alpha^u, \left(L_p^{max}\right)_\alpha^u, \left(\beta_p^{max}\right)_\alpha^u \ , \forall\alpha \in [0,1]\right\}$. Then, models (7), (9) and (12) provide the trapezoidal fuzzy numbers R_p^{max}, L_p^{max} and β_p^{max} as follows:

$$\left\{\left(R_p^{max}\right)_0^l, \left(R_p^{max}\right)_1^l, \left(R_p^{max}\right)_1^u, \left(R_p^{max}\right)_0^u\right\}$$
$$\left\{\left(L_p^{max}\right)_0^l, \left(L_p^{max}\right)_1^l, \left(L_p^{max}\right)_1^u, \left(L_p^{max}\right)_0^u\right\}$$
$$\left\{\left(\beta_p^{max}\right)_0^l, \left(\beta_p^{max}\right)_1^l, \left(\beta_p^{max}\right)_1^u, \left(\beta_p^{max}\right)_0^u\right\}.$$

The minimum values of the objectives, i.e., R_p^{min}, L_p^{min} and β_p^{min}, are obtained by minimizing the objective functions in the previous problems (models (7) to (12)); thus, the minimal values of the anti-ideal solutions for each objective are as follows:

$$\left\{\left(R_p^{min}\right)_0^l, \left(R_p^{min}\right)_1^l, \left(R_p^{min}\right)_1^u, \left(R_p^{min}\right)_0^u\right\}$$
$$\left\{\left(L_p^{min}\right)_0^l, \left(L_p^{min}\right)_1^l, \left(L_p^{min}\right)_1^u, \left(L_p^{min}\right)_0^u\right\}$$
$$\left\{\left(\beta_p^{min}\right)_0^l, \left(\beta_p^{min}\right)_1^l, \left(\beta_p^{min}\right)_1^u, \left(\beta_p^{min}\right)_0^u\right\}$$

When the ideal and anti-ideal solutions are obtained, we can determine the possibility distributions of each fuzzy criterion, namely, $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$.

2.3 *Construction of the fuzzy possibility distribution*

The possibility distribution of the fuzzy parameter can be obtained using the decomposition theorem. To determine this fuzzy distribution, the method of [53] was used. The lower limit of the possibility fuzzy distribution for each objective is obtained for the specific values of α, other objectives are defuzzified, and the solutions $\left\{\left(R_p^*\right)_\alpha^l, \left(L_p^*\right)_\alpha^l, \text{and } \left(\beta_p^*\right)_\alpha^l \ , \forall\alpha \in [0,1]\right\}$ obtained from a fuzzy multi-objective model are reformulated as a deterministic single-objective model. For example, the return objective will be obtained as follows:

$$\text{Max} \quad \lambda \tag{13}$$

$$s.t. \ \lambda \leq \frac{\left[\sum_{j=1}^m (r_{j1} + (r_{j2} - r_{j1})\alpha)x_j - \left(R_p^{min}\right)_\alpha^l\right]}{\left[\left(R_p^{max}\right)_\alpha^l - \left(R_p^{min}\right)_\alpha^l\right]} \tag{13.1}$$

$$\lambda \leq \frac{\left[\sum_{j=1}^m l_j^* x_j - \bar{L}_p^{min}\right]}{\left[\bar{L}_p^{max} - \bar{L}_p^{min}\right]} \tag{13.2}$$

$$\lambda \leq \frac{\left[\sum_{j=1}^m b_j^* x_j - \bar{\beta}_p^{min}\right]}{\left[\bar{\beta}_p^{max} - \bar{\beta}_p^{min}\right]} \tag{13.3}$$

And constraints (4)-(6) (13.4)

For liquidity and risk, we have equations (14) - (14-4) and (15) - (15-4), respectively:

$$\text{Max}\phi \tag{14}$$

$$s.t.\ \phi \leq \frac{\left[\sum_{j=1}^{m}(l_{j1}+(l_{j2}-l_{j1})\alpha)x_i-\left(L^{min}_{p}\right)^{l}_{\alpha}\right]}{\left[\left(L^{max}_{p}\right)^{l}_{\alpha}-\left(L^{min}_{p}\right)^{l}_{\alpha}\right]} \tag{14.1}$$

$$\phi \leq \frac{\left[\sum_{j=1}^{m} b_j^* x_j-\bar{\beta}_p^{min}\right]}{\left[\bar{\beta}_p^{max}-\bar{\beta}_p^{min}\right]} \tag{14.2}$$

$$\phi \leq \frac{\left[\sum_{j=1}^{m} r_j^* x_j-\bar{R}_p^{min}\right]}{\left[\bar{R}_p^{max}-\bar{R}_p^{min}\right]} \tag{14.3}$$

And constraints (4)-(6) (14.4)

$$\text{Max}\gamma \tag{15}$$

$$s.t.\ \gamma \leq \frac{\left[\sum_{j=1}^{m}(b_{j1}+(b_{j2}-b_{j1})\alpha)x_j-\left(\beta^{min}_{p}\right)^{l}_{\alpha}\right]}{\left[\left(\beta^{max}_{p}\right)^{l}_{\alpha}-\left(\beta^{min}_{p}\right)^{l}_{\alpha}\right]} \tag{15.1}$$

$$\gamma \leq \frac{\left[\sum_{j=1}^{m} l_j^* x_j-\bar{L}_p^{min}\right]}{\left[\bar{L}_p^{max}-\bar{L}_p^{min}\right]} \tag{15.2}$$

$$\gamma \leq \frac{\left[\sum_{j=1}^{m} r_j^* x_j-\bar{R}_p^{min}\right]}{\left[\bar{R}_p^{max}-\bar{R}_p^{min}\right]} \tag{15.3}$$

And constraints (4)-(6) (15.4)

where λ, ϕ, and γ represent the compromise degree for satisfying the objectives of the return, liquidity, and fuzzy risk, respectively, while the coefficients at the α level are obtained. $\bar{R}_p^{max}$, $\bar{L}_p^{max}$, and $\bar{\beta}_p^{max}$ ($\bar{R}_p^{min}$, $\bar{L}_p^{min}$, and $\bar{\beta}_p^{min}$, respectively) are determined by maximizing (minimizing) the expected values of $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$ (see section 2-3), and they are obtained under constraints (4)-(6). In addition, to construct the upper limit of the fuzzy distribution for each objective, the solutions of $\left\{\left(R_p^*\right)^u_\alpha, \left(L_p^*\right)^u_\alpha, \text{and } \left(\beta_p^*\right)^u_\alpha, \forall\alpha \in [0,1]\right\}$ are obtained by solving the following mathematical models:

$$\text{Max}\ \ \lambda \tag{16}$$

$$s.t.\ \lambda \leq \frac{\left[\sum_{j=1}^{m}(r_{j4}-(r_{j4}-r_{j3})\alpha)x_j-\left(R^{min}_{p}\right)^{u}_{\alpha}\right]}{\left[\left(R^{max}_{p}\right)^{u}_{\alpha}-\left(R^{min}_{p}\right)^{u}_{\alpha}\right]} \tag{16.1}$$

$$\lambda \leq \frac{\left[\sum_{j=1}^{m} l_j^* x_j-\bar{L}_p^{min}\right]}{\left[\bar{L}_p^{max}-\bar{L}_p^{min}\right]} \tag{16.2}$$

$$\lambda \leq \frac{\left[\sum_{j=1}^{m} b_j^* x_j-\bar{\beta}_p^{min}\right]}{\left[\bar{\beta}_p^{max}-\bar{\beta}_p^{min}\right]} \tag{16.3}$$

And constraints (4)-(6) (16.4)

$$\text{Max}\phi \tag{17}$$

$$s.t.\ \phi \leq \frac{\left[\sum_{j=1}^{m}(l_{j4}-(l_{j4}-l_{j3})\alpha)x_j-\left(L^{min}_{p}\right)^{l}_{\alpha}\right]}{\left[\left(L^{max}_{p}\right)^{u}_{\alpha}-\left(L^{min}_{p}\right)^{u}_{\alpha}\right]} \tag{17.1}$$

$$\phi \leq \frac{\left[\sum_{j=1}^{m} b_j^* x_j-\bar{\beta}_p^{min}\right]}{\left[\bar{\beta}_p^{max}-\bar{\beta}_p^{min}\right]} \tag{17.2}$$

$$\phi \leq \frac{\left[\sum_{j=1}^{m} r_j^* x_j - \bar{R}_p^{min}\right]}{\left[\bar{R}_p^{max} - \bar{R}_p^{min}\right]} \tag{17.3}$$

And constraints (4)-(6) (17.4)

$$\text{Max}\gamma \tag{18}$$

$$s.t.\ \gamma \leq \frac{\left[\sum_{j=1}^{m}(b_{j4}-(b_{j4}-b_{j3})\alpha)x_j - \left(\beta^{min}_{\ \ p}\right)^u_\alpha\right]}{\left[\left(\beta^{max}_{\ \ p}\right)^u_\alpha - \left(\beta^{min}_{\ \ p}\right)^u_\alpha\right]} \tag{18.1}$$

$$\gamma \leq \frac{\left[\sum_{j=1}^{m} l_j^* x_j - \bar{L}_p^{min}\right]}{\left[\bar{L}_p^{max} - \bar{L}_p^{min}\right]} \tag{18.2}$$

$$\gamma \leq \frac{\left[\sum_{j=1}^{m} r_j^* x_j - \bar{R}_p^{min}\right]}{\left[\bar{R}_p^{max} - \bar{R}_p^{min}\right]} \tag{18.3}$$

And constraints (4)-(6) (18.4)

When the fuzzy possibility distributions of $\tilde{R}_p$, $\tilde{L}_p$, and $\tilde{\beta}_p$ are constructed, the next step is to calculate the expected intervals for the objectives of return, liquidity, and risk in the portfolio, which are denoted by $EI(\tilde{R}_p^*)$, $EI(\tilde{L}_p^*)$, and $EI(\tilde{\beta}_p^*)$, respectively.

