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Generalized operational matrices to analyze inhomogeneous multi-term fractional differential equations : An efficient direct method

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Abstract

Fractional calculus has gained popularity as a modeling and control tool for dynamical systems. For systems with commensurate and non-commensurate generalized fractional orders, some control theory techniques have been proposed. In this study, a novel iterative method for dealing with fractional order systems is provided. An orthogonal function set, known as hybrid function (HF) set is linearly integrated with the sample and hold function (SHF) set and triangular function (TF) set, is a novel technique for evaluating fractional order systems, used to represent fractional order plants. To illustrate how the suggested approach has been validated, a few instances are given. A few numerical examples are provided, along with relevant graphs and tables, to validate the proposed approach.

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1. Introduction

Since real-world systems typically have fractional orders, the mathematical model and the actual system may differ significantly. This is especially true for integer-order descriptions of real-world systems that make a number of assumptions. The generalization of the typical derivation and integration to an undetermined real or complex order is the idea behind non-integer calculus. Although for a long time, fractional calculus was thought of as a mathematics of the 21st century, its applicability in numerous domains has solidified its interest.

It has been used in many engineering and scientific fields, such as biotechnology **[1, 2]**, advanced control **[3-7]**, long transmission line modeling **[8, 9]**, acoustics **[10]**, electrochemistry **[11-13]**, robotics **[14, 15]** and many more. Currently, the bulk of studies on fractional order systems use the transfer function form because of how simple it is to present, however it has drawbacks. Only fractional order systems of equal order can be represented using the transfer function-based technique.

The following is how the paper is set up: A few important definitions related to fractional integral and derivatives are presented in **Section 2**. **Sections 3** reviewed the types of mathematical models used to represent fractional order systems.

The orthogonal hybrid function **[16, 17]** set has been introduced in **Section 4**, which is then used to derive generalized integral operational matrices and presented in **Section 5**. The proposed direct solution of the fractional order systems has been developed and demonstrated in **Section 6**. Finally, **Section 7** applies the HF domain integration operational matrices in the solution of FDEs. The efficacy of the proposed approach is discussed in detail in **Section 8**.

2. Important Definitions of Fractional Integral and Derivatives:

Numerous definitions of fractional derivatives have been produced by scholars and may be found in publications such as Reimann-Liouville (R-L) **[18]**, Caputo **[19]**, Grunwald-Letnikov **[20]**, and others. Despite the fact that they are displayed in various ways, they all contain the same fundamental components and can, under certain circumstances, be transferred from one to another.

Recalling the R-L integration of an order α (where α is in between 0 and 1) with,

(t_0, t) is the interval considered for integral operation.

$\Gamma(n) = (n-1)!$, is a gamma function, and
T be the total time under consideration.

For $t_0 = 0$, the fractional order integration is symbolically expressed as I^{α}.

Important features of R-L fractional integration are listed below; these will be re-proved in **Section 5** using orthogonal HF domain and generalized integral operational matrices. Lastly, these are required here and are,

$$I_t^{\alpha} I_t^{\beta} f(t) = I_t^{\alpha+\beta} f(t) \tag{1}$$

$$I_t^{\alpha} t^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n}, \tag{2}$$

where n may be a fraction or an integer

Similarly, a function $f(t)$ can be written as the factional derivative of order α, which is

$$D_t^{\alpha} f(t) = I_t^{n-\alpha} D_t^n f(t), \tag{3}$$

where α may be a fraction or an integer.

For a condition of $(n-1) \le \alpha < n$, and $t > 0$.

3. Mathematical Representation of Fractional Order Systems

A differential equation known as a fractional differential equation (FDE) can be used to represent a fractional order system. On the other hand, fractional order systems with a single input and a single output (SISO-LTI) can also be represented in the form of a transfer function.

Let us consider a differential equation for a particular case of commensurate system,

$$y(t) + \sum_{j=1}^{n} a_j D^{\alpha_j} y(t) = \sum_{j=1}^{m} b_j D^{\beta_j} u(t) \tag{4}$$

where, $a_j, b_j \in \mathbf{R}$,
$\alpha_j, \beta_j \in \mathbf{R}^+$,

$u(t) \in \mathbf{R}$, is input applied to the above system, and
$y(t) \in \mathbf{R}$, is the output of the system.

Applying Laplace transform to equation (4), we obtain

$$Y(s)+\sum_{j=1}^{n} a_j s^{\alpha_j} Y(s)=\sum_{j=1}^{m} b_j s^{\beta_j} U(s) \tag{5}$$

The fractional order system of equation (5) can also be written as, in the form of a transfer function:

$$\frac{Y(s)}{U(s)}=\frac{\sum_{j=1}^{m} b_j s^{\beta_j}}{1+\sum_{j=1}^{n} a_j s^{\alpha_j}} \tag{6}$$

4. Background of Orthogonal Hybrid Function (HF) Set: [16, 17]

The HF set, often known as a hybrid function set, is a linear combination of the SHF set $S_k(t)$ and the right-handed TF set $T_k(t)$. The (k +1)-th component of the m-set hybrid function $\mathbf{H}_{(m)}(t)$ has the following mathematical definition:

$$H_k(t)=a_k S_k(t)+b_k T_k(t) \text{ for } 0\le t<T \text{ and } 0\le k\le 1 \tag{7}$$

where, a_k and b_k are the (k +1)-th weighed constants.

The definition of (k +1)-th component of SHF set S_k is

$$S_k(t)=\begin{cases}1 & \text{when } kh\le t<(k+1)h\\ 0 & \text{elsewhere}\end{cases} \text{ for } k=0,\ 1,\ 2,\ldots,\ (m-1) \tag{8}$$

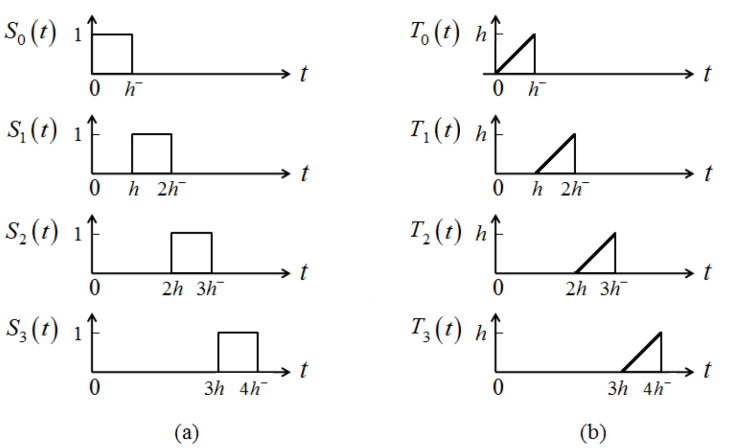


Figure 1

SHF sets $S_{(4)}$ and RHTF sets $T_{(4)}$ with four component functions to create orthogonal HF sets.

The definition of (k +1)-th component of right-handed TF set T_k is

$$T_k(t) = \begin{cases} (t-kh)/h, & \text{when } kh \le t < (k+1)h \\ 0, & \text{elsewhere} \end{cases} \text{ for } 0 \le k \le (m-1) \tag{9}$$

where, T is the observation time, and m components employed, and the sampling period is denoted by h (= T/m) in seconds.

Figure 1 displays $\mathbf{S}_{(m)}(t)$ and $\mathbf{T}_{(m)}(t)$, where arbitrarily 4 segments have been selected. We have created an orthogonal HF set using this collection of orthogonal functions.

5. Derivation of Generalized HF Domain Operational Matrix for Integration

The R-L integral, is employed in the HF domain to build the fractional integration matrices, as following:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(n)} \int_0^t (t-t_1)^{n-1} f(t_1) \mathrm{d}t_1 \quad \text{for } n = 1, 2, \ \ldots \text{ and } 0 \le t < T \tag{10}$$

The expanded form of the function $f(t_1)$ in the HF domain in equation (10) results in the R-L integral of order α becoming

$$\begin{aligned} I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} f(t_1) \mathrm{d}t_1 \\ &\approx \frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} [\mathbf{C}_{\mathrm{S}}^{\mathrm{T}} \mathbf{S}_{(m)}(t_1) + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}} \mathbf{T}_{(m)}(t_1)] \mathrm{d}t_1 \\ &= \mathbf{C}_{\mathrm{S}}^{\mathrm{T}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} \mathbf{S}_{(m)}(t_1) \, \mathrm{d}t_1 + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} \mathbf{T}_{(m)}(t_1) \, \mathrm{d}t_1 \end{aligned} \tag{11}$$

5-A. *Fractional Integration of Sample-and-Hold Functions (SHF):*

Let us consider $f_1(t) = t^{\alpha-1}$ and $f_2(t) = S_k(t)$ in the SHF part of equation (11) for $\alpha > 0$.

