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Minimal time paths with constant normal accelerations

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Abstract

For navigating vehicles, new two dimensional curves are proposed using kinematical principles. The vehicle is subject to a normal acceleration which is constant during motion. The minimal time paths with the restriction of constant normal acceleration are found using the principles of variational calculus. The minimization process leads to a nonlinear ordinary differential equation with third order. The path equation is transformed into a dimensionless form first. In this form, the equation contains a path parameter which is non-dimensional. The differential equation system is solved numerically by applying a fourth order Runge-Kutta algorithm. By adjusting the path parameter within the differential equation, the final destination point can be reached from the given initial conditions for the system. A shooting like algorithm is necessary to determine the exact value of the specific path parameter to reach a given final location. The velocity functions corresponding to the specific paths are also calculated. The curves enable smooth transitions from a straight path to a curved path for land, aerial and marine vehicles moving in two dimensional horizontal or vertical motions.

Subject Classification (2020): *34A34, 65K10, 65L05.*

Keywords: *Minimal time, Variational calculus, Acceleration, Vehicle navigation, Numerical solutions.*

1. Introduction

During navigation, it is impossible for vehicles to track straight paths always. Due to the non-planar surfaces, the highways or railways contain curved paths. Although there are lesser obstacles for vehicles moving in sea or air, they

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occasionally navigate through curves when maneuvering, avoiding vehicles and structures, departing or arriving harbors or airports. The transition must be smooth from a straight to a curved path for such vehicles. Sharp curves which introduce high normal acceleration components should be avoided since this acceleration component is responsible from turning over or side slips. Normal acceleration component mainly determines the comfort during a travel and safety requires this component to be lower than a fixed value. Therefore, curves which maintain constant normal acceleration values during navigation are of technological importance in transportation.

To ensure smooth transitions from straight to curved paths, many different functions are employed in the design of highways and railways. The so called “clothoid” is one of the most commonly used such curves [1-7]. Unmanned aerial vehicles follow clothoidal curves [2]. For such curves, the curvature and arc length are linearly related with each other. Various approximations of the parametric integral form of the clothoid curves were given in the literature, such as s-power series [3] or arc spline approximations [4]. When the speed of the vehicle is constant, constant angular accelerations are possible [5]. Algorithms were given to connect oriented circumferences via clothoids [6]. Initial value problems were solved numerically by clothoid parameterization [7]. Parabolas, Bloss curves, cosinusoids and sinusoids may be some alternatives to the clothoid curves [8]. In determining the curvature of the axis of railways, the moving chord method was presented [9]. A detailed review of the railway tracks was presented in [10].

In all the mentioned curves [1-10] commonly used, the normal acceleration component is not constant during navigation. It is required that this component must remain under a safe value to avoid side slips or turn overs. New curves for navigation are therefore proposed which maintains this acceleration component fixed under a safe value [11-14]. The curves need additional constraints for uniqueness and special restrictions in the tangential component of acceleration are imposed such as constant tangential deceleration [11-13] and drag induced tangential deceleration [14].

In this work, instead of imposing an additional constraint on the tangential acceleration of such curves [11-14], the constraint of time minimization is used. The two conditions, i.e., the constant normal acceleration component and the minimum time, then uniquely determine the path from a given initial point to a final destination. Variational calculus [15] is used in minimizing the time functional which leads to an ordinary differential equation with third order and high nonlinearity. Variational calculus was developed after the well-known 300 year problem of brachistochrone [16-20]. The problem investigates the minimal time path for a descending mass within a 2D vertical space under its own weight and contrary to the expectations, the path turns out to be a cycloid rather than being a straight line. In our problem, the minimal paths are not straight lines also. The differential equation system is made non-dimensional which leads to a dimensionless path parameter. Since exact analytical solutions are not available, the system is solved numerically by employing a fourth order Runge-Kutta algorithm (Matlab subroutine Ode45). For a given set of initial conditions, the path parameter should be adjusted so that the curve reaches the exact destination. The determination of the path parameter requires a shooting-like technique with trial and errors. Curves and corresponding velocity functions are depicted for various path parameters in the figures. The analysis is basically a kinematic analysis which does not consider the causes (forces, moments etc.) of motion. Therefore, although the curves are mainly constructed for a horizontal two dimensional space, they can also be employed in vertical two dimensional spaces as well. Possible applications of the study include design of curved portions for highways and their inlets and exits, and railroads. The curves may also find practical applications in transforming the marine and aerial vehicles from straight to curved routes and vice versa.