2.4 *Determination of the expected distances and expected values*

To obtain the expected interval $(\tilde{R}_p^*)$, which corresponds to $\tilde{R}_p$, the following formula was used (Mansour et al., 2019) [44]:

$$EI(\tilde{R}_p^*) = \left[R_p^l\, , R_p^u\,\right] = \left[\frac{(\sum_{\alpha=0}^{1-h}(R_p^*)^{\ l}_\alpha + \sum_{\alpha=h}^{1}(R_p^*)^{\ l}_\alpha)\times h}{2}\, , \frac{(\sum_{\alpha=0}^{1-h}(R_p^*)^u_\alpha + \sum_{\alpha=h}^{1}(R_p^*)^u_\alpha)\times h}{2}\right] \tag{19.1}$$

where h is equal to the step size corresponding to the cutting value α. Additionally, by generalizing this formula for $\tilde{L}_p$, its expected interval, $\tilde{L}_p^*$, is obtained as follows:

$$EI(\tilde{L}_p^*) = \left[L_p^l\, , L_p^u\,\right] = \left[\frac{(\sum_{\alpha=0}^{1-h}(L_p^*)^{\ l}_\alpha + \sum_{\alpha=h}^{1}(L_p^*)^{\ l}_\alpha)\times h}{2}\, , \frac{(\sum_{\alpha=0}^{1-h}(L_p^*)^u_\alpha + \sum_{\alpha=h}^{1} L_p^*)^u_\alpha)\times h}{2}\right] \tag{19.2}$$

We propose a new formula to determine the expected interval $(\tilde{\beta}_p^*)$ of the parameter $\tilde{\beta}_p$ as follows:

$$EI(\tilde{\beta}_p^*) = \left[\beta_p^l\, , \beta_p^u\,\right]$$
$$= \left[\beta_g - \left|\frac{(\beta_p^*)_0^l + (\beta_p^*)_1^l + (\beta_p^*)_1^u + (\beta_p^*)_0^u}{4} - \beta_g\right|\, , \beta_g + \left|\frac{(\beta_p^*)_0^l + (\beta_p^*)_1^l + (\beta_p^*)_1^u + (\beta_p^*)_0^u}{4} - \beta_g\right|\right] \tag{19.3}$$

where β_g is equal to the goal value of $\tilde{\beta}_p$. If $\left|\frac{(\beta_p^*)_0^l + (\beta_p^*)_1^l + (\beta_p^*)_1^u + (\beta_p^*)_0^u}{4} - \beta_g\right|$ is equal to β_p^*, then equation (19-3) changes as follows:

$$EI(\tilde{\beta}_p^*) = \left[\beta_p^l\, , \beta_p^u\,\right] = \left[\beta_g - \ \beta_p^*, \beta_g + \beta_p^*\right] \tag{19.4}$$

The expected intervals $\tilde{r}_i$, $\tilde{l}_i$, and $\tilde{b}_i$, which are shown by $EI(\tilde{r}_i)$, $EI(\tilde{l}_i)$, and $EI(\tilde{b}_i)$, respectively, are calculated as follows [54]:

$$EI(\tilde{r}_i) = [r_i^l, r_i^u] = \left[\frac{r_{i1}+r_{i2}}{2}, \frac{r_{i3}+r_{i4}}{2}\right], \tag{20.1}$$

$$EI(\tilde{l}_i) = [l_i^l, l_i^u] = \left[\frac{l_{i1}+l_{i2}}{2}, \frac{l_{i3}+l_{i4}}{2}\right], \tag{20.2}$$

$$EI(\tilde{b}_i) = [b_i^l, b_i^u] = \left[\frac{b_{i1}+b_{i2}}{2}, \frac{b_{i3}+b_{i4}}{2}\right], \tag{20.3}$$

In the next section, a goal programming model with fuzzy interval values and using the concept of satisfaction functions is proposed for the financial portfolio selection problem.

3. Construction of satisfaction limits and diversification of the investment portfolio

The main objective of investors in selecting an investment portfolio in financial markets is to choose a portfolio with the highest return, liquidity, and reliability. In other words, different objectives are involved in choosing an investment portfolio. For example, several researchers have suggested a set of objectives that investors can consider to determine an investment portfolio, including ([17]; [55]; [1]; [56]; [37] and [43]). In this research, three objectives—return, liquidity, and risk—are considered.

3.1. *Modeling the investment portfolio goals*

In financial markets, especially in emerging financial markets, the investor always faces the challenge of choosing an efficient portfolio among a set of choices that should be justified. Considering this fact, it is possible to design satisfaction functions for the objectives considered in the problem, i.e., return, liquidity, and risk, based on the investor's preferences and according to the most favorable possible value of each objective. In the proposed approach, in addition to considering the satisfaction functions of return, liquidity, and risk, the goal values of the objectives, the interval values of the objectives for each asset, and the expert's comments are considered to determine a suitable portfolio.

3.1.1 Constructing the satisfaction function for investment portfolio returns

The purpose of the investor in selecting an investment portfolio is to gain profit or future income through the price growth of the assets in the portfolio or the asset interest. According to the investor's preferences regarding the most favorable return of the investment portfolio, a limit should be defined for the return objective. In other words, the investor is always trying to achieve the maximum return of the investment portfolio, so an upper limit for the expected interval of the return, R_p^u, is suggested that is the most desirable return of the investment portfolio or its goal value. On the other hand, since the parameters related to the returns of different assets are fuzzy, their expected intervals can be used instead of the expected values. According to these descriptions, the relevant constraints are suggested as follows:

$$\sum_{j=1}^{m} y_{ij}x_i + n_1 \geq R_p^u \quad i = 1, \forall j \tag{21}$$

$$y_{1\,j} + e_{1j1} - e_{1j2} = r_j^u \;, \forall j \tag{21.1}$$

$$r^l \leq y_{1j} \leq r^u \qquad \forall j \tag{21.2}$$

where n_1 is the negative deviation from the goal value of the portfolio return, y_{1j} is the return of the *j-th* asset in the investment portfolio, and e_{1j1} and e_{1j2} are the negative and positive deviations from the upper limit of the expected interval, respectively, for each asset. Therefore, constraint 21-1 states that the return of each asset is equal to the upper limit of its expected interval, i.e., r^u, considering the negative and positive deviations, and constraint 21-2 states that the return should fall within the expected interval.

The investor's satisfaction with the return goal can be measured by a satisfaction function denoted by $f_R(\delta_R)$ when it falls within the expected interval of the objective. The satisfaction function is measured in the interval $[R_p^l, R_p^u]$. $f_R(\delta_R)$ can be written as follows:

$$f_R(\delta_R) = f_R^-(n_1) \tag{22}$$

According to the definition of the return goal, achieving more than the goal level does not reduce the satisfaction function, so only its negative deviation, i.e., n_1, is included in the satisfaction function.

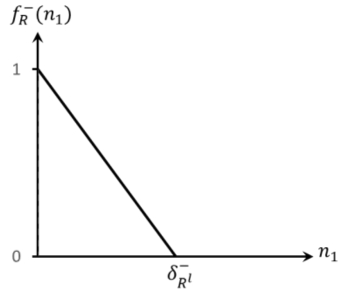

Figure 2
Satisfaction function of investment portfolio return

The investor is completely satisfied when $\tilde{R}_p \geq R_p^u$. Therefore, when $\tilde{R}_p$ moves toward its lower limit, that is, R_p^l, the investor's satisfaction with the return objective decreases, and if the investment portfolio has $\tilde{R}_p$ less than R_p^L, then the investor is completely dissatisfied, and the portfolio is rejected. In Figure 2, the changes in the satisfaction function of portfolio returns are denoted. According to Figure 2, the investor is completely satisfied with the portfolio when the negative deviation from its goal level, R_p^u, is equal to zero. In other words, when n_1 is equal to zero, the satisfaction degree of the portfolio return is maximized and is equal to one. In addition, since a portfolio with a return greater than $\delta_{R^l}^-$ does not fall within the interval $n_1 \in [0, \delta_{R^l}^-]$, it will be rejected by the investor.

The mathematical programming to maximize the satisfaction function of the portfolio return, $f_R(\delta_R)$, can be written as follows:

$$\max f_R(\delta_R) = \mu_1 \tag{23}$$

$$s.t. \quad \mu_1 + \frac{n_1}{\delta_{R^l}^-} = 1 \tag{23.1}$$

$$0 \leq n_1 \leq \delta_{R^l}^- \tag{23.2}$$

3.1.2 Construction of the satisfaction function for the liquidity of the investment portfolio

The liquidity of the investment portfolio, which is indicated by $\tilde{L}_p$ and is equal to $\tilde{L}_p = \sum_{i=1}^{n} \tilde{l}_j x_j$, where $\tilde{l}_i$ indicates the liquidity of the i-th investigated asset, which is equal to the ratio of the traded volume to the total market volume of an asset in one day. This definition of liquidity can be interpreted as the amount of trading of an asset compared to its total number of trades on different days. In other words, this ratio indicates the possibility of converting an asset into cash without a significant reduction in its value, which has a direct relationship with its desirability, and the higher its value is, the more desirable it is, and the greater the liquidity of that asset. ([55]; [57]; [1]).