Then integration of (k +1)-th component of SHF set in equation (11) can be shown as

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} S_k(t_1) \, \mathrm{d}t_1 = \frac{1}{\Gamma(\alpha)} \int_0^t f_1(t-t_1) f_2(t_1) \, \mathrm{d}t_1 \tag{12}$$

Taking the Laplace transform of equation (12), we have

$$\mathcal{L}\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} S_k(t_1) \, \mathrm{d}t_1 \right] = \frac{1}{\Gamma(\alpha)} F_1(s) F_2(s) \tag{13}$$

where, $F_1(s) = \mathcal{L}[t^{\alpha-1}] = \dfrac{\Gamma(\alpha)}{s^{\alpha}}$, and

$$F_2(s) = \mathcal{L}[S_k(t)] = \frac{1}{s}[\mathrm{e}^{-khs} - \mathrm{e}^{-(k+1)hs}]$$

Therefore, equation (13) can be expressed as

$$\mathcal{L}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} S_k(t_1)\, \mathrm{d}t_1\right] = \frac{1}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}[e^{-khs} - e^{-(k+1)hs}] \tag{14}$$

Taking the inverse Laplace transform of equation (14), we get

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} S_k(t_1)\, \mathrm{d}t_1 = \frac{1}{\Gamma(\alpha+1)}[A - B] \tag{15}$$

where, $A - B \triangleq (t-kh)^{\alpha} u(t-kh) - (t-(k+1)h)^{\alpha} u(t-(k+1)h)$

Similarly, t^{α} can be expanded as

$$t^{\alpha} u(t) \approx [c_0 c_1 \cdots c_{(m-1)}]\mathbf{S}_{(m)}(t) + [(c_1 - c_0)(c_2 - c_1)\cdots(c_m - c_{(m-1)})]\mathbf{T}_{(m)}(t) \tag{16}$$

with, $c_i \triangleq k^{\alpha} h^{\alpha}$

$c_{(i+1)} \triangleq (k+1)^{\alpha} h^{\alpha}$ and so on.

$$c_{(k+1)} - c_k \triangleq [h^{\alpha}(k+1)^{\alpha} - h^{\alpha} k^{\alpha}], \text{ for } 0 \le k \le (m-1).$$

Using equation (16), the first term A of equation (15) can be shown in HF domain as

$$\begin{aligned}(t-kh)^{\alpha} u(t-kh) \approx [\underbrace{0\ 0\ \cdots\ 0}_{k\text{-terms}}\ c_0\ \ c_1\ \ \cdots\ \ c_{(m-k-1)}]\mathbf{S}_{(m)}(t) \\ + [\underbrace{0\ 0\cdots 0}_{k\text{-terms}}\ (c_1 - c_0)\ (c_2 - c_1)\cdots(c_{(m-i)} - c_{(m-k-1)})]\mathbf{T}_{(m)}(t)\end{aligned} \tag{17}$$

Similarly, the second term B of equation (15) can be presented in HF domain as

$$\begin{aligned}(t-(k+1)h)^{\alpha} u(t-(k+1)h) \approx [\underbrace{0\ 0\ \cdots\ 0}_{(k+1)\text{ terms}}\ c_0\ \ c_1\ \ \cdots\ \ c_{(m-k-2)}]\mathbf{S}_{(m)}(t) \\ + [\underbrace{0\ 0\ \cdots\ 0}_{(k+1)\text{ terms}}\ (c_1 - c_0)\ (c_2 - c_1)\ \cdots\ (c_{(m-k-1)} - c_{(m-k-2)})]\mathbf{T}_{(m)}(t)\end{aligned} \tag{18}$$

Finally, equation (15) can be presented as

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} S_k(t_1)\,\mathrm{d}t_1 = \frac{1}{\Gamma(\alpha+1)}[0\ 0\ \cdots\ 0\ c_0\ (c_1-c_0)\ (c_2-c_1)\ \cdots \\ \cdots\ (c_{(m-k-1)}-c_{(m-k-2)})]\mathbf{S}_{(m)}(t)$$

$$+\frac{1}{\Gamma(\alpha+1)}[0\ \ 0\ \cdots\ 0\ \ (c_1-c_0)\ \ \{(c_2-c_1)-(c_1-c_0)\} \\ \cdots\ \{(c_{(m-k)}-c_{(m-k-1)})-(c_{(m-k-1)}-c_{(m-k-2)})\}]\mathbf{T}_{(m)}(t) \tag{19}$$

where, $(c_{(k+2)}-c_{(k+1)})-(c_{(k+1)}-c_k)=h^\alpha[(k+2)^\alpha-2(k+1)^\alpha+k^\alpha]$
for $0\le k\le(m-k-2)$

Integrating each sample-and-hold component function using equation (19) and express the first part of equation (11) in following vector-matrix form:

$$\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1}\mathbf{S}_{(m)}(t_1)\,\mathrm{d}t_1$$

$$\approx \mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\left[\!\!\left[\frac{c_0}{\Gamma(\alpha+1)}\quad\frac{(c_1-c_0)}{\Gamma(\alpha+1)}\quad\cdots\quad\cdots\quad\frac{1}{\Gamma(\alpha+1)}(c_{(m-1)}-c_{(m-2)})\right]\!\!\right]\mathbf{S}_{(m)}(t)$$

$$+\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\left[\!\!\left[\frac{1}{\Gamma(\alpha+1)}(c_1-c_0)\quad\frac{1}{\Gamma(\alpha+1)}\{(c_2-c_1)-(c_1-c_0)\}\quad\cdots\quad\cdots\right.\right.$$

$$\left.\left.\cdots\quad\frac{1}{\Gamma(\alpha+1)}\{(c_m-c_{(m-1)})-(c_{(m-1)}-c_{(m-2)})\}\right]\!\!\right]\mathbf{T}_{(m)}(t)$$

$$=\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\frac{1}{\Gamma(\alpha+1)}h^\alpha\left[\!\!\left[f_0\quad f_1\quad f_2\quad\cdots\quad f_{(m-1)}\right]\!\!\right]\mathbf{S}_{(m)}(t)$$

$$+\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\frac{1}{\Gamma(\alpha+1)}h^\alpha\left[\!\!\left[g_0\quad g_1\quad g_2\quad\cdots\quad g_{(m-1)}\right]\!\!\right]\mathbf{T}_{(m)}(t) \tag{20}$$

where, $f_0\triangleq h^{-\alpha}c_0=0$, and $f_{(k+1)}=[(k+1)^\alpha-k^\alpha]$,

$g_0\triangleq h^{-\alpha}(c_1-c_0)=1$, and

$$g_{(k+1)}=[(k+2)^\alpha-2(k+1)^\alpha+k^\alpha],\ \text{for}\ 0\le k\le(m-2)$$

As a result, we are able to create the matrices for fractional integration using the SHF coefficient's integration, as $\mathbf{P}_{\mathrm{FSS}}^{\alpha}$ and $\mathbf{P}_{\mathrm{FST}}^{\alpha}$. The following compact form can be used to express these two square matrices:

$$\mathbf{P}_{\text{FSS}}^{\alpha} \triangleq \frac{1}{\Gamma(\alpha+1)} h^{\alpha} \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_{(m-1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{h^{\alpha}}{\Gamma(\alpha+1)} & \frac{h^{\alpha}}{\Gamma(\alpha+1)}(2^{\alpha}-1) & \cdots & \frac{h^{\alpha}}{\Gamma(\alpha+1)}\{(m-1)^{\alpha}-(m-2)^{\alpha}\} \end{bmatrix} \tag{21}$$

and $$\mathbf{P}_{\text{FST}}^{\alpha} \triangleq \frac{1}{\Gamma(\alpha+1)} h^{\alpha} \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_{(m-1)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{h^{\alpha}}{\Gamma(\alpha+1)} & \frac{h^{\alpha}}{\Gamma(\alpha+1)}(2^{\alpha}-2\times 1^{\alpha}+0) & \cdots & \frac{h^{\alpha}}{\Gamma(\alpha+1)}\{m^{\alpha}-2(m-1)^{\alpha}+(m-2)^{\alpha}\} \end{bmatrix} \tag{22}$$

5-B: *Fractional Integration of Triangular Functions (TF)*

Let us consider $f_3(t) = t^{\alpha-1}$ and $f_4(t) = T_k(t)$ in the TF part of equation (11) for $\alpha > 0$.