2. Mathematical model

The curve lays in a two dimensional horizontal space (Figure 1). The path starts from origin with no initial slope and radius of curvature ρ_0 . The initial velocity is v_0. The initial radius must attain a high value to ensure that the normal acceleration component is below a safe value. The vehicle has variable velocity v^*. s is the coordinate along the path. Cartesian coordinates are employed in the analysis.

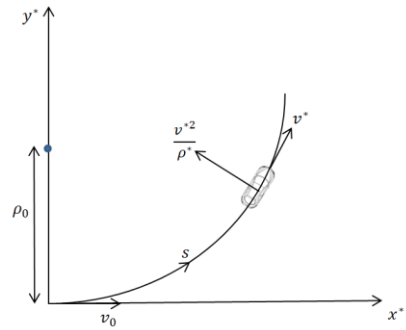

Figure 1
The Motion in the Curve

The curves are designed so that the normal component of acceleration [21] is constant throughout the motion

$$\frac{v^{*2}}{\rho^*} = \frac{v_0^2}{\rho_0} . \tag{1}$$

From calculus [22], the radius of curvature and the total curve length are

$$\frac{1}{\rho^*} = \frac{y^{*\prime\prime}}{(1+y^{*\prime 2})^{3/2}} , s = \int_0^{x^*} \sqrt{1+y^{*\prime 2}}dx^*, \tag{2}$$

where $(\,)' = \frac{d}{dx^*}$. The incremental time is

$$dt = \frac{ds}{v^*} , \tag{3}$$

with *ds* being

$$ds = \sqrt{1 + y^{*\prime 2}}dx^* \; . \tag{4}$$

Solving v^* from (1), substituting (2) and (4) into (3) and integrating, the total time elapsed is

$$t = \frac{\sqrt{\rho_0}}{v_0}\int \frac{(y^{*\prime\prime})^{1/2}}{(1+y^{*\prime 2})^{1/4}}dx^*. \tag{5}$$

Defining the function

$$F = \frac{(y^{*\prime\prime})^{1/2}}{(1+y^{*\prime 2})^{1/4}} \; , \tag{6}$$

taking the variation of (5) and equating to zero leads to the Euler equation [15]

$$\frac{\partial F}{\partial y^*} - \frac{d}{dx^*}\left(\frac{\partial F}{\partial y^{*\prime}}\right) + \frac{d^2}{dx^{*2}}\left(\frac{\partial F}{\partial y^{*\prime\prime}}\right) = 0 \; , \tag{7}$$

which can be integrated once since $\frac{\partial F}{\partial y^*} = 0$

$$\frac{d}{dx^*}\left(\frac{\partial F}{\partial y^{*\prime\prime}}\right) - \frac{\partial F}{\partial y^{*\prime}} = c \; . \tag{8}$$

Substituting (6) into (8), performing the necessary algebra, the differential equation for minimal time path is

$$(1 + y^{*\prime 2})y^{*\prime\prime\prime} - y^{*\prime}y^{*\prime\prime 2} + k(y^{*\prime\prime})^{3/2}(1 + y^{*\prime 2})^{5/4} = 0 \tag{9}$$

which is highly nonlinear and third order. k is a constant related to the integration constant *c*. Without loss of generality, the initial conditions for the problem can be imposed as

$$y^*(0) = 0, \quad y^{*\prime}(0) = 0, \quad y^{*\prime\prime}(0) = \frac{1}{\rho_0} \; . \tag{10}$$