Since the investor is trying to reach the maximum amount of liquidity in the portfolio as the objective of portfolio liquidity, L_p^u is determined as the most desirable amount of liquidity in the investment portfolio. Therefore, we consider the goal value of the portfolio's liquidity objective equal to L_p^u and write the corresponding constraint as well as the constraints of the fuzzy interval values for the liquidity as follows:

$$\sum_{j=1}^{m} y_{ij} x_i + n_2 \geq L_p^u \quad i = 2, \forall j \tag{24}$$

$$y_{2\,j} + e_{2j1} - e_{2j2} = l_j^u \quad , \forall j \tag{24.1}$$

$$l^l \leq y_{2j} \leq l^u \quad \forall j \tag{24.2}$$

where n_2 is the negative deviation from the goal value of liquidity, y_{2j} is the liquidity of asset j in the portfolio, and e_{2j1} and e_{2j2} are the negative and positive deviations from the upper limit of the expected interval of liquidity for each asset, respectively. Therefore, the constraint of 24-1 denotes that the liquidity of each asset is equal to the upper limit of its expected interval, l_j^u, considering negative and positive deviations, and the constraint of 24-2 expresses the limits of the liquidity variable, which is equal to its expected interval.

The investor's satisfaction regarding the change in the goal value of portfolio liquidity is represented by a satisfaction function, $f_L(\delta_L)$. In fact, the investor's satisfaction with the liquidity is measured by this satisfaction function when it changes in the interval $\left[L_p^l, L_p^u\right]$. Considering that only a negative deviation from liquidity causes a decrease in the satisfaction level, only a negative deviation from this goal, i.e., n_2, is included in the satisfaction function. Therefore, $f_L(\delta_L)$ is written as follows:

$$f_L(\delta_L) = f_L^-(n_2) \tag{25}$$

The function $f_L(\delta_L)$ is depicted in Figure 3. According to this figure, when the negative deviation from the goal level of L_p^u is equal to zero, the investor is completely satisfied, and as much as the liquidity of the portfolio moves toward the lower limit of the expected interval, then L_p^l the level of satisfaction decreases. In addition, since a portfolio with liquidity n_2 values greater than $\delta_{L^l}^-$ does not fall within the interval of $n_2 \in [0, \delta_{L^l}^-]$, it is rejected by the investor.

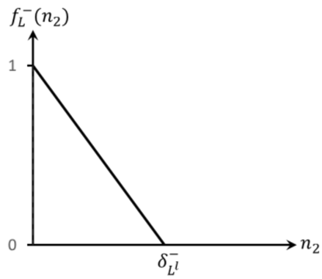

Figure 3
Liquidity satisfaction function of the investment portfolio

The mathematical programming to maximize the satisfaction level of the liquidity for the portfolio, $f_L(\delta_L)$, can be written as follows:

$$\max f_L(\delta_L) = \mu_2 \tag{26}$$

s.t.

$$\mu_2 + \frac{n_2}{\delta_{L^l}^-} = 1 \tag{26.1}$$

$$0 \leq \delta_L^- \leq \delta_{L^l}^- \tag{26.2}$$

3.1.3 Construction of a satisfaction function for investment portfolio risk

According to the definition of financial investment portfolio risk [44], the investment portfolio risk is obtained as follows based on the linear regression equation:

$$\beta_j = \frac{\sum_{z=1}^{\acute{z}} (Re_j^t - \overline{Re}_j^t)(T_z^t - \bar{T}^t)}{\sum_{z=1}^{\acute{z}} (T_z^t - \bar{T}^t)^2} \qquad \forall j \tag{27}$$

where Re_{zi}^t is the z-th day return of asset i in period t, T_z^t is the daily return of the total market index on day z in period t, $\bar{T}^t$ is the average daily return of the market index in period t, and $\overline{Re}_i^t$ is the average daily return of asset i in period t. In other words, the correlation between the return of an asset and the return of the entire market index is measured as the portfolio risk. As a result, the lower correlation of an asset with the market index indicates the independent performance of that asset from the market. To apply the total market index in decision making and with regard to an active investment strategy, it is assumed that the investor accepts the efficient market assumption and follows a passive investment strategy [17]. Therefore, the value of one is suggested as the goal value of the portfolio risk (i.e., β_g) to choose an investment portfolio based on the total market index. Considering this goal value for risk helps to form a portfolio with balanced risk by selecting assets with risk values less than or greater than one. Therefore, the objective function of risk is written as follows:

$$\sum_{j=1}^{m} y_{ij} x_i + n_i - p_i = \beta_g \quad i = 3, \forall j \tag{28}$$

$$y_{3\,j} + e_{3j1} - e_{3j2} = b_{j^*}^* \ , \ \forall j \tag{28.1}$$

$$b^l \le y_{3j} \le b^u \quad \forall j \tag{28.2}$$

where n_3 and p_3 are positive and negative deviations from the goal level of risk, respectively; y_{3j} is the amount of risk of the j_{th} asset in the portfolio; and e_{3j1} and e_{3j2} are negative and positive deviations from the lower limit of the expected interval for the risk, respectively, such that the goal value of the expected interval is $b_{j^*}^*$. Therefore, constraint 28-1 is equal to the risk of each asset in comparison with its goal value, $b_{j^*}^*$, considering its negative and positive deviations, and constraint 28-2 denotes the risk variable, which should fall within the expected interval.

Since the goal value of portfolio risk (i.e., β_g) was considered equal to one, the expected interval of portfolio risk is equal to $[1 - \beta_p^*, 1 + \beta_p^*]$. The investor's satisfaction concerning the change in the risk level of the investment portfolio can be expressed by designing a risk satisfaction function, $f_\beta(\delta_\beta)$. In fact, $f_\beta(\delta_\beta)$ is the level of satisfaction with the portfolio risk in the interval $[1 - \beta_p^*, 1 + \beta_p^*]$. Therefore, the risk satisfaction function is written as follows:

$$f_\beta(\delta_\beta) = f_\beta^-(n_3) + f_\beta^+(p_3) \tag{29}$$

Since the objective of the risk is to reach the value of one, both negative and positive deviations should be minimized concerning this goal value. As a result, n_3 and p_3 are used in the objective function $f_\beta(\delta_\beta)$. The risk satisfaction function is shown in Figure 4. Based on this figure, when the two-way deviations from the goal value are equal to zero, the investor is completely satisfied with this condition, and the level of satisfaction decreases by increasing the deviation from the goal value. In addition, it is obvious that a portfolio with a risk value greater than $1 + \beta_p^*$ and less than $1 - \beta_p^*$ is rejected by the investor.

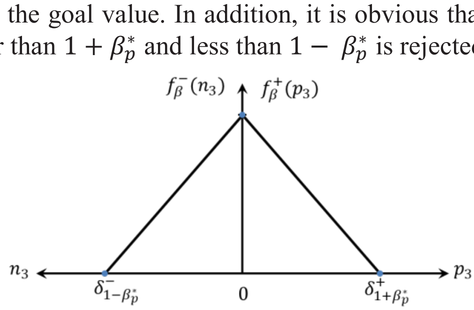

Figure 4
Risk satisfaction function of the investment portfolio

The portfolio risk cannot be lower or higher than the goal simultaneously. Therefore, by defining the binary variables $\rho_{\beta 1}$ and $\rho_{\beta 2}$ (to choose the deviation from one side), mathematical programming to maximize the satisfaction degree for the risk objective can be written as follows:

$$\max f_\beta(\delta_\beta) = \mu_3 \tag{30}$$

s.t.

$$\mu_3 + \rho_{\beta 1}\left(\frac{n_3}{\delta^-_{1-\beta^*_p}}\right) + \rho_{\beta 2}\left(\frac{p_3}{\delta^+_{1+\beta^*_p}}\right) = 1 \tag{30.1}$$

$$0 \leq n_3 \leq \rho_{\beta 1}\delta^-_{1-\beta^*_p} \tag{30.2}$$

$$0 \leq p_3 \leq \rho_{\beta 2}\delta^+_{1+\beta^*_p} \tag{30.3}$$

$$\rho_{\beta 1} + \rho_{\beta 2} = 1 \tag{30.4}$$

$$\rho_{\beta 1}, \rho_{\beta 2} \in \{0,1\} \tag{30.5}$$

3.2 *Investment portfolio diversification constraints*

In addition to the constraints related to the objectives of return, liquidity, and risk, the following constraints are considered:

The diversity of the portfolio and the selection of several assets from different sectors are considered primary criteria for determining an efficient portfolio. For example, ([17]; [56]; [13]; [14]) stated that the diversity of the number of assets is favorable for selecting a portfolio as long as their variation control will be possible, so considering the experience of the market experts, an upper and lower limit should be considered. On the other hand, the total market value of an asset (calculated by multiplying the number of assets by its price) is an essential criterion in selecting a portfolio, and this criterion helps investors estimate the future of that asset and its price. Thus, the higher this measure is, the greater the investors' interest and trust in that asset. Therefore, the greater the percentage of the total market index included, the more it should be considered for selecting the investment portfolio.