Like **Section 5-A**, for the TF set in equation (11) can be written as

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} T_k(t_1)\, \mathrm{d}t_1 = \frac{1}{\Gamma(\alpha)} \int_0^t f_3(t-t_1) f_4(t_1)\, \mathrm{d}t_1 \tag{23}$$

From equation (23), we get

$$\mathcal{L}\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} T_k(t_1)\, \mathrm{d}t_1 \right] = \frac{1}{\Gamma(\alpha)} F_3(s) F_4(s) \tag{24}$$

where, $F_3(s) = \mathcal{L}[t^{\alpha-1}] = \dfrac{\Gamma(\alpha)}{s^{\alpha}}$, and

$$F_4(s) = \mathcal{L}[T_k(t)] = \frac{1}{h}\frac{1}{s^2}[e^{-khs} - e^{-(k+1)hs}] - \frac{1}{s} e^{-(k+1)hs}$$

Therefore, equation (24) is as

$$\mathcal{L}\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} T_k(t_1)\, \mathrm{d}t_1 \right] = \frac{1}{h} \frac{1}{\Gamma(\alpha+2)} \frac{\Gamma(\alpha+2)}{s^{\alpha+2}} [e^{-khs} - e^{-(k+1)hs}]$$
$$- \frac{1}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} e^{-(k+1)hs} \tag{25}$$

Equation (25), yields

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} T_k(t_1)\, \mathrm{d}t_1 = \frac{1}{h} \frac{1}{\Gamma(\alpha+2)} [C-D] - \frac{1}{\Gamma(\alpha+1)} E \tag{26}$$

where, $C - D \triangleq (t-kh)^{\alpha+1}u(t-kh) - (t-(k+1)h)^{\alpha+1}u(t-(k+1)h)$, and

$$E \triangleq (t-(k+1)h)^{\alpha}u(t-(k+1)h)$$

In HF domain $t^{\alpha+1}$ can be presented as

$$t^{\alpha+1}u(t) \approx [d_0 \quad d_1 \quad \cdots \quad d_{(m-1)}]\mathbf{S}_{(m)}(t) + \left[(d_1-d_0) \quad (d_2-d_1) \quad \cdots \quad (d_m - d_{(m-1)})\right]\mathbf{T}_{(m)}(t) \tag{27}$$

where,

$$d_k \triangleq k^{\alpha+1}h^{\alpha+1}$$

$$d_{(k+1)} - d_k = h^{\alpha+1}[(k+1)^{\alpha+1} - k^{\alpha+1}], \text{ for } 0 \le k \le (m-1).$$

As a result of equation (27), the HF domain expression for the first term C in equation (26) is

$$(t-kh)^{\alpha+1}u(t-kh) \approx [\underbrace{0 \;\; 0 \;\; \cdots \;\; 0}_{k-\text{terms}} \;\; d_0 \;\; d_1 \;\; \cdots \;\; d_{(m-k-1)}]\mathbf{S}_{(m)}(t)$$
$$+ [\underbrace{0 \;\; 0 \;\; \cdots \;\; 0}_{k-\text{terms}} \;\; (d_1-d_0) \;\; (d_2-d_1) \;\; \cdots \;\; (d_{(m-k)} - d_{(m-k-1)})]\mathbf{T}_{(m)}(t) \tag{28}$$

Similarly, in equation (26), the second term D in HF domain, is as

$$(t-(k+1)h)^{\alpha+1} \approx [0 \;\; 0 \;\; \cdots \;\; 0 \;\; \underbrace{d_0 \;\; d_1 \;\; \cdots \;\; d_{(m-k-2)}}_{(k+1)\text{ terms}}]\mathbf{S}_{(m)}(t)$$
$$+ [\underbrace{0 \;\; 0 \;\; \cdots \;\; 0}_{(k+1)\text{ terms}} \;\; (d_1-d_0) \;\; (d_2-d_1) \;\; \cdots \;\; (d_{(m-k-1)} - d_{(m-k-2)})]\mathbf{T}_{(m)}(t) \tag{29}$$

The third term E in equation (26), is expressed in HF domain, as

$$(t-(k+1)h)^{\alpha} \approx [\underbrace{0 \;\; 0 \;\; \cdots \;\; 0}_{(k+1)\text{ terms}} \;\; c_0 \;\; c_1 \;\; \cdots \;\; c_{(m-k-2)}]\mathbf{S}_{(m)}(t)$$
$$+ [\underbrace{0 \;\; 0 \;\; \cdots \;\; 0}_{(k+1)\text{ terms}} \;\; (c_1-c_0) \;\; (c_2-c_1) \;\; \cdots \;\; (c_{(m-k-1)} - c_{(m-k-2)})]\mathbf{T}_{(m)}(t) \tag{30}$$

Using equations (28), (29) and (30), equation (26) is expressed as

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1}T_k(t_1)\,\mathrm{d}t_1 = \frac{1}{h}\frac{1}{\Gamma(\alpha+2)}[\underbrace{0 \;\; 0 \;\; \cdots \;\; 0}_{k\text{-terms}} \;\; d_0 \;\; d_1 \;\; \cdots \;\; d_{(m-k-1)}]\mathbf{S}_{(m)}(t)$$
$$+ \frac{1}{h}\frac{1}{\Gamma(\alpha+2)}[\underbrace{0 \;\; 0 \;\; \cdots \;\; 0}_{k\text{-terms}} \;\; (d_1-d_0) \;\; (d_2-d_1) \;\; \cdots \;\; (d_{(m-k)} - d_{(m-k-1)})]\mathbf{T}_{(m)}(t)$$

$$-\frac{1}{h}\frac{1}{\Gamma(\alpha+2)}[\underbrace{0\ \ 0\ \ \cdots\ \ 0}_{(k+1)\text{terms}}\ d_0\ d_1\ \ \cdots\ \ d_{(m-k-2)}\ ]\mathbf{S}_{(m)}(t)$$

$$-\frac{1}{h}\frac{1}{\Gamma(\alpha+2)}[\underbrace{0\ \ 0\ \cdots\ 0}_{(k+1)\text{terms}}\ (d_1-d_0)\ \ (d_2-d_1)\ \cdots\ (d_{(m-k-1)}-d_{(m-k-2)})]\mathbf{T}_{(m)}(t)$$

$$-\frac{1}{\Gamma(\alpha+1)}[\underbrace{0\ \ 0\ \cdots\ 0}_{(k+1)\text{terms}}c_0\ \ c_1\ \ \cdots\ c_{(m-k-2)}\ ]\mathbf{S}_{(m)}(t)$$

$$-\frac{1}{\Gamma(\alpha+1)}[\underbrace{0\ \ 0\ \cdots\ 0}_{(k+1)\text{terms}}\ (c_1-c_0)\ \ (c_2-c_1)\ \cdots\ (c_{(m-k-1)}-c_{(m-k-2)})]\mathbf{T}_{(m)}(t)$$

where,

$$\text{and}\quad \left.\begin{aligned} d_{(k+1)}-d_k &= [h^{\alpha+1}(k+1)^{\alpha+1}-h^{\alpha+1}k^{\alpha+1}],\\ c_{(k+1)}-c_k &= [h^{\alpha}(k+1)^{\alpha}-h^{\alpha}k^{\alpha}] \end{aligned}\right\}\ \text{for } 0\le k\le(m-1).$$

