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Exploring novel Fermat polynomials and their transformative properties

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Abstract

In this research, we present a novel family of polynomials that are derived from Fermat numbers. Alongside these newly introduced polynomials, we provide a series of important identities and properties, including the Binet formula and generating functions, which are fundamental to understanding their structure. We also show that Fermat polynomials can be efficiently expressed using matrix representations, enabling an exploration of fundamental properties derived from the invariance of the determinants of these matrices. This approach

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provides new perspectives on the internal connections within these polynomials. In addition, many different transforms have been considered with the help of polynomials of these new number sequences, and these transforms can play a role in many areas from secure key generation to differential equation solutions. These transformations shed light on the deeper structural properties of the Fermat polynomials and reveal intricate connections between these polynomials and other well-established mathematical constructs, enhancing our understanding of their broader implications in the field of number theory and beyond.

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1. Introduction

This research focuses on sequences of positive integers, particularly Fermat (and Mersenne) integer sequences that adhere to specific recurrence relations. We present several established identities linked to these sequences and their related polynomials [1]. Among all integer sequences, Fibonacci (and Lucas) sequences are arguably the most renowned. Prominent researchers, including Hoggatt and Koshy, along with numerous others, have extensively analyzed Fibonacci sequences and polynomials derived from integer sequences and developed innovative formulas that link Mersenne and Fermat numbers with a class of generalized Fibonacci numbers. Additionally, the booklet introduces the interesting properties of Fibonacci and Lucas numbers, offering students a chance to explore how numerous mathematical generalizations can arise from simple concepts. [2,3]. This study builds upon some of their foundational findings. In a recent study [4], researchers established new connections between Mersenne and Fermat numbers and a class of generalized Fibonacci numbers, which includes familiar sequences like Fibonacci and Pell numbers. By investigating the relationships between Fermat and generalized Fibonacci polynomials, they derived these new formulas. Their approach involved using power series representations and inversion techniques.

This study focuses on Fermat numbers and the polynomials associated with them, which adhere to second-order recurrence relations. We establish key identities for these numbers and highlight the notable properties of both Fermat numbers and their corresponding polynomials. Furthermore, we present generating functions, the Binet formula, and summation formulas that are connected to these polynomials. The study also examines matrix representations of Fermat polynomials, with a particular focus on the determinants of these matrices. The Hankel transform, attributed to Herman Hankel and also referred to as the Fourier-Bessel transform, frequently appears in combinatorial analysis through the computation of Hankel determinants. Barry [5] conducted research on the Hankel, Catalan, and binomial transform of these sequences and polynomials, producing significant insights. This approach provides new perspectives on the internal connections within these polynomials. In addition, many different transforms have been considered with the help of polynomials belonging to these new number sequences defined, and these transforms are Cryptography: Analysis of sequences and secure key generation.

Data Compression: Development of compression algorithms by taking advantage of the structural properties of sequences. Sequential Analysis: Investigation of transformation properties of special sequences such as Fibonacci, Lucas, Fermat or Mersenne.

Generating Functions: Mathematical modeling and solution of differential equations motivated by these findings, we analyze these transformations within the framework of Fermat number sequences and their polynomials, determining explicit coefficient expressions for their second-order recurrence relations.

Tsang, Theorey, and Stein [6] focused on numbers of the form $2^{2^n}+1$, known as Fermat numbers, named after Pierre de Fermat (1601–1665), they initially explored them. Although the question of whether there are infinitely many primes of this form remains unresolved, their paper examined fundamental properties of Fermat numbers, including their primality and divisibility. Additionally, numbers of the form $2^n-1$are briefly discussed. These numbers play a crucial role in various mathematical fields, particularly number theory and combinatorics. Despite their significance, research on Fermat numbers remains relatively sparse. This study aims to pave the way for future research on Fermat numbers and their polynomials. Similarly, Mersenne numbers, defined as $\mathcal{M}_n=2^n-1$, [7,8]. Kuloğlu [9] a novel class of numbers and polynomials, Gaussian Fermat numbers and polynomials presented. This research delved into their properties, connections to other number sequences, and establishes several new identities. Also, author introduced a novel mathematical construct: Gaussian Fermat numbers and polynomials. This paper examined their mathematical properties, including relationships with other number sequences and the derivation of various identities. In [10], authors present the hyperbolic k-Mersenne and k-Mersenne-Lucas octonions and explore their algebraic properties. They provide several interrelations, including well-known identities.

Mersenne numbers $\mathcal{M}_n$ can be given by

$$\mathcal{M}_n=3\mathcal{M}_{n-1}-2\mathcal{M}_{n-2}\,, n\geq 2 \tag{1}$$

with $\mathcal{M}_0=0, \mathcal{M}_1=1$.

Mersenne polynomials $\mathcal{M}(x)$ can be defined by

$$\mathcal{M}_n(x)=3x\mathcal{M}_{n-1}(x)-2\mathcal{M}_{n-2}(x)\ n\geq 2 \tag{2}$$

with $\mathcal{M}_0(x)=0, \mathcal{M}_1(x)=1$.

We get Mersenne numbers putting $x=1$ in (2).

Some terms of $\mathcal{M}_n(x)$ are as follows:

$$\mathcal{M}_2(x)=3x$$
$$\mathcal{M}_3(x)=9x^2-2$$
$$\mathcal{M}_4(x)=27x^3-12x$$

The characteristic equation of $\mathcal{M}_n(x)$ is

$$x^2 - 3x + 2 = 0.$$

It has the following roots

$$p(x) = \frac{3x - \sqrt{9x^2 - 8}}{2}$$

$$q(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}.$$

It can be easily seen that $p(1) = 1$ and $q(1) = 2$.