To consider the diversification of the investment portfolio, the following constraints are considered:

I. First, it is suggested to add a constraint for the minimum and maximum value that is desirable for investment in each asset, and according to the experts' comments, this constraint is written as follows:

$$f^l \mho_j \leq x_j \leq f^u \mho_j \qquad \forall j = 1, \dots, m \tag{31}$$

where f^l and f^u represent the minimum and maximum allowed value of selecting an asset, respectively, and $\mho_j$ is defined as a binary variable that is equal to one if the *j-th* asset is included in the portfolio for $j = 1, \dots, n$ and 0 otherwise.

II. To comply with the criterion of choosing assets with a higher total market index, using the weighting formula of [58] and expert comments, constraint (32) is suggested for categorizing the total market.

$$\sum_{v=c_k}^{\acute{c}_k} x_v \leq w_v. \ \forall v \in 1, \dots, f \tag{32}$$

$$w_v = \frac{(m+1)-k_v}{\sum_{v=1}^{f} k_v}. \ \forall v \in 1, \dots, f \tag{33}$$

where $\sum_{v=c_k}^{\acute{c}_k} x_v$ is the sum of the assets of the *v-th* category and w_v is the selection portion from the *v-th* category, which is calculated using formula (33). The parameter f is the number of categories, and k_v is the number of assets in the *v-th* category.

III. In addition, the minimum and maximum number of assets in a portfolio are considered constraints because a low number of assets increases the risk. If it is high, then perfect control of the portfolio is lacking. Therefore, the following constraint is suggested:

$$\acute{M} \leq \sum_{j=1}^{m} \mho_j \leq M \tag{34}$$

where $\acute{M}$ and M are the lower limit and upper limit, respectively, which are obtained using expert comments.

IV. The constraint of the budget involved in assets: the sum of the invested proportions in the assets is precisely equal to one (i.e., $\sum_{j=1}^{m} x_j = 1$).

V. The constraint of no short selling, i.e., $x_j \geq 0$ for every $j = 1, \dots, m$, will also be considered.

In the next section, the weights of the objective functions are determined using the fuzzy criterion-based method.

4. Determining the weight of goals using the fuzzy base criterion method (F-BCM)

In the previous sections, the satisfaction functions and goal values for each of the studied objectives in the investment portfolio selection problem were discussed. However, the weights of each of the objectives are not equal. In this research, it is suggested to use the fuzzy-based criterion method expressed by [47] to determine the weight of the objectives. The F-BCM method performs fuzzy pairwise comparisons using linguistic variables by decision makers or experts to determine the weights of each objective. Among the advantages of this method compared to methods such as AHP, we can mention the reduction of pairwise comparisons between the decision criteria of the problem and, as a result, the reduction of calculations. Table 1 shows the linguistic sets of nonfuzzy and fuzzy triangular numbers [47]. The matrix of fuzzy pairwise comparisons between the objectives of the problem will be as follows.

$$A = \begin{bmatrix} (1,1,1) & \tilde{a}_{R,\beta} & \tilde{a}_{R,L} \\ \tilde{a}_{\beta,R} & (1,1,1) & \tilde{a}_{\beta,L} \\ \tilde{a}_{L,R} & \tilde{a}_{L,\beta} & (1,1,1) \end{bmatrix}$$

Table 1
Linguistic variables and related fuzzy numbers

Linguistic sets	TFNs	Nonfuzzy numbers (Saaty, 1977)
Equally importance (EqI)	(1, 1, 1)	1
Moderate importance (MI)	(2, 3, 4)	3
Strong importance (SI)	(4, 5, 6)	5
Very strong importance (VSI)	(6, 7, 8)	7
Extreme importance (ExI)	(8, 9, 9)	9
Intermediate values (IV)	(1, 2, 3)	2
	(3, 4, 5)	4
	(5, 6, 7)	6
	(7, 8, 9)	8
Reciprocal number (RN)	(u-1, m-1,l-1)	x-1

where for example, $\tilde{a}_{R,\beta}$ represents the relative preference of objective R to β, and it is obvious that $\tilde{a}_{\beta,R}$ represents the preference of objective β to R, which is represented by a triangular fuzzy number. All the elements of the diagonal part in the matrix are equal to $(1,1,1)$. The steps of the F-BCM for weighting the objectives are as follows:

Step 1) The expert chooses one objective as the basic objective.

Step 2) Based on Table 1, pairwise comparisons of the basic objective are performed with the other objectives, and the vector resulting from these comparisons is as follows:

$$\tilde{A}_B = (\tilde{a}_{B,R}, \tilde{a}_{B,L}, \tilde{a}_{B,\beta}) \tag{35}$$

where $\tilde{A}_B$ represents the comparison vector of the basic objective with respect to other objectives such that $\tilde{a}_{B,R},\ \tilde{a}_{B,L}, \tilde{a}_{B,\beta}$ represents the fuzzy preference of the basic objective compared to the return, liquidity, and risk objectives. Clearly, $\tilde{a}_{B,B} = (1,1,1)$.

In this step, the optimal fuzzy weights $\widetilde{\omega}_R$, $\widetilde{\omega}_L$, and $\widetilde{\omega}_\beta$, which are the weights of the objectives, return, liquidity, and portfolio risk, are determined. The optimal fuzzy weight satisfies the equation $\frac{\widetilde{\omega}_B}{\widetilde{\omega}_i} = \tilde{a}_{B,i}$. The maximum absolute differentiation $\left|\frac{\widetilde{\omega}_B}{\widetilde{\omega}_i} - \tilde{a}_{B,i}\right|$ must be minimized for all values of i. Considering that the weights of the objectives are nonnegative (i.e., $\widetilde{\omega}_i \geq 0$), the optimal fuzzy weights can be obtained as follows:

$$\begin{aligned} &\text{Min max} \left|\frac{\widetilde{\omega}_B}{\widetilde{\omega}_i} - \tilde{a}_{B,i}\right| \quad \forall i \\ &s.t. \quad \textstyle\sum_{i=1}^{3} R(\widetilde{\omega}_i) = 1 \\ &\qquad l_i^\omega \leq m_i^\omega \leq u_i^\omega \\ &\qquad l_i^\omega \geq 0 \quad . \forall i 1,2,3 \end{aligned} \tag{36}$$

where $R(\widetilde{\omega}_i)$ or the ranking of fuzzy triangular numbers is obtained using the graded mean integration formula (GIMR) introduced by [59] (of [47]). The formula is as follows:

$$R(\widetilde{\omega}_i) = \frac{l_i+4m_i+u_i}{6} \tag{37}$$

Model (36) can be rewritten as a nonlinear mathematical problem as follows:

$$\begin{aligned} &\min \tilde{\xi} \\ s.t. &\left|\frac{\widetilde{\omega}_B}{\widetilde{\omega}_i} - \tilde{a}_{B,i}\right| \leq \tilde{\xi} \qquad \forall t \\ &\textstyle\sum_{i=1}^{3} R(\widetilde{\omega}_i) = 1 \\ &l_i^{\omega} \leq m_i^{\omega} \leq u_i^{\omega} \\ &l_i^{\omega} \geq 0 \quad . \forall i = 1,2,3 \end{aligned} \tag{38}$$

where $\widetilde{\omega}_B = (l_B^{\omega}, m_B^{\omega}, u_B^{\omega})$, $\widetilde{\omega}_i = (l_i^{\omega}, m_i^{\omega}, u_i^{\omega})$, and $\tilde{\xi} = (l^{\xi}, m^{\xi}, u^{\xi})$ are triangular fuzzy numbers and $\tilde{\xi} = (f^*, f^*, f^*)$, and the mathematical model (38) can be written as follows:

$$\begin{aligned} &\min \tilde{\xi} \\ s.t. &\left|\frac{(l_B^{\omega}, m_B^{\omega}, u_B^{\omega})}{(l_i^{\omega}, m_i^{\omega}, u_i^{\omega})} - (l_{Bi}, m_{Bi}, u_{Bi})\right| \leq (f^*, f^*, f^*) \\ &\textstyle\sum_{i=1}^{3} R(\widetilde{\omega}_i) = 1 \\ &l_i^{\omega} \leq m_i^{\omega} \leq u_i^{\omega} \\ &l_i^{\omega} \geq 0 \quad . \forall i = 1,2,3 \end{aligned} \tag{39}$$

To increase the accuracy of the results, it is suggested that the model be resolved according to the opinions of several other experts. For example, the model can be solved by considering two other experts. Finally, after defuzzifying the weights obtained from the objectives (using equation (37)), since the experience of each expert in the financial markets will also influence his/her decision-making, by using weighted averaging, it is suggested that the final weight of each objective be obtained as follows:

$$\acute{\omega}_i = \textstyle\sum_{i=1}^{3} \omega_i * \frac{h_e}{h_{total}} \qquad \forall e = 1,2,3 \tag{40}$$

where h_e is the experience of expert e, h_{total} is the sum of the experiences of all three experts, and $\acute{\omega}_i$ is the final weight used in the proposed investment portfolio selection model.

In the next section, the proposed model for selecting investment portfolios is presented.