Using equation (23), integrating each triangular function component function and express the second part of equation (11) in following vector-matrix form:

$$\begin{aligned}
&\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\frac{1}{\Gamma(\alpha)}\int_0^t(t-t_1)^{\alpha-1}\mathbf{T}_{(m)}(t_1)\,\mathrm{d}t_1\\
&\quad\approx\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\frac{1}{h}\left[\!\!\left[\frac{d_0}{\Gamma(\alpha+2)}\ \ \frac{(d_1-d_0)}{\Gamma(\alpha+2)}\ \ \cdots\ \ \cdots\ \ \frac{1}{\Gamma(\alpha+2)}(d_{(m-1)}-d_{(m-2)})\right]\!\!\right]\mathbf{S}_{(m)}(t)\\
&\quad+\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\left[\!\!\left[\frac{(d_1-d_0)}{\Gamma(\alpha+2)}\ \ \frac{1}{\Gamma(\alpha+2)}\{(d_2-d_1)-(d_1-d_0)\}\ \ \cdots\ \ \cdots\right.\right.\\
&\qquad\left.\left.\cdots\ \ \frac{1}{\Gamma(\alpha+2)}\{(d_m-d_{(m-1)})-(d_{(m-1)}-d_{(m-2)})\}\right]\!\!\right]\mathbf{T}_{(m)}(t)\\
&\quad-\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\left[\!\!\left[0\ \ \frac{c_0}{\Gamma(\alpha+1)}\ \ \frac{c_1}{\Gamma(\alpha+1)}\ \ \cdots\ \ \cdots\ \ \frac{c_{(m-2)}}{\Gamma(\alpha+1)}\right]\!\!\right]\mathbf{S}_{(m)}(t)\\
&\quad-\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\left[\!\!\left[0\ \ \frac{1}{\Gamma(\alpha+1)}(c_1-c_0)\ \ \cdots\ \ \cdots\ \ \frac{1}{\Gamma(\alpha+1)}(c_{(m-1)}-c_{(m-2)})\right]\!\!\right]\mathbf{T}_{(m)}(t)\\
&\quad=\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\frac{1}{h}\frac{1}{\Gamma(\alpha+2)}h^{\alpha+1}\left[\!\!\left[w_0\ \ w_1\ \ w_2\ \ \cdots\ \ w_{(m-1)}\right]\!\!\right]\mathbf{S}_{(m)}(t)\\
&\quad+\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\frac{1}{h}\frac{1}{\Gamma(\alpha+2)}h^{\alpha+1}\left[\!\!\left[x_0\ \ x_1\ \ x_2\ \ \cdots\ \ x_{(m-1)}\right]\!\!\right]\mathbf{T}_{(m)}(t)\\
&\quad-\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\left[\!\!\left[y_0\ \ y_1\ \ y_2\ \ \cdots\ \ y_{(m-1)}\right]\!\!\right]\mathbf{S}_{(m)}(t)
\end{aligned}$$

$$-\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\frac{1}{\Gamma(\alpha+1)}h^{\alpha}\left[\!\left[z_0 \quad z_1 \quad z_2 \quad \cdots \quad z_{(m-1)}\right]\!\right]\mathbf{T}_{(m)}(t) \tag{32}$$

where, $w_0 \triangleq h^{-(\alpha+1)}d_0 = 0$, and $w_{(k+1)} = [(k+1)^{\alpha} - k^{\alpha}]$,

$x_0 \triangleq h^{-(\alpha+1)}(d_1 - d_0) = 1$, and $x_{(k+1)} = [(k+2)^{(\alpha+1)} - 2(k+1)^{(\alpha+1)} + k^{(\alpha+1)}]$,

$$y_0 = 0, \text{ and } y_{(k+1)} = k^{\alpha},$$

$$z_0 = 0, \text{ and } z_{(k+1)} = [(k+1)^{\alpha} - k^{\alpha}], \text{ for } 0 \le k \le (m-2)$$

Consequently, we can express the fractional order integration operational matrices as $\mathbf{P}_{\mathrm{FTS}}^{\alpha}$ and $\mathbf{P}_{\mathrm{FTT}}^{\alpha}$, which are derived from the integration of the TF portion. The compact form of these two square matrices is as follows.

$$\mathbf{P}_{\mathrm{FTS}}^{\alpha} \triangleq \frac{1}{\Gamma(\alpha+2)}h^{\alpha}\left[\!\left[(w_0 - y_0) \quad (w_1 - y_1) \quad (w_2 - y_2) \quad \cdots \quad (w_{(m-1)} - y_{(m-1)})\right]\!\right]$$

$$= \frac{1}{\Gamma(\alpha+2)}\left[\!\left[0 \quad h^{\alpha}\{1^{\alpha+1} - 0^{\alpha+1} - (\alpha+1).0^{\alpha}\} \quad h^{\alpha}\{2^{\alpha+1} - 1^{\alpha+1} - (\alpha+1).1^{\alpha}\} \quad \cdots \right.\right.$$

$$\left.\left. \cdots \quad h^{\alpha}\{(m-1)^{\alpha+1} - (m-2)^{\alpha+1} - (\alpha+1).(m-2)^{\alpha}\}\right]\!\right] \tag{33}$$

$$\text{and } \mathbf{P}_{\mathrm{FTT}}^{\alpha} \triangleq \frac{1}{\Gamma(\alpha+2)}\left[\!\left[h^{\alpha}(x_0 - z_0) \quad h^{\alpha}(x_1 - z_1) \quad \cdots \quad \cdots \quad h^{\alpha}(x_{(m-1)} - z_{(m-1)})\right]\!\right]$$

$$= \frac{1}{\Gamma(\alpha+2)}\left[\!\left[h^{\alpha} \quad h^{\alpha}\{(2^{\alpha+1} - 1^{\alpha+1}) - (1^{\alpha+1} - 0^{\alpha+1}) - (\alpha+1).(1^{\alpha} - 0^{\alpha})\} \quad \cdots \right.\right.$$

$$\left.\left. \cdots \quad h^{\alpha}\{(m^{\alpha+1} - (m-1)^{\alpha+1}) - ((m-1)^{\alpha+1} - (m-2)^{\alpha+1}) - (\alpha+1).((m-1)^{\alpha} - (m-2)^{\alpha})\}\right]\!\right] \tag{34}$$

6. HF Domain Fractional Order Integration Matrices for Time Functions

The approximate HF domain form of a function is displayed in equation (11). The time function's fractional integration in the HF domain with regard to t can be written as

$$I^{\alpha}f(t) \approx [\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathbf{P}_{\mathrm{FSS}}^{\alpha} + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathbf{P}_{\mathrm{FTS}}^{\alpha}]\mathbf{S}_{(m)}(t) + [\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathbf{P}_{\mathrm{FST}}^{\alpha} + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathbf{P}_{\mathrm{FTT}}^{\alpha}]\mathbf{T}_{(m)}(t) \tag{35}$$

To validate equation (35), we shall fractionally integrate a time function in the next section.

6-A. *Illustration 1:*

Using the orthogonal HF set and the block pulse functions, we carry out the fractional integration of $f_1(t) = t^{0.7}$ for orders of 0.4 (when $0 < \alpha < 1$) and 1.4 (when $\alpha > 1$), assuming 8 segments, with 1 s duration.

The exact solution for the fractional integration of $f(t) = t^n$ is as,

$$(I^{\alpha} f)(t^n) = \frac{\Gamma(n+1)}{\Gamma(\alpha + n + 1)} t^{\alpha+n} \tag{36}$$

where, α and n can be any integer or fraction.