It is more suitable to cast the equation in a dimensionless form. Defining the dimensionless variables as

$$x = \frac{x^*}{\rho_0} \; , \; y = \frac{y^*}{\rho_0}, \tag{11}$$

the dimensionless system is

$$(1 + y'^2)y''' - y'y''^2 + \varepsilon(y'')^{3/2}(1 + y'^2)^{5/4} = 0 \tag{12}$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \tag{13}$$

where

$$\varepsilon = k\sqrt{\rho_0} \tag{14}$$

is the path parameter. Defining the dimensionless velocity

$$v = \frac{v^*}{v_0} \; , \tag{15}$$

from (1) and (2), the unitless velocity in terms of the path function reads

$$v = \frac{(1+y'^2)^{3/4}}{(y'')^{1/2}} \; . \tag{16}$$

3. Numerical solutions

Since the equation is highly nonlinear and does not possess an immediate exact solution in closed form functions, numerical solution of the problem is sought using an adaptive step size Runge-Kutta algorithm Matlab Ode45.

Equation (12) and the initial conditions (13) are expressed as a first order differential system by defining $y_1 = y$, $y_2 = y'$, $y_3 = y''$,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= \frac{y_2 y_3^2 - \varepsilon y_3^{3/2}(1+y_2^2)^{5/4}}{1+y_2^2} \end{aligned} \tag{17}$$

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 1, \tag{18}$$

In Figure 2, numerical solutions corresponding to three different path parameters are given.

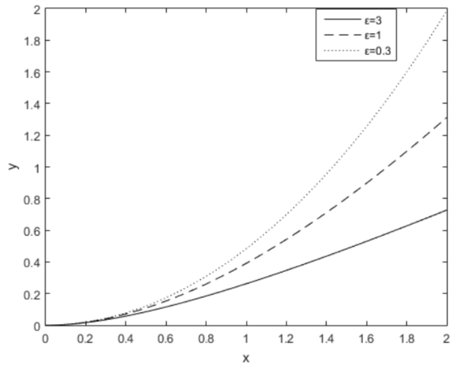

Figure 2
Minimal time curves for various path parameters

Since the path parameter ε is proportional to the square root of the initial radius of curvature, i.e., equation (14), a higher path parameter corresponds to a higher initial radius of curvature which results in lower values of the curve function. The velocities of the vehicles corresponding to these paths are given in Figure 3 as a function of the spatial coordinate.

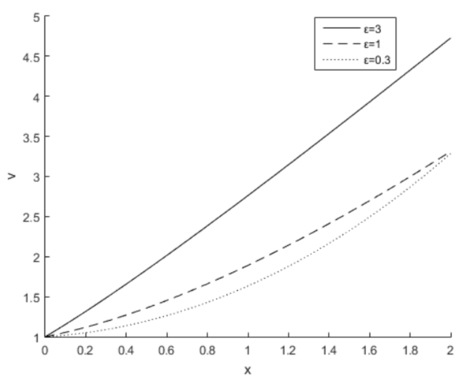

Figure 3
Velocity variation for various path parameters

When navigating through a specific path, the vehicle has to adjust its speed according to the given values in Figure 3 to maintain constant normal acceleration with minimization in time.

As can be expected from the very definition of the path parameter, as the parameter becomes large, the paths resemble more of a straight line (Figure 4).