5. Goal programming model for the problem of choosing an investment portfolio in emerging financial markets

Based on the method of [38], the multi-objective optimization model with fuzzy interval variables to determine the values invested in the assets of an emerging financial market is proposed as follows:

$$\max \ IOS = \textstyle\sum_{i=1}^{n} \acute{\omega}_i \mu_i$$

$$-\sum_{j=1}^{m}\sum_{k=1}^{2}\acute{\omega}_1 Q_j\left(\frac{e_{1jk}}{\sum_{j=1}^{m} r_j^u}\right)-\sum_{j=1}^{m}\sum_{k=1}^{2}\acute{\omega}_2 Q_j\left(\frac{e_{2jk}}{\sum_{j=1}^{m} l_j^u}\right)-\sum_{j=1}^{m}\sum_{k=1}^{2}\acute{\omega}_3 Q_j\left(\frac{e_{3jk}}{\sum_{j=1}^{m} b_{j^*}^*}\right) \tag{41}$$

$$s.t\ \ \sum_{j=1}^{m} y_{ij}x_i+n_i \geq R_p^u \ \ ,i=1 \tag{41.1}$$

$$\sum_{j=1}^{m} y_{ij}x_i+n_i \geq L_p^u \ \ ,i=2 \tag{41.2}$$

$$\sum_{j=1}^{m} y_{ij}x_i+n_i-p_i=\beta_g \ \ ,i=3 \tag{41.3}$$

$$\mu_1+\frac{n_1}{\delta_{R^l}^-}=1 \tag{41.4}$$

$$\mu_2+\frac{n_2}{\delta_{L^l}^-}=1 \tag{41.5}$$

$$\mu_3+\rho_{\beta 1}\left(\frac{n_3}{\delta_{1-\beta_p^*}^-}\right)+\rho_{\beta 2}\left(\frac{p_3}{\delta_{1+\beta_p^*}^+}\right)=1 \tag{41.6}$$

$$y_{1j}+e_{1j1}-e_{1j2}=r_j^u \quad \forall j \tag{41.7}$$

$$y_{2j}+e_{2j1}-e_{2j2}=l_j^u \quad \forall j \tag{41.8}$$

$$y_{3j}+e_{3j1}-e_{3j2}=b_{j^*}^* \quad \forall j \tag{41.9}$$

$$r^l \leq y_{1j} \leq r^u \quad \forall j \tag{41.10}$$

$$l^l \leq y_{2j} \leq l^u \quad \forall j \tag{41.11}$$

$$b^l \leq y_{3j} \leq b^u \quad \forall j \tag{41.12}$$

$$0 \leq n_1 \leq \delta_{R^l}^- \tag{41.13}$$

$$0 \leq n_2 \leq \delta_{L^l}^- \tag{41.14}$$

$$0 \leq n_3 \leq \rho_{\beta 1}\delta_{1-\beta_p^*}^- \tag{41.15}$$

$$0 \leq p_3 \leq \rho_{\beta 2}\delta_{1+\beta_p^*}^+ \tag{41.16}$$

$$\rho_{\beta 1}+\rho_{\beta 2}=1 \tag{41.17}$$

$$\sum_{j=1}^{n} x_j=1 \tag{41.18}$$

$$\acute{M} \leq \sum_{j=1}^{m} \mho_j \leq M \tag{41.19}$$

$$f^l\mho_j \leq x_j \leq f^u\mho_j\ ,\forall j=1,\dots,m \tag{41.20}$$

$$\sum_{v=c_k}^{\acute{c}_k} x_v \leq w_v.\ \forall v \in 1,\dots,f \tag{41.21}$$

$$\rho_{\beta 1}\,,\rho_{\beta 2}\,,\mho_j \ \in \{0,1\} \quad \forall j=1,\dots,m \tag{41.22}$$

$$x_j,\delta_\beta^-,\delta_\beta^+ \geq 0 \tag{41.23}$$

Equations (41-1) - (41-3) are the constraints of return, liquidity, and risk objectives, respectively, and equations (41-4) - (41-6) are the constraints of the satisfaction functions (see section 3-1). Equations (41-7) - (41-12) are the constraints of goal values for each asset, and equations (41-13) - (41-17) are the constraints for the acceptable deviations of the satisfaction functions (see section 3-1). Equations (41-18) - (41-21) are the constraints of the budget allocated to each sector and each available asset, the lack of short selling, and the minimum and maximum amount of the selected asset in the investment portfolio (see section 3-2).

In objective function (41), two parts are included: the satisfaction level for the objectives with positive weighted values and the deviations of each objective for each asset with negative weighted values (k=1 negative deviation and k=2 positive deviation), which ultimately maximizes the total satisfaction level of the objective function. The linear normalization method was used to adjust the values for each objective. This approach leads to a balanced objective function for each type of deviation. For example, a minor deviation in the return of an asset in the case of non-normalization causes the objective function to become zero or even negative. However, normalization reduces the satisfaction level of the portfolio by only a small percentage.

6. Case study: Application in the crypto-currency market

The optimization model will be applied to select an investment portfolio in the crypto-currency market. Since the first 110 currencies of the crypto-currency market account for almost 98% of the market volume[1]; thus, among the 110 mentioned currencies, those with historical data from January 1, 2018, to September 30, 2022, except for stable currencies, 27 currencies (see Table 2) were used in this study. To construct fuzzy numbers for the parameters of each crypto-currency, the price and the volume of transactions with daily changes over 365 days were considered. The trapezoid fuzzy numbers r_1, r_2, r_3, r_4 and l_1, l_2, l_3, l_4 should be determined to obtain the return and liquidity. First, the average total return and the daily liquidity of 1734 days for each currency were calculated and considered r_1 fuzzy return and l_1 fuzzy liquidity of the *i-th* currency, and confidence levels of 90, 95, and 99%, which are r_2, r_3, r_4 and l_2, l_3, l_4 , respectively, were calculated, and trapezoidal fuzzy liquidity was obtained for each currency. Finally, to determine the risk trapezoidal fuzzy number, equation (27) is used to calculate b_1, and for b_2, b_3, b_4, the confidence levels of 90, 95, and 99% are calculated. Table 2 shows the values of the return, liquidity, and trapezoidal fuzzy risk of 27 selected currencies, and the expected intervals, i.e., $[r_i^l, r_i^u]$, $[b_i^l, b_i^u]$ and $[l_i^l, l_i^u]$, are determined for the considered crypto-currencies (see Table 3).

Table 2
Financial data of goals in currencies

Currencies	x_i Variable	Fuzzy ($\tilde{r}_i$) Return	Fuzzy ($\tilde{l}_i$) Liquidity	Fuzzy ($\tilde{b}_i$) risk
		(r_1, r_2, r_3, r_4)	(l_1, l_2, l_3, l_4)	(b_1, b_2, b_3, b_4)
BTC	x_1	(0.096, 4.297, 6.64, 11.129)	(0.096, 0.187, 0.245, 0.365)	(0.908, 0.923, 0.927, 0.935)
ETH	x_2	(0.163, 5.829, 8.422, 13.636)	(0.229, 0.527, 0.756, 1.003)	(1.115, 1.138, 1.145, 1.158)
BNB	x_3	(0.372, 5.96, 8.764, 18.983)	(0.068, 0.129, 0.161, 0.213)	(0.998, 1.057, 1.073, 1.104)
XRP	x_4	(0.093, 5.842,	(0.131, 0.268,	(1.044, 1.099,

[1] https://coinmarketcap.com/

		8.854, 21.094)	0.38, 0.746)	1.114, 1.143)
ADA	x_5	(0.15, 6.812, 10.042, 18.167)	(0.082, 0.195, 0.271, 0.438)	(1.145, 1.188, 1.2, 1.222)
DOGE	x_6	(0.47, 5.878, 10.412, 26.192)	(0.181, 0.488, 0.668, 1.017)	(1.039, 1.089, 1.103, 1.13)
TRX	x_7	(0.465, 6.774, 10.615, 22.906)	(0.426, 1.168, 1.396, 1.691)	(-0.047, 0.035, 0.058, 0.101)
LTC	x_8	(0.058, 5.841, 8.567, 14.704)	(0.46, 1.101, 1.315, 1.671)	(1.114, 1.147, 1.156, 1.173)
ETC	x_9	(0.195, 6.186, 9.544, 22.036)	(0.705, 1.674, 2.296, 3.175)	(1.109, 1.159, 1.173, 1.2)
LINK	x_{10}	(0.396, 8.527, 11.241, 21.096)	(0.159, 0.333, 0.407, 0.623)	(1.129, 1.182, 1.197, 1.225)
XLM	x_{11}	(0.114, 6.013, 9.098, 17.945)	(0.13, 0.31, 0.42, 0.587)	(1.066, 1.119, 1.134, 1.162)
XMR	x_{12}	(0.095, 5.777, 8.103, 13.985)	(0.127, 0.223, 0.419, 0.615)	(1.035, 1.081, 1.094, 1.118)
BCH	x_{13}	(0.015, 5.89, 8.873, 17.86)	(0.353, 0.745, 0.864, 1.153)	(1.237, 1.272, 1.282, 1.3)
MANA	x_{14}	(0.429, 8.036, 11.854, 26.473)	(0.208, 0.449, 0.555, 0.925)	(1.102, 1.158, 1.173, 1.202)
EOS	x_{15}	(0.09, 6.395, 9.603, 18.571)	(0.514, 1.113, 1.399, 2.055)	(1.22, 1.261, 1.272, 1.294)
KCS	x_{16}	(0.286, 6.494, 10.536, 23.438)	(0.051, 0.121, 0.15, 0.243)	(1.041, 1.1, 1.116, 1.148)
ZEC	x_{17}	(0.053, 6.711, 9.171, 17.118)	(0.467, 0.928, 1.203, 1.904)	(1.089, 1.138, 1.151, 1.177)
MIOTA	x_{18}	(0.049, 6.651, 9.604, 17.168)	(0.024, 0.042, 0.057, 0.11)	(1.151, 1.196, 1.208, 1.232)
NEO	x_{19}	(0.062, 6.488, 9.985, 16.453)	(0.318, 0.701, 0.892, 1.196)	(1.19, 1.228, 1.239, 1.259)
DASH	x_{20}	(-0.014, 5.915, 8.603, 15.905)	(0.394, 0.794, 0.98, 1.497)	(1.093, 1.14, 1.154, 1.179)
BAT	x_{21}	(0.194, 7.172, 10.622, 20.602)	(0.203, 0.443, 0.61, 1.047)	(1.087, 1.141, 1.156, 1.185)
ENJ	x_{22}	(0.381, 7.832, 11.812, 23.991)	(0.145, 0.264, 0.403, 0.959)	(1.156, 1.21, 1.225, 1.253)
LRC	x_{23}	(0.3, 8.2, 12.803, 29.016)	(0.146, 0.298, 0.378, 0.845)	(1.197, 1.248, 1.262, 1.29)
DCR	x_{24}	(0.103, 6.187, 9.345, 16.688)	(0.072, 0.173, 0.575, 0.696)	(1.008, 1.067, 1.083, 1.114)
XEM	x_{25}	(0.008, 6, 9.336, 19.582)	(0.052, 0.092, 0.116, 0.198)	(1.036, 1.093, 1.109, 1.139)
BTG	x_{26}	(0.074, 5.691, 9.06, 21.4)	(0.084, 0.219, 0.271, 0.46)	(1.133, 1.185, 1.2, 1.228)
GNO	x_{27}	(0.154, 6.875, 9.882, 17.673)	(0.011, 0.024, 0.032, 0.05)	(0.968, 1.03, 1.048, 1.081)