Here equation (36), when applied to the function $f_1(t) = t^{0.7}$ in the HF domain, yields the following straight expansion of the integration for order 0.4:

$$\begin{aligned}(I^{0.4} f)(t^{0.7}) \approx &\left[0.00000000 \quad 0.08815743 \quad 0.18896958 \quad 0.29518364\right.\\ &\left.0.40506516 \quad 0.51775691 \quad 0.63273998 \quad 0.74966415\right]\mathbf{S}_{(8)}(t)\\ &+\left[0.08815743 \quad 0.10081215 \quad 0.10621406 \quad 0.10988152\right.\\ &\left.0.11269175 \quad 0.11498307 \quad 0.11692417 \quad 0.11861203\right]\mathbf{T}_{(8)}(t)\end{aligned} \tag{37}$$

Direct expansion of the function $f_1(t)$ is as

$$\begin{aligned}f_1(t) = t^{0.7} \approx &\left[0.00000000 \quad 0.23325825 \quad 0.37892914 \quad 0.50329416\right.\\ &\left.0.61557221 \quad 0.71964119 \quad 0.81760377 \quad 0.91076354\right]\mathbf{S}_{(8)}(t)\\ &+\left[0.23325825 \quad 0.14567089 \quad 0.12436501 \quad 0.11227805\right.\\ &\left.0.10406898 \quad 0.09796258 \quad 0.09315977 \quad 0.08923646\right]\mathbf{T}_{(8)}(t)\end{aligned} \tag{38}$$

Substituting the values of $\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}$ and $\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}$ from equation (38) in equation (35), we perform fractional integration of 0.4 order on $f_1(t)$ in HF domain to obtain

$$\begin{aligned}(I^{0.4} f_1)(t) \approx &\left[0.00000000 \quad 0.08173728 \quad 0.18501403 \quad 0.29206846\right.\\ &\left.0.40242587 \quad 0.51543447 \quad 0.63064774 \quad 0.74774872\right]\mathbf{S}_{(8)}(t)\\ &+\left[0.08173728 \quad 0.10327674 \quad 0.10705443 \quad 0.11035741\right.\\ &\left.0.11300860 \quad 0.11521327 \quad 0.11710098 \quad 0.11875318\right]\mathbf{T}_{(8)}(t)\end{aligned} \tag{39}$$

The results are graphically compared in **Figure 2** and **Figure 3**.

The quantitative study that was conducted using the relevant operational matrices to achieve MISEs of equal band in fractional integration is displayed in **Table 1**.

7. Solving Multi-Order Fractional Differential Equation by Application of HF Domain Generalized Integration Operational Matrices

Here is a fractional non-homogeneous differential equation of multiple orders.

$$\mathrm{D}^{\alpha} f(t) + a\mathrm{D}^{\beta} f(t) + b\mathrm{D}^{\gamma} f(t) + cf(t) = d(t) \text{ for } \alpha \geq 2,\ 1 < \beta < 2 \text{ and } 0 < \gamma < 1 \tag{40}$$

where, the coefficients are considered as positive constants.

Integrating equation (40) in the order of α, we get

$$\begin{aligned} f(t) + a\mathrm{I}^{\alpha-\beta} f(t) + b\mathrm{I}^{\alpha-\gamma} f(t) + c\mathrm{I}^{\alpha} f(t) &= \mathrm{I}^{\alpha} d(t) + f(0)u(t)[1 + a\mathrm{I}^{\alpha-\beta} + b\mathrm{I}^{\alpha-\gamma}] \\ &+ f'(0)\left[\int u(t)\mathrm{d}t + a\mathrm{I}^{\alpha-\beta+1}u(t)\right] + f''(0)\mathrm{I}^{2}u(t) \end{aligned} \tag{41}$$

Assume the solution fractional differential equation $f(t)$ of (40) in HF domain is as,

$$f(t) \approx \mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathbf{S}_{(m)}(t) + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathbf{T}_{(m)}(t)$$

Also, $u(t)$ can be presented as $u(t) = \mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathbf{S}_{(m)}(t)$

When we expand every time function in the *m*-term HF domain, we get

$$\begin{aligned} &\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathbf{S} + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathbf{T} + a[\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{\alpha-\beta}\mathbf{S} + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathrm{I}^{\alpha-\beta}\mathbf{T}] + b[\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{\alpha-\gamma}\mathbf{S} + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathrm{I}^{\alpha-\gamma}\mathbf{T}] \\ &+ c[\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{\alpha}\mathbf{S} + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathrm{I}^{\alpha}\mathbf{T}] = [\mathbf{D}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{\alpha}\mathbf{S} + \mathbf{D}_{\mathrm{T}}^{\mathrm{T}}\mathrm{I}^{\alpha}\mathbf{T}] + f(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathbf{S} + af(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{\alpha-\beta}\mathbf{S} \\ &+ bf(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{\alpha-\gamma}\mathbf{S} + f'(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{1}\mathbf{S} + a\dot{f}(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{\alpha-\beta+1}\mathbf{S} + f''(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathrm{I}^{2}\mathbf{S} \end{aligned} \tag{42}$$

where, I^{α} indicates fractional integration of order α.

Using the HF domain generalized integration operational matrices in equation (42), we obtain

$$\begin{aligned} &\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}\mathbf{S} + \mathbf{C}_{\mathrm{T}}^{\mathrm{T}}\mathbf{T} + a\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FSS}}^{\alpha-\beta}\mathbf{S} + \mathbf{P}_{\mathrm{FST}}^{\alpha-\beta}\mathbf{T}] + a\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FTS}}^{\alpha-\beta}\mathbf{S} + \mathbf{P}_{\mathrm{FTT}}^{\alpha-\beta}\mathbf{T}] \\ &+ b\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FSS}}^{\alpha-\gamma}\mathbf{S} + \mathbf{P}_{\mathrm{FST}}^{\alpha-\gamma}\mathbf{T}] + b\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FTS}}^{\alpha-\gamma}\mathbf{S} + \mathbf{P}_{\mathrm{FTT}}^{\alpha-\gamma}\mathbf{T}] + c\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FSS}}^{\alpha}\mathbf{S} + \mathbf{P}_{\mathrm{FST}}^{\alpha}\mathbf{T}] \\ &+ c\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FTS}}^{\alpha}\mathbf{S} + \mathbf{P}_{\mathrm{FTT}}^{\alpha}\mathbf{T}] = \mathbf{D}_{\mathrm{S}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FSS}}^{\alpha}\mathbf{S} + \mathbf{P}_{\mathrm{FST}}^{\alpha}\mathbf{T}] + \mathbf{D}_{\mathrm{T}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FTS}}^{\alpha}\mathbf{S} + \mathbf{P}_{\mathrm{FTT}}^{\alpha}\mathbf{T}] + f(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}\mathbf{S} \\ &+ af(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FSS}}^{\alpha-\beta}\mathbf{S} + \mathbf{P}_{\mathrm{FST}}^{\alpha-\beta}\mathbf{T}] + bf(0)\mathbf{U}_{\mathrm{S}}^{\mathrm{T}}[\mathbf{P}_{\mathrm{FSS}}^{\alpha-\gamma}\mathbf{S} + \mathbf{P}_{\mathrm{FST}}^{\alpha-\gamma}\mathbf{T}] \end{aligned}$$

$$+f'(0)\mathbf{U}_S^T[\mathbf{P}_{FSS}^1\mathbf{S}+\mathbf{P}_{FST}^1\mathbf{T}]+af'(0)\mathbf{U}_S^T[\mathbf{P}_{FSS}^{\alpha-\beta+1}\mathbf{S}+\mathbf{P}_{FST}^{\alpha-\beta+1}\mathbf{T}]$$
$$+f''(0)\mathbf{U}_S^T[\mathbf{P}_{FSS}^2\mathbf{S}+\mathbf{P}_{FST}^2\mathbf{T}] \tag{43}$$

Equating the like coefficients of SHF part of equation (43), we have

$$\mathbf{C}_S^T[\mathbf{I}+a\mathbf{P}_{FSS}^{\alpha-\beta}+b\mathbf{P}_{FSS}^{\alpha-\gamma}+c\mathbf{P}_{FSS}^{\alpha}]+\mathbf{C}_T^T[a\mathbf{P}_{FTS}^{\alpha-\beta}+b\mathbf{P}_{FTS}^{\alpha-\gamma}+c\mathbf{P}_{FTS}^{\alpha}]$$
$$=[\mathbf{D}_S^T\mathbf{P}_{FSS}^{\alpha}+\mathbf{D}_T^T\mathbf{P}_{FTS}^{\alpha}]+f(0)\mathbf{U}_S^T[\mathbf{I}+a\mathbf{P}_{FSS}^{\alpha-\beta}+b\mathbf{P}_{FSS}^{\alpha-\gamma}]$$
$$+f'(0)\mathbf{U}_S^T[\mathbf{P}_{FSS}^1+a\mathbf{P}_{FSS}^{\alpha-\beta+1}]+f''(0)\mathbf{U}_S^T\mathbf{P}_{FSS}^2 \tag{44}$$