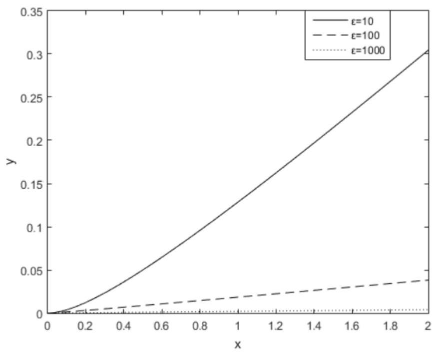

Figure 4
Minimal time curves for large path parameters

The technological problem is to reach from the initial point of origin $y(0)=0$ to a specific final point $y(x_f)=y_f$. This requires finding the right path parameter ε, for which the end point condition is satisfied. This can be done by a procedure similar to the shooting technique, that is one assumes an initial path parameter ε_0, integrates the differential equation up to x_f, and finds a value y_{0f}. Comparing y_{0f} with the desired y_f value, another ε_1 value is selected and the procedure is repeated until the end point condition is satisfied. This can be done by trial or error or by more sophisticated methods.

Notice that $y_f=y(x_f;\varepsilon)$ and assuming ε_c the correction to the initial guess

$$\varepsilon_1=\varepsilon_0+\varepsilon_c\ , \tag{19}$$

substituting into y_f and expanding in a Taylor series

$$y(x_f;\varepsilon_0+\varepsilon_c)\cong y_f\cong y(x_f;\varepsilon_0)+\varepsilon_c\frac{dy}{d\varepsilon}(x_f;\varepsilon_0) \tag{20}$$

solving for ε_c and substituting into (19) yields

$$\varepsilon_1=\varepsilon_0+\frac{y_f-y(x_f;\varepsilon_0)}{\frac{dy}{d\varepsilon}(x_f;\varepsilon_0)}\ . \tag{21}$$

The iteration continues until a satisfactory result is obtained. To calculate the derivative approximately

$$\frac{dy}{d\varepsilon}=\frac{\Delta y}{\Delta\varepsilon}=\frac{y(x_f;\varepsilon_0+\delta)-y(x_f;\varepsilon_0)}{\delta}\ , \tag{22}$$

select δ extremely small. The iteration equation is

$$\varepsilon_{n+1}=\varepsilon_n+\frac{y_f-y(x_f;\varepsilon_n)}{\frac{\Delta y}{\Delta\varepsilon}(x_f;\varepsilon_n)}\ ,\ n=0,1,2,\ldots \tag{23}$$

As an example, the iterations for path parameter corresponding to $y(2) = 1$ is given in Table 1. Starting from an initial guess of $\varepsilon_0 = 2$, after four iterations with $\delta = 0.001$, the final point is reached with four digit precision after four iterations. Hence to arrive at $y(2) = 1$, the path parameter should be $\varepsilon = 1.7407$.

Table 1
Iteration of path parameters

Path parameter	Final Location
$\varepsilon_0 = 2$	0.9267
$\varepsilon_1 = 1.7557$	0.9955
$\varepsilon_2 = 1.7407$	1.0001
$\varepsilon_3 = 1.7410$	1.0001
$\varepsilon_4 = 1.7413$	1.0000

4. Concluding remarks

Minimal time paths under the constraint of constant normal accelerations are derived for vehicles navigating in a two dimensional space. Controlling the component of normal acceleration and keeping under a safe value is extremely important for vehicles to avoid possible accidents. The minimization process is achieved by employing variational calculus principles. The mathematical model turns out to be a highly nonlinear ordinary differential equation of third order which is made dimensionless. In the dimensionless form, the physical problem depends on a single path parameter in contrast with the original problem depending on three physical parameters. The number of figures is reduced and the solutions are expressed in a more compact form when dimensionless quantities are used. In order to reach a specific final destination point, the numerical solutions should be repeated similar to the shooting technique in which the boundary conditions are expressed as suitable initial conditions. In our problem, shooting is necessary for the path parameter of the differential equation, not in the initial conditions. A systematic way of reducing the computations in trial and error was also given for the problem. The specific minimal time path problem posed is new and may find applications in navigation of land, marine and aerial vehicles.

5. Declarations

Ethical Approval: Not applicable.

Availability of Supporting Data: There is no accompanying additional data.

Competing Interests: No competing interests declared.

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Author Contribution: M. P. is the single author responsible from all stages of the work.

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