Table 3
Expected intervals of goals

Currencies	**Expected Interval return**	**Expected Interval liquidity**	**Expected Interval risk**
BTC	$2.197 \leq y_{11} \leq 8.885$	$0.141 \leq y_{31} \leq 0.305$	$0.916 \leq y_{21} \leq 0.931$
ETH	$2.996 \leq y_{12} \leq 11.029$	$0.378 \leq y_{32} \leq 0.879$	$1.126 \leq y_{22} \leq 1.151$

BNB	$3.166 \leq y_{13} \leq 13.874$	$0.098 \leq y_{33} \leq 0.187$	$1.027 \leq y_{23} \leq 1.088$
XRP	$2.967 \leq y_{14} \leq 14.974$	$0.199 \leq y_{34} \leq 0.563$	$1.071 \leq y_{24} \leq 1.128$
ADA	$3.481 \leq y_{15} \leq 14.104$	$0.138 \leq y_{35} \leq 0.354$	$1.166 \leq y_{25} \leq 1.211$
DOGE	$3.174 \leq y_{16} \leq 18.302$	$0.334 \leq y_{36} \leq 0.843$	$1.064 \leq y_{26} \leq 1.116$
TRX	$3.619 \leq y_{17} \leq 16.761$	$0.797 \leq y_{37} \leq 1.543$	$-0.006 \leq y_{27} \leq 0.080$
LTC	$2.950 \leq y_{18} \leq 11.635$	$0.781 \leq y_{38} \leq 1.493$	$1.131 \leq y_{28} \leq 1.164$
ETC	$3.190 \leq y_{19} \leq 15.790$	$1.189 \leq y_{39} \leq 2.736$	$1.134 \leq y_{29} \leq 1.186$
LINK	$4.461 \leq y_{110} \leq 16.169$	$0.246 \leq y_{310} \leq 0.515$	$1.156 \leq y_{210} \leq 1.211$
XLM	$3.063 \leq y_{111} \leq 13.522$	$0.220 \leq y_{311} \leq 0.503$	$1.093 \leq y_{211} \leq 1.148$
XMR	$2.936 \leq y_{112} \leq 11.044$	$0.175 \leq y_{312} \leq 0.517$	$1.058 \leq y_{212} \leq 1.106$
BCH	$2.953 \leq y_{113} \leq 13.366$	$0.549 \leq y_{313} \leq 1.009$	$1.255 \leq y_{213} \leq 1.291$
MANA	$4.232 \leq y_{114} \leq 19.163$	$0.328 \leq y_{314} \leq 0.740$	$1.130 \leq y_{214} \leq 1.188$
EOS	$3.243 \leq y_{115} \leq 14.087$	$0.814 \leq y_{315} \leq 1.727$	$1.241 \leq y_{215} \leq 1.283$
KCS	$3.390 \leq y_{116} \leq 16.987$	$0.086 \leq y_{316} \leq 0.197$	$1.070 \leq y_{216} \leq 1.132$
ZEC	$3.382 \leq y_{117} \leq 13.144$	$0.697 \leq y_{317} \leq 1.553$	$1.113 \leq y_{217} \leq 1.164$
MIOTA	$3.350 \leq y_{118} \leq 13.386$	$0.033 \leq y_{318} \leq 0.083$	$1.173 \leq y_{218} \leq 1.220$
NEO	$3.275 \leq y_{119} \leq 13.219$	$0.509 \leq y_{319} \leq 1.044$	$1.209 \leq y_{219} \leq 1.249$
DASH	$2.951 \leq y_{120} \leq 12.254$	$0.594 \leq y_{320} \leq 1.238$	$1.117 \leq y_{220} \leq 1.166$
BAT	$3.683 \leq y_{121} \leq 15.612$	$0.323 \leq y_{321} \leq 0.828$	$1.114 \leq y_{221} \leq 1.171$
ENJ	$4.106 \leq y_{122} \leq 17.902$	$0.205 \leq y_{322} \leq 0.681$	$1.183 \leq y_{222} \leq 1.239$
LRC	$4.250 \leq y_{123} \leq 20.910$	$0.222 \leq y_{323} \leq 0.611$	$1.222 \leq y_{223} \leq 1.276$
DCR	$3.145 \leq y_{124} \leq 13.017$	$0.123 \leq y_{324} \leq 0.635$	$1.037 \leq y_{224} \leq 1.099$
XEM	$3.004 \leq y_{125} \leq 14.459$	$0.072 \leq y_{325} \leq 0.157$	$1.064 \leq y_{225} \leq 1.124$
BTG	$2.883 \leq y_{126} \leq 15.230$	$0.152 \leq y_{326} \leq 0.365$	$1.159 \leq y_{226} \leq 1.214$
GNO	$3.514 \leq y_{127} \leq 13.778$	$0.017 \leq y_{327} \leq 0.041$	$0.999 \leq y_{227} \leq 1.064$

The minimum and maximum amounts of each currency are equal to $f^l = 0.01$ and $f^u = 0.20$, respectively, based on expert comments, and the number of selected currencies used in portfolio selection should be between 4 and 8, i.e., $4 \leq \sum_{i=1}^{27} \mho_i \leq 8$. Three categories are considered (m=3) based on the market value in the mentioned sample; thus, according to equation (33), the maximum amounts of investment in these three categories are 0.5, 0.33 and 0.17($w_1 = 0.5$,$w_2 = 0.33$ and $w_3 = 0.17$), respectively.

In this case study, the following procedure is employed. First, the ideal and anti-ideal solutions for each of the objectives, i.e., return, risk, and liquidity, are determined. Second, the fuzzy possibility distributions of each objective are obtained. Third, the expected intervals and satisfaction functions of the objectives are obtained. Then, the weighting of the objectives is carried out using the F-BCM method. Finally, the obtained values are substituted in the proposed model, and the results are presented.

Using equations (7) - (1-12), ideal and anti-ideal values are determined for each of the objectives in the investment portfolio (see Table 4).

Table 4
Ideal and anti-ideal values of the problem objectives

α	$(R_p^{max})^l$	$(R_p^{max})^u$	$(R_p^{min})^l$	$(R_p^{min})^u$	$(\beta_p^{max})^l$	$(\beta_p^{max})^u$	$(\beta_p^{min})^l$	$(\beta_p^{min})^u$	$(L_p^{max})^l$	$(L_p^{max})^u$	$(L_p^{min})^l$	$(L_p^{min})^u$
0	0.426	25.297	0.047	14.149	0.506	2.053	0.053	0.222	1.173	1.249	0.779	0.874
0.2	1.782	22.453	1.152	12.948	0.635	1.937	0.064	0.206	1.181	1.245	0.79	0.869
0.4	3.193	19.609	2.244	11.747	0.764	1.821	0.076	0.191	1.189	1.24	0.8	0.863
0.6	4.619	16.765	3.33	10.547	0.895	1.705	0.088	0.175	1.197	1.236	0.811	0.857
0.8	6.049	13.933	4.415	9.334	1.026	1.589	0.099	0.16	1.206	1.231	0.821	0.852
1	7.48	11.218	5.498	8.106	1.157	1.473	0.111	0.145	1.215	1.227	0.832	0.846

Using models (13) - (18-4) and the values obtained in Table 4 and based on the values of $\bar{R}_p^{max} = 10.954$, $\bar{R}_p^{min} = 6.991$, $\bar{\beta}_p^{max} = 1.216$, $\bar{\beta}_p^{min} = 0.833$, $\bar{L}_p^{max} = 1.296$, and $\bar{L}_p^{min} = 0.133$ (see section 2-3), the fuzzy distributions of return, risk, and liquidity are determined (see Table 5 and Figure 5 (5-a, 5-b, 5-c)).