In equation (44), let us define

$$\mathbf{I}+a\mathbf{P}_{FSS}^{\alpha-\beta}+b\mathbf{P}_{FSS}^{\alpha-\gamma}+c\mathbf{P}_{FSS}^{\alpha}\triangleq\mathbf{W}$$

$$a\mathbf{P}_{FTS}^{\alpha-\beta}+b\mathbf{P}_{FTS}^{\alpha-\gamma}+c\mathbf{P}_{FTS}^{\alpha}\triangleq\mathbf{X}$$

$$[\mathbf{D}_S^T\mathbf{P}_{FSS}^{\alpha}+\mathbf{D}_T^T\mathbf{P}_{FTS}^{\alpha}]+f(0)\mathbf{U}_S^T[\mathbf{I}+a\mathbf{P}_{FSS}^{\alpha-\beta}+b\mathbf{P}_{FSS}^{\alpha-\gamma}]+f'(0)\mathbf{U}_S^T[\mathbf{P}_{FSS}^1+a\mathbf{P}_{FSS}^{\alpha-\beta+1}]$$
$$+f''(0)\mathbf{U}_S^T\mathbf{P}_{FSS}^2\triangleq\mathbf{E}$$

Then equation (44) may be written as

$$\mathbf{C}_S^T\mathbf{W}+\mathbf{C}_T^T\mathbf{X}=\mathbf{E} \tag{45}$$

Rearranging the coefficients of TF part of equation (43), we can write

$$\mathbf{C}_S^T[a\mathbf{P}_{FST}^{\alpha-\beta}+b\mathbf{P}_{FST}^{\alpha-\gamma}+c\mathbf{P}_{FST}^{\alpha}]+\mathbf{C}_T^T[\mathbf{I}+a\mathbf{P}_{FTT}^{\alpha-\beta}+b\mathbf{P}_{FTT}^{\alpha-\gamma}+c\mathbf{P}_{FTT}^{\alpha}]$$
$$=[\mathbf{D}_S^T\mathbf{P}_{FST}^{\alpha}+\mathbf{D}_T^T\mathbf{P}_{FTT}^{\alpha}]+f(0)\mathbf{U}_S^T[a\mathbf{P}_{FST}^{\alpha-\beta}+b\mathbf{P}_{FST}^{\alpha-\gamma}]$$
$$+f'(0)\mathbf{U}_S^T[\mathbf{P}_{FST}^1+a\mathbf{P}_{FST}^{\alpha-\beta+1}]+f''(0)\mathbf{U}_S^T\mathbf{P}_{FST}^2 \tag{46}$$

In equation (46), let us define

$$a\mathbf{P}_{FST}^{\alpha-\beta}+b\mathbf{P}_{FST}^{\alpha-\gamma}+c\mathbf{P}_{FST}^{\alpha}\triangleq\mathbf{Y}$$

$$\mathbf{I}+a\mathbf{P}_{FTT}^{\alpha-\beta}+b\mathbf{P}_{FTT}^{\alpha-\gamma}+c\mathbf{P}_{FTT}^{\alpha}\triangleq\mathbf{Z}$$

$$[\mathbf{D}_S^T\mathbf{P}_{FST}^{\alpha}+\mathbf{D}_T^T\mathbf{P}_{FTT}^{\alpha}]+f(0)\mathbf{U}_S^T[a\mathbf{P}_{FST}^{\alpha-\beta}+b\mathbf{P}_{FST}^{\alpha-\gamma}]+f'(0)\mathbf{U}_S^T[\mathbf{P}_{FST}^1+a\mathbf{P}_{FST}^{\alpha-\beta+1}]$$
$$+f''(0)\mathbf{U}_S^T\mathbf{P}_{FST}^2\triangleq\mathbf{F}$$

So, equation (46) can be expressed as

$$\mathbf{C}_S^T\mathbf{Y}+\mathbf{C}_T^T\mathbf{Z}=\mathbf{F} \tag{47}$$

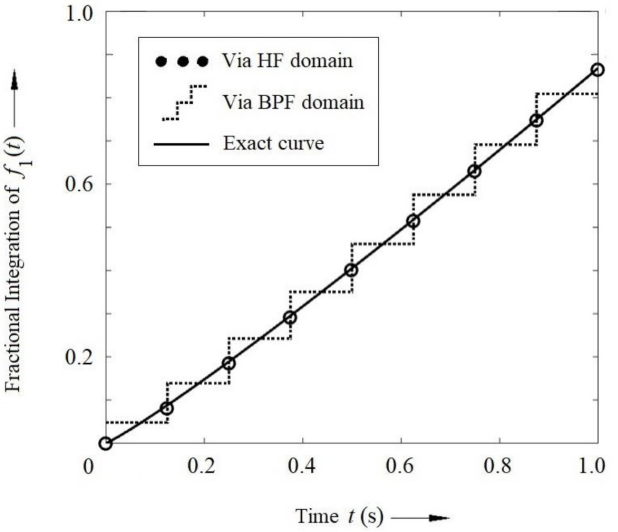


Figure 2

Comparison of $I^{\alpha}[f_1(t)]$ both in HF and BPF domain, for 8 number of segments, T = 1 s and α = 0.4, with its exact integration.

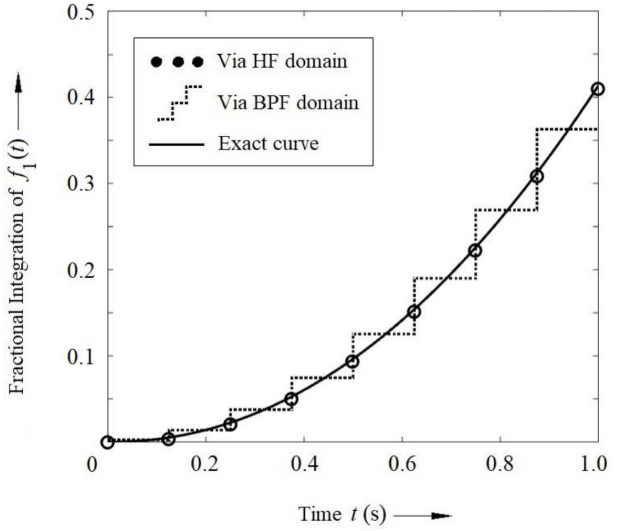


Figure 3

Comparison of $I^{\alpha}[f_1(t)]$ both in HF and BPF domain, for 8 number of segments, T = 1 s and α = 1.4, with its exact integration.

Table 1

Comparing the number of segments (m) needed for integration of $f_1(t)$ to achieve the same range of MISE for the HF and BPF-based approaches.

Integration for the Order of	Fractional operational matrices in the HF domain			Generalized operational matrices in the BPF domain		
	MISE	Required number of segments m	Elapsed time (s)	MISE	Required number of segments m	Elapsed time (s)
0.4	7.67e-06	8	0.130610	7.21e-06	92	21.433658
1.4	2.24e-06	8	0.211263	2.15e-06	76	18.705492

Getting the matrix equations (45) and (47) solved for $\mathbf{C}_{\mathrm{S}}^{\mathrm{T}}$ and $\mathbf{C}_{\mathrm{T}}^{\mathrm{T}}$, we get

$$\mathbf{C}_{\mathrm{S}}^{\mathrm{T}} = [\mathbf{F} - (\mathbf{E} - \mathbf{F}\mathbf{Y}^{-1}\mathbf{W})(\mathbf{X} - \mathbf{Z}\mathbf{Y}^{-1}\mathbf{W})^{-1}\mathbf{Z}]\mathbf{Y}^{-1} \tag{48}$$

and

$$\mathbf{C}_{\mathrm{T}}^{\mathrm{T}} = (\mathbf{E} - \mathbf{F}\mathbf{Y}^{-1}\mathbf{W})(\mathbf{X} - \mathbf{Z}\mathbf{Y}^{-1}\mathbf{W})^{-1} \tag{49}$$

8. Numerical Examples:

Illustration 2: Let us consider an initial value issue of fractional order homogeneous system.

$$\mathrm{D}^{\alpha} f(t) - f(t) = 0, \text{ for } 0 < \alpha < 2 \tag{50}$$

with $f(0) = 1,\ f'(0) = 0.$

The initial condition $f'(0)$ is applicable for the case of $\alpha > 1$. The accurate solution of (50) is

$$f(t) = \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(\alpha k + 1)}.$$

We estimate the answer and contrast it with the precise solution using the technique described in the preceding section. MISE **[21]** is used to compare the findings produced in the block pulse function (BPF) domain and in the orthogonal HF domain to the given fractional differential equation (FDE) developed using the suggested approach. The quantitative comparison of the solutions for the orders of $\alpha = 0.5$ and 1.5, respectively,

together with their MISEs, are shown in **Tables 2(a)** and **(b)**. For two distinct sets of fractional order α, **Figures 4(a)** and **(b)** provide a graphical comparison of the solutions.