Table 5
Objective distribution functions

α	$\tilde{R}_p^* = \max \sum_{i=1}^{27} \tilde{r}_i x_i$	$\tilde{L}_p^* = \max \sum_{i=1}^{27} \tilde{l}_i x_i$	$\tilde{\beta}_p^* = \max \sum_{i=1}^{27} \tilde{b}_i x_i$
0	(0.321 , 22.977)	(0.400 , 1.702)	(1.089 , 1.170)
0.2	(1.618 , 20.471)	(0.504 , 1.598)	(1.097 , 1.165)
0.4	(2.956 , 17.965)	(0.608 , 1.494)	(1.106 , 1.160)
0.6	(4.303 , 15.459)	(0.712 , 1.390)	(1.115 , 1.155)
0.8	(5.654 , 12.962)	(0.817 , 1.286)	(1.124 , 1.151)
1	(7.004 , 10.524)	(0.921 , 1.184)	(1.133 , 1.146)

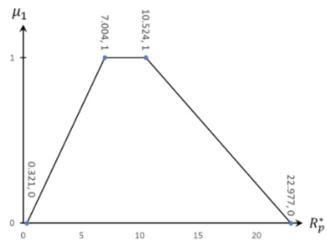

Figure 5-a: Distribution function of portfolio return

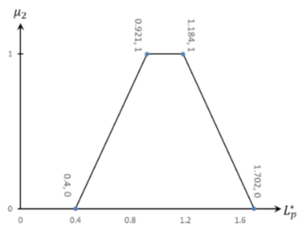

Figure 5-b: Distribution function of portfolio liquidity

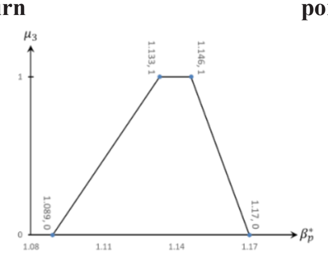

Figure 5-c: Distribution function of portfolio risk

Figure 5
Distribution functions of objectives in the portfolio

Using equations (19-1) - (19-2) and (19-3) and considering a constant parameter h= 0.2, the expected intervals of return, risk, and liquidity of the investment portfolio are calculated as follows.

$$EI(\tilde{R}_p^*) = [R_p^l, R_p^u]$$
$$= \left[\frac{(\Sigma_{\alpha=0}^{0.8}(R_p^*)_\alpha^l + \Sigma_{\alpha=0.2}^{1}(R_p^*)_\alpha^l)\times 0.2}{2}, \frac{(\Sigma_{\alpha=0}^{0.8}(R_p^*)_\alpha^u + \Sigma_{\alpha=0.2}^{1}(R_p^*)_\alpha^u)\times 0.2}{2}\right] = [3.639, 16.721]$$
$$EI(\tilde{L}_p^*) = [L_p^l, L_p^u]$$
$$= \left[\frac{(\Sigma_{\alpha=0}^{0.8}(L_p^*)_\alpha^l + \Sigma_{\alpha=0.2}^{1}(L_p^*)_\alpha^l)\times 0.2}{2}, \frac{(\Sigma_{\alpha=0}^{0.8}(L_p^*)_\alpha^u + \Sigma_{\alpha=0.2}^{1} L_p^*)_\alpha^u)\times 0.2}{2}\right] = [0.660, 1.442]$$
$$EI(\tilde{\beta}_p^*) = [\beta_p^l, \beta_p^u] = [\beta_g - \beta_p^*, \beta_g + \beta_p^*] = [0.866, 1.134]$$

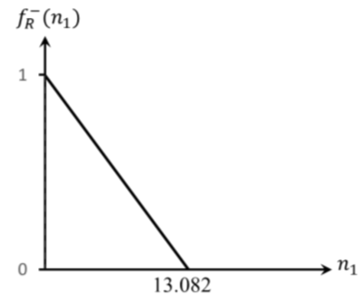

Figure 6
Fuzzy return satisfaction function of the investment portfolio

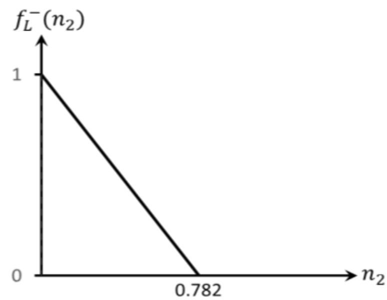

Figure 7
Fuzzy liquidity satisfaction function of the investment portfolio

According to the output of the expected interval for portfolio returns obtained based on equation (19-1), the upper limit of this interval, $R_p^u = 16.721$, is equal to the goal value of the fuzzy return. The satisfaction function of the fuzzy return is shown in Figure 6. When the deviation from the goal value of this objective is equal to zero, the function $f_R^-(n_1)$ is at its maximum value, i.e., one. The level of investor satisfaction in this objective falls within the interval of $0 \le \delta_R^- \le 13.082$, which decreases with increasing n_1. It is clear that the portfolio with a negative deviation from the goal value or n_1 values greater than 13.082 is rejected. Therefore, its analytical model is as follows:

$$\begin{aligned} &\max f_R(\delta_R) = \mu_1 \\ &s.t. \quad \mu_1 + \frac{n_1}{13.082} = 1 \\ &0 \le n_1 \le 13.082 \end{aligned} \tag{42}$$

According to equation (19-2), the goal value of the liquidity is equal to L_p^u, that is, 1.442. Figure 7 shows the satisfaction function of liquidity. According to this figure, when the negative deviation from 1.442 is equal to zero ($n_2 = 0$), the investor is completely satisfied with the liquidity of the portfolio. The investor's satisfaction decreases in the interval $n_2 \in [0, 0.782]$. It is clear that the investment portfolio with liquidity values less than 0.660 (i.e., $n_2 > 0.782$) is rejected. Therefore, the mathematical model of the satisfaction function of liquidity is defined as follows:

$$\max f_L(\delta_L) = \mu_2$$
$$\text{s.t. } \mu_2 + \frac{n_2}{0.782} = 1$$
$$0 \le n_2 \le 0.782 \tag{43}$$

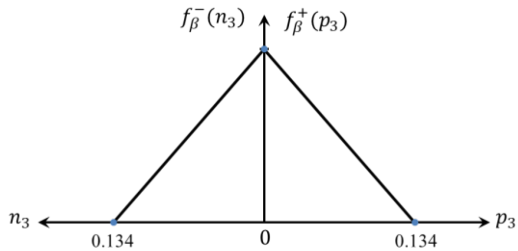

Figure 8
The fuzzy risk satisfaction function of the investment portfolio

Finally, for the fuzzy risk of the investment portfolio ($\tilde{\beta}_p$), according to equation (19-4) and the values in Table 5, the expected interval of investor satisfaction falls within the interval $\tilde{\beta}_p \in [0.866, 1.134]$. The variations in the satisfaction function of the fuzzy risk are shown in Figure 8. According to this figure, the investor is completely satisfied with this objective when the negative and positive deviations from the goal value equal zero. Investor satisfaction decreases with increasing deviations p_2 and n_2. In addition, it is clear that a risk level greater than 1.134 and less than 0.866 is rejected. Therefore, the mathematical model of the risk satisfaction function is as follows:

$$\max f_\beta(\delta_\beta) = \mu_3$$
$$s.t. \quad \mu_3 + \rho_{\beta 1}\left(\frac{n_3}{0.134}\right) + \rho_{\beta 2}\left(\frac{p_3}{0.134}\right) = 1$$
$$0 \le n_3 \le 0.134\rho_{\beta 1}$$
$$0 \le p_3 \le 0.134\rho_{\beta 2}$$
$$\rho_{\beta 1} + \rho_{\beta 2} = 1$$
$$\rho_{\beta 1}, \rho_{\beta 2} \in \{0,1\} \tag{44}$$

The expected intervals of the return, liquidity, and fuzzy risk for each crypto-currency were calculated using equations (20-1) - (20-3), and the obtained values are shown in Table 3.

Table 6
Experts' opinions on weighting objectives

Decision makers	reputation (year)	return	liquidity	risk
D.M$_1$	5	(3,4,5)	(1,2,3)	(1,1,1)
D.M$_2$	9	(5,6,7)	(2,3,4)	(1,1,1)
D.M$_3$	8	(4,5,6)	(1,2,3)	(1,1,1)

Among the three objectives of return, liquidity, and risk, the risk objective has been chosen as the basic objective based on the comments of three experts with 5, 9, and 8 years of experience in the field of financial markets. Each expert made fuzzy-based comparisons using the linguistic data in Table 1. Table 6 shows the triangular fuzzy numbers of the fuzzy comparisons. Therefore, for example, according to the comment of the first expert, the fuzzy-based comparison vector will be as follows:

$$\tilde{A}_B = ((3,4,5),(1,2,3),(1,1,1))$$

By substituting the values of this vector in model (39), the optimal fuzzy weights of the objectives can be obtained by solving the following nonlinear optimization model:

$$\begin{aligned}
&\min \tilde{\xi} \\
&s.t.\ \left|\frac{(l_\beta,m_\beta,u_\beta)}{(l_R,m_R,u_R)} - (3,4,5)\right| \le (f^*,f^*,f^*) \\
&\left|\frac{(l_\beta,m_\beta,u_\beta)}{(l_L,m_L,u_L)} - (1,2,3)\right| \le (f^*,f^*,f^*) \\
&\tfrac{1}{6}\left(l_R + 4m_R + u_R + l_L + 4m_L + u_L + l_\beta + 4m_\beta + u_\beta\right) = 1 \\
&l_R \le m_R \le u_R \\
&l_L \le m_L \le u_L \\
&l_\beta \le m_\beta \le u_\beta \\
&l_R, l_L, l_\beta \ge 0 \\
&f^* \ge 0
\end{aligned} \tag{45}$$

By solving model (45), the optimal fuzzy weights of each objective are obtained as follows:

$$\omega_R = (0.117,0.146,0.146)\quad \omega_L = (0.194,0.292,0.438)$$
$$\omega_\beta = (0.438,0.583,0.583)$$

Using formula (37), the optimal fuzzy weights obtained for each objective are converted into nonfuzzy numbers as follows:

$$\omega_R = 0.141 \qquad \omega_L = 0.300 \qquad \omega_\beta = 0.559$$

In the same way, model (39) is solved for the other experts, and optimal weights are obtained. Finally, using equation (40), the final weights of the objectives, $\acute{\omega}_t$, are obtained. Table 7 shows the results and the final weights of the objectives:

Table 7
Final weighting of objectives based on the opinions of all three experts

	Return	Liquidity	Risk	ω_R	ω_L	ω_β
DM_1	(0.117,0.146, 0.146)	(0.194,0.292, 0.438)	(0.438,0.583, 0.583)	0.141	0.300	0.559
DM_2	(0.097,0.114, 0.114)	(0.170,0.227, 0.284)	(0.568,0.681, 0.681)	0.111	0.227	0.662
DM_3	(0.099,0.119, 0.119)	(0.198,0.297, 0.475)	(0.475,0.594, 0.594)	0.116	0.310	0.574
Final weights				0.120	0.274	0.606

Considering the final weights of each of the objectives ($\acute{\omega}_\beta = 0.606, \acute{\omega}_L = 0.274, \acute{\omega}_R = 0.120$) and equations (42) - (44), the multi-objective mathematical model with fuzzy interval values for the objectives is obtained as follows:

$$\max\ IOS = 0.120\mu_1 + 0.274\mu_2 + 0.606\mu_3 - 0.120\sum_{j=1}^{27}\sum_{k=1}^{2} Q_j\left(\frac{e_{1jk}}{\sum_{j=1}^{27} r_j^u}\right) - 0.274\sum_{j=1}^{27}\sum_{k=1}^{2} Q_j\left(\frac{e_{2jk}}{\sum_{j=1}^{27} l_j^u}\right) - 0.606\sum_{j=1}^{27}\sum_{k=1}^{2} Q_j\left(\frac{e_{3jk}}{\sum_{j=1}^{27} b_j^*}\right)$$

$$s.t\ \ \sum_{j=1}^{27} y_{1j}x_i + n_i \geq 16.721$$
$$\sum_{j=1}^{27} y_{2j}x_i + n_i \geq 1.442$$
$$\sum_{j=1}^{27} y_{3j}x_i + n_i - p_i = 1$$
$$\mu_1 + \frac{n_1}{13.082} = 1$$
$$\mu_2 + \frac{n_2}{0.782} = 1$$
$$\mu_3 + \rho_{\beta 1}\left(\frac{n_3}{0.134}\right) + \rho_{\beta 2}\left(\frac{p_3}{0.134}\right) = 1$$
$$y_{1j} + e_{1j1} - e_{1j2} = r^u \quad \forall j$$
$$y_{2j} + e_{2j1} - e_{2j2} = l^u \quad \forall j$$
$$y_{3j} + e_{3j1} - e_{3j2} = b^* \quad \forall j$$
$$r^l \leq y_{1j} \leq r^u \quad \forall j$$
$$l^l \leq y_{2j} \leq l^u \quad \forall j$$
$$b^l \leq y_{3j} \leq b^u \quad \forall j$$
$$0 \leq n_1 \leq 13.082$$
$$0 \leq n_2 \leq 0.782$$
$$0 \leq n_3 \leq 0.134\rho_{\beta 1}$$
$$0 \leq p_3 \leq 0.134\rho_{\beta 2}$$
$$\rho_{\beta 1} + \rho_{\beta 2} = 1$$
$$\sum_{j=1}^{27} x_j = 1$$

$$4 \le \sum_{j=1}^{n} \mho_j \le 8$$
$$0.01\mho_j \le x_j \le 0.20\mho_j \,, \forall j$$
$$\sum_{v=1}^{9} x_v \le 0.5$$
$$\sum_{v=10}^{18} x_v \le 0.33$$
$$\sum_{v=19}^{27} x_v \le 0.17$$
$$\rho_{\beta 1} \,, \rho_{\beta 2} \,, \mho_j \in \{0,1\} \quad \forall j$$
$$x_j, \delta_\beta^-, \delta_\beta^+ \ge 0 \tag{46}$$

By solving the nonlinear programming model (46) in the Games 24.1.2 software with the COUENNE solver, the optimal solutions and the selected values for the expected intervals are shown in Figure 9 and Table 8. Figure 9 shows the optimal ratios, x_j, for the investment in each selected crypto-currency, and Table 8 shows the satisfaction level for the objectives and the total satisfaction level of the determined investment portfolio (value of the objective function). The investor's satisfaction levels for return, liquidity, and risk are 16.721, 1.442, and 1, respectively, which will lead to a 100% total satisfaction degree. Therefore, it can be concluded that the investor has taken advantage of the excellent profitability in the emerging financial market of crypto currencies, a liquid and low-risk portfolio that brings a 100% satisfaction degree.

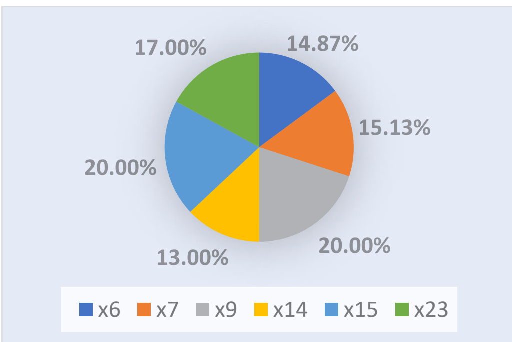

Figure 9
Investment ratios in selected currencies

Table 8
Levels of satisfaction and access to goals

Goals	Reachable levels of goals	Positive and negative deviations	degree of satisfaction
Returns	16,721	$n_1=0$	100%
liquidity	1,442	$n_2=0$	100%
Risk	1	n_3, $p_3=0$	100%
Overall satisfaction degree value (objective function)	-	-	100%

Figure 9 clearly shows that the total ratios of selected assets from each sector are equal to the ratios determined based on the experts' comments, so the total ratios of the first sector are 50%, those of the second sector are 33%, and those of the third sector are 17%. In addition to the investment ratios in each sector, the investment ratios in each asset have also been determined, as shown in Figure 9. Each selected asset falls within one and twenty percent of the investment portfolio. The number of selected assets in the portfolio is also based on the experts' comments, i.e., between 4 and 8 assets (6 selected assets), which, as previously stated. The reason is to control the variation of the portfolio and monitor the investment risk. As shown in Table 8, the values of positive and negative deviations are equal to zero for all the return, risk, and liquidity objectives. These zero deviations indicate that all return, liquidity, and risk objectives have reached their goal levels. Thus, it is concluded that the model can determine a portfolio with significant profits and low risk around its goal value, which causes the portfolio risk to be under control. Furthermore, it provides adequate liquidity. Therefore, according to the obtained results, it can be claimed that the investment portfolio of crypto currencies obtained from the proposed model has the necessary efficiency for investment.

7. Conclusion and discussion

Financial markets, especially emerging financial markets, always have uncertainty, and the decision to select a suitable investment portfolio is a difficult problem. Goal programming and its combination with fuzzy theory, multi-objective decision making, and satisfaction functions can be among the best methods for solving the problem of selecting an investment portfolio in the presence of uncertainty by considering different objectives, such as return, liquidity, and risk. Among the advantages of the method presented in this research, the following can be mentioned:

a) This paper presents a decision-making method that allows all investors to choose the best policy by taking into account expert comments and considering different objectives.
b) A decision support tool for improving the solutions of the portfolio selection problem
c) The ability to generalize the application of the proposed model in other small and large financial markets to determine an investment portfolio.
d) The use of fuzzy interval values, as well as their combination with satisfaction functions, to make a decision in the presence of uncertainty

Activity in financial markets does not need only to determine an investment portfolio. Trading in these markets can be more beneficial. In addition to the fact that trading requires more time allocation, it is necessary to plan a framework and rules for decision making. One of the classical methods is to apply a suitable and efficient strategy for portfolio selection in the trading field. Therefore, the proposed approach can be extended in this field by defining other appropriate objectives. On the other hand, in future research, the proposed model can be applied to other financial markets.

8. Management perspectives and practical conclusions

According to the findings in sections 6 and 7, the following are considered practical results and management perspectives.

I. The risk level of the investment portfolio is the most critical factor in selecting the investment portfolio.
II. The use of fuzzy interval values in the selection of the investment portfolio, in addition to controlling the conditions of uncertainty, leads to achieving the maximum satisfaction level.

Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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