Figure 5 shows that the samples (the dots) obtained for different orders using equations (49) and (50) are almost exactly the same as the samples. **Figure 6** compares the changes in mean integral square errors (MISEs) in the fractional order system of **Example 2** for increasing m (m = 5 to 50), with T = 1 sec. It is noted that the MISEs caused by the BPF-based approach are much higher than the approach proposed via HF domain.

Illustration 3: Consider the non-homogeneous Bagley-Torvik equation,

$$\mathrm{D}^2 f(t) + \mathrm{D}^{1.5} f(t) + f(t) = 2 + t^2 + \frac{4}{\Gamma(0.5)} t^{0.5} \tag{51}$$

with $f(0) = 0,\ f'(0) = 0.$

The analytical answer to (51) is as $f(t) = t^2$.

In terms of the MISEs, the outcomes of the suggested HF domain method are contrasted with the outcome of the conventional BPF domain solution. The graphical comparison of **Example 3**'s FDE solution for 8 segments and T = 1 second is displayed in **Figure 7**.

It illustrates how the suggested approach's error contribution to the FDE solution is significantly smaller than that of the BPF-based answer. Furthermore, as observed in the HF domain solution's direct expansion, which is derived from equations (48) and (49), closely resembles the precise curve. It has been observed that the BPF domain approach must take into account m = 180 number of segments ($\mathrm{MISE}_{\mathrm{BPF}}$ = 0.00000343) in order to reach the range of precision ($\mathrm{MISE}_{\mathrm{HF}}$ = 0.00000347) attained by the HF-based approach, whereas the proposed strategy has just taken into consideration m = 8 number of segments.

The ongoing changes in MISE for **Example 3** with T = 1 sec throughout system analysis are shown in **Table 3**. It is seen that the BPF-based strategy results in significantly more errors than the HF domain approach. The cumulative error of the solution based on BPF is identical to the one based on Haar wavelets.

The graphical comparison of MISEs for the BPF and HF domain techniques, taking into account the same number of segments in the analysis of **Example 3** with T = 1 sec, is shown in **Figure 8**. Once more, it should be highlighted that the BPF-based technique causes errors that are far higher than those in the HF domain solution.

Illustration 4: Consider an inhomogeneous fractional order differential equation,

$$\mathrm{D}^2 f(t) + \mathrm{D}^{0.5} f(t) + f(t) = t^3 + 6t + \frac{3.2}{\Gamma(0.5)} t^{2.5} \tag{52}$$

with $f(0) = 0$ and $f'(0) = 0$, having an accurate analytical answer of $f(t) = t^3$.

Table 2

The HF domain answer of Example 2's fractional order system is compared to the actual samples and the MISE, with m = 4, for T = 1 s.

(a) for the order of α = 0.5

Time t (s)	Exact solution	Using HF domain solution [Equations (48) and (49)]		Using BPF domain solution [21]	
		Samples from HF analysis	MISE	Samples from BPF analysis	MISE
0.00	1.00000000	1.00000000	0.00327212	1.60288876	0.08650595
0.25	1.95235837	1.90433314		2.40345060	
0.5	2.77417952	2.73058885		3.32444887	
0.75	3.76575711	3.72472451		4.46618573	
1.00	5.00327630	4.96998568			

(b) for the order of α = 1.5

Time t (s)	Exact solution	Using HF domain solution [Equations (48) and (49)]		Using BPF domain solution [21]	
		Samples from HF analysis	MISE	Samples from BPF analysis	MISE
0.00	1.00000000	1.00000000	0.00001537	1.03908264	0.00551744
0.25	1.09667342	1.09770660		1.18758783	
0.5	1.28766128	1.29032018		1.42302016	
0.75	1.56440568	1.56933313		1.75043350	
1.00	1.93948726	1.94761595			

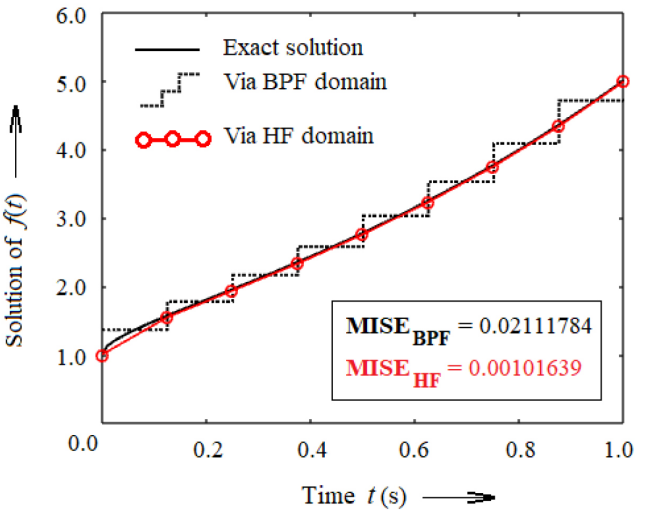


Figure 4 (a)

Example 2's exact solution for 8 segments, T = 1 s, and α = 0.5 is compared to the HF domain and BPF domain solutions.

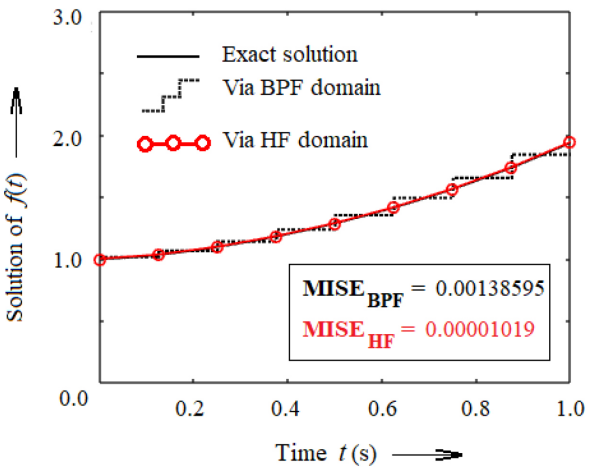


Figure 4(b)

Example 2's exact solution for 8 segments, T = 1 s, and α = 1.5 is compared to the HF domain and BPF domain solutions.

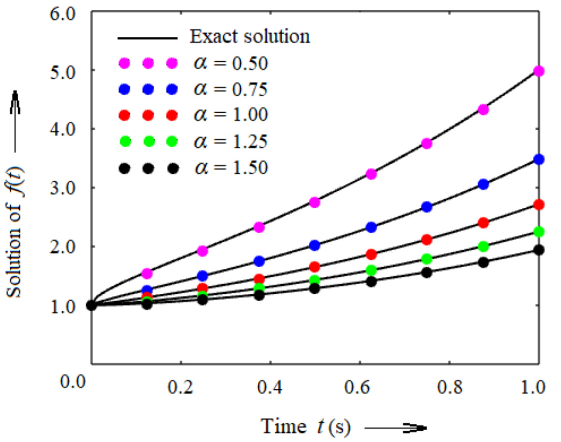


Figure 5

A comparison between the Example 2's precise answer and the HF domain solution for 8 segments, T = 1 s, and few choices of α (= 0.5, 0.75, 1.0, 1.25, and 1.5). The HF domain technique yields samples that are extremely close to the exact answer for all values.

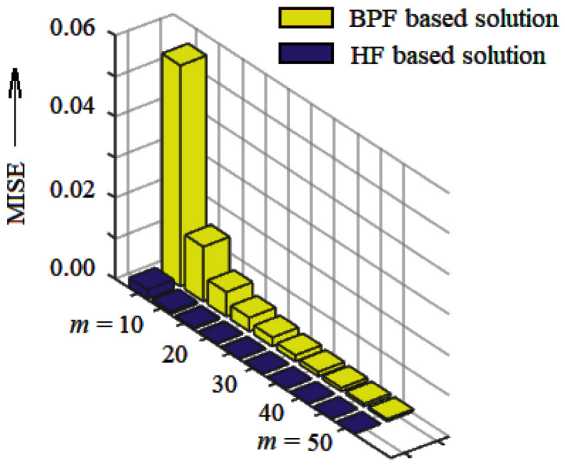


Figure 6

Example 2's HF and BPF domain solution, shows changes in MISE with increasing value of m for T = 1 s, with α = 0.5. It demonstrates that, in comparison to BPF-based solutions, the HF-based technique yields substantially more accurate results.

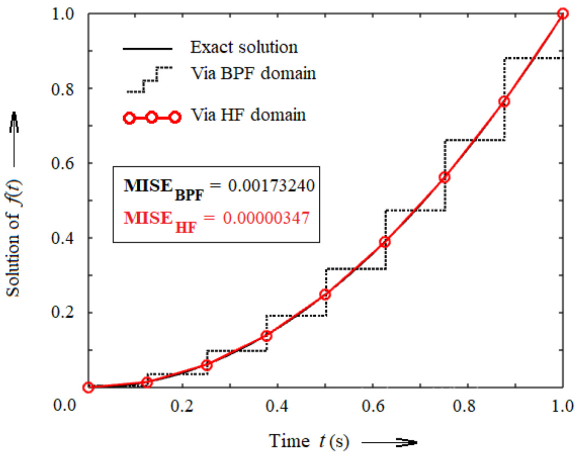


Figure 7

A visual comparison between Example 3's exact answer and the HF domain and BPF domain solutions for 8 segments and T = 1 second.

This problem was discussed in **[22]** and **[23]**, which has been investigated here with the approach presented in this paper and observed some important facts to mention.

The solution obtained using HF based approach is compared with the solution using Haar wavelet collocation method **[22]**, and through other six methods discussed in **[23]**. Both the orthogonal functions, *i.e.* the BPF and Haar wavelet provides staircase approximations of time functions, and they are connected through similarity matrix. Thus, The results obtained using the BPF method will be same as the solution obtained using Haar wavelet. So only one solution out of the BPF and Haar wavelet approach is displayed in **Table 2**.

Lastly, **Table 2** shows the quantitative comparison of absolute errors resulting from solving the fractional differential equation (FDE) using the six approaches discussed in **[23]**, the orthogonal HF based approach, and the Haar wavelet collocation method.

Table 2 shows that, of all the strategies offered, the HF-based strategy has the lowest absolute error. Put otherwise, we can deduce that the orthogonal HF based strategy needs fewer segments to achieve a comparable range of accuracy when solving the fractional differential equation in **Example 4**.

Figure 9 displays the graphical comparison between solutions achieved using the HF and BPF-based approaches and the FDE's exact solution.

Table 3

MISE's comparison for the BPF and HF based analysis of the system of Example 3, for different number of number of sub-intervals with T = 1 sec.

No. of sub-intervals used, m	MISE using BPF approach	Elapsed time (in sec)	MISE using HF approach	Elapsed time (in sec)	Remarks
4	0.00688341	0.821165	0.00005826	0.064507	$MISE_{HF}$ is always less than $MISE_{BPF}$
10	0.00110960	1.706859	0.00000141	0.102023	
20	0.00027769	2.927566	0.00000009	0.288284	
30	0.00012344	4.133522	0.00000002	0.596860	
40	0.00006944	5.633020	0.00000000	0.885054	
50	0.00004444	6.657734	0.00000000	1.313799	

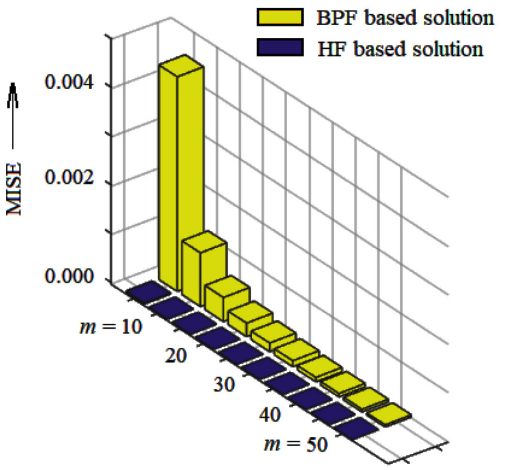


Figure 8

Variations in MISE as the value of m increases in Example 3's HF and BPF domain solution with its exact solution, for T = 1 s. It demonstrates that, in comparison to BPF-based solutions, the HF-based technique yields substantially more accurate results.

Table 2

Quantitative comparison of absolute error on analyzing system of Example 4 using proposed HF domain approach with other methods presented in [22] and [23], for $T = 1$ sec.

h	HF based Proposed Approach	Haar Wavelet Collocation Method [21]	Other methods in [22]					
			Method 1a	Method 1b	Method 2	Method 3	Method 1a (2)	Method 3 (2)
1/32	5.9810e –08	7.6183e –06	1.86e –02	1.60e –02	1.27e –03	1.86e –02	4.61e –04	4.61e –04
1/64	3.7402e –09	9.5319e –07	6.62e –03	5.85e –03	4.32e –04	6.62e –03	8.96e –05	8.96e –05
1/128	2.3380e –10	1.1918e –07	2.35e –03	2.12e –03	1.49e –04	2.35e –03	1.80e –05	1.80e –05
1/256	1.4613e –11	1.4900e –08	8.36e –04	7.59e –04	5.14e –05	8.36e –04	3.76e –06	3.76e –06
1/512	4.7078e –12	1.8626e –09	2.96e –04	2.71e –04	1.79e –05	2.96e –04	8.14e –07	8.14e –07

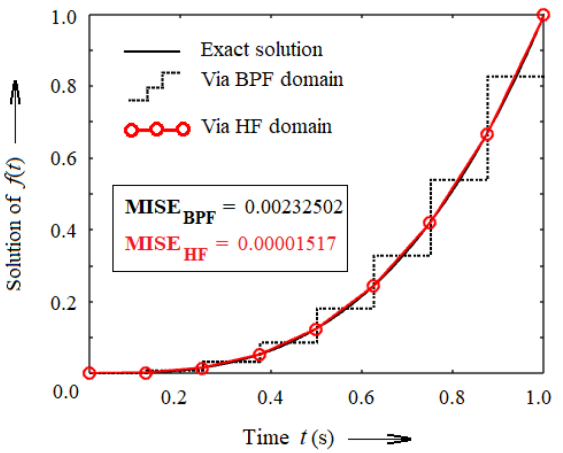


Figure 9

Example 4's HF domain and BPF domain solutions are compared to its exact solution for 8 segments and $T = 1$ second.

9. Conclusion

A generalized formulation of the hybrid function based operational matrices for integration of fractional order has been derived. These matrices are applied to find a mostly accurate solution of the fractional order systems under study. A few cases are thoroughly examined and contrasted with alternative approaches that have been previously

suggested by various researchers. The effectiveness of the suggested strategy has been demonstrated in the orthogonal hybrid function domain, which combines triangular and sample-and-hold function sets.

The solution obtained using the proposed method has challenged all other approaches by its (i) easier algorithm, (ii) lesser mathematical burden, (iii) require much lesser elapsed time, (iv) a sample-based analysis, (v) easier to implement digitally, and the most importantly (vi) comparatively providing the best accuracy over the previously presented approaches. The entire study presents a problem-solving mechanism that exemplifies the effectiveness of the proposed technique. Future directions for this approach could include fractional order optimum control, parameter estimation of fractional-order systems, and many more.

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