Binet Formula for $\mathcal{M}_n(x)$ is

$$\mathcal{M}_n(x) = 2^n - 1.$$

The interest in integer sequences, which began with the discovery of the Fibonacci sequence and many generalizations of which have been defined, continues to increase as these sequences find applications in other scientific fields. Many properties of sequences have been given by establishing relationships with concepts such as matrices, polynomials and quaternions. One of these sequences is the Fermat sequence.

In this study, we introduce a new kind of polynomials that are derived from Fermat numbers, expanding the existing research on these numbers. The proposed Fermat polynomials exhibit several interesting properties and identities, which include well-known formulations such as the Binet formula and their corresponding generating functions. These findings enrich the mathematical understanding of Fermat numbers and their associated polynomials by providing a deeper insight into their structure and behavior.

Furthermore, we explore the matrix representations of Fermat polynomials, highlighting the significance of the determinants of these matrices. By examining the constancy of these determinants, we reveal important propositions that lead to a more profound comprehension of the algebraic properties of Fermat polynomials. This matrix approach offers an alternative perspective on how these polynomials can be analyzed and manipulated in various mathematical contexts.

In addition to these foundational results, we also investigate the application of several transformations to the Fermat sequence and polynomials. Specifically, we focus on the Hankel and Catalan transformations, which play crucial roles in the study of sequences and polynomials. These transformations allow us to explore the relationships between Fermat polynomials and other well-known mathematical constructs, opening up potential avenues for further research in both theoretical and applied mathematics.

To avoid repetition and to shorten the length of the equations, we will denote the left side of the equations with LHS.

2. New fermat polynomials

Fermat numbers have drawn the interest of many mathematicians [11]. The following recurrence relation pertains to Fermat numbers.

$$\mathcal{F}_n = 3\mathcal{F}_{n-1} - 2\mathcal{F}_{n-2}, n \geq 2 \tag{3}$$

with $\mathcal{F}_0 = 2, \mathcal{F}_1 = 3$.

Definition 2.1: The family of Fermat polynomials, $\mathcal{F}_n(x)$, is defined by a second order recurrence relation as given in (1) and we can define these polynomial as

$$\mathcal{F}_n(x) = 3x\mathcal{F}_{n-1}(x) - 2\mathcal{F}_{n-2}(x), n \geq 2 \tag{4}$$

with $\mathcal{F}_0(x) = 2, \mathcal{F}_1(x) = 3$.

We get Fermat numbers putting $x = 1$ in (3).

We show some terms of $\mathcal{F}_n(x)$ as follows:

$$\mathcal{F}_2(x) = 9x - 4$$
$$\mathcal{F}_3(x) = 27x^2 - 12x - 6$$
$$\mathcal{F}_4(x) = 81x^3 - 36x^2 - 36x + 8$$
$$\mathcal{F}_5(x) = 243x^4 - 108x^3 - 162x^2 + 48x + 12$$

The characteristic equation of $\mathcal{F}_n(x)$ is

$$x^2 - 3x + 2 = 0.$$

It has the roots as

$$a(x) = \frac{3x - \sqrt{9x^2 - 8}}{2}$$
$$b(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}.$$

It can be easily seen that $a(1) = 1$ and $b(1) = 2$.

Theorem 2.2: (Binet Formula)

Binet formula for $\mathcal{F}_n(x)$

$$\mathcal{F}_n(x) = c_1 a^n(x) + c_2 b^n(x)$$

where

$$c_1 = \frac{2b(x) - 3}{b(x) - a(x)},$$
$$c_2 = \frac{-2a(x) + 3}{b(x) - a(x)}.$$

Proof: For $n = 0$ we have

$$\mathcal{F}_0(x) = c_1 + c_2 = 2$$

For $n = 1$ we have

$$\mathcal{F}_1(x) = c_1 a(1) + c_2 b(1) = 3,$$

where $a(1) = 1$ and $b(1) = 2$.

If we solve equations then we can find the coefficients c_1, c_2.

For $x = 1$ we easily get the Binet formula for $\mathcal{F}_n(x)$

$$\mathcal{F}_n(x) = 2^n + 1$$

Theorem 2.3: $\mathcal{F}(x)$ *has the following the generating function*

$$\mathcal{F}(x) = \frac{-3x+2}{2x^2-3x+1}$$

Proof:

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} f_n(x).x^n = f_0(x) + f_1(x)x + \cdots + f_n(x)x^n + \cdots$$

$$\mathcal{F}(x) = f_0(x) + f_1(x)x + \sum_{n=2}^{\infty} f_n(x).x^n$$

$$= f_0(x) + f_1(x)x + \sum_{n=2}^{\infty}(3f_{n-1}(x) - 2f_{n-2}(x))x^n$$

$$= f_0(x) + f_1(x)x + 3(f_1(x)x^2 + f_2(x)x^3 + \cdots) - 2(f_0(x)x^2 + f_1(x)x^3 + \cdots)$$

$$= f_0(x) + f_1(x)x + 3x(f_1(x)x^1 + f_2(x)x^2 + \cdots) - 2x^2(f_0(x) + f_1(x)x^1 + \cdots)$$

$$\mathcal{F}(x) = 2 + 3x + 3x(\mathcal{F}(x) - f_0(x)) - 2x^2.\mathcal{F}(x)$$

$$(2x^2 - 3x + 1)\mathcal{F}(x) = 2 - 3x$$

and finally, we get the generating function for $\mathcal{F}(x)$ as follows

$$\mathcal{F}(x) = \frac{-3x+2}{2x^2-3x+1}$$

Theorem 2.4: *The exponential generating function for* $\mathcal{F}(x)$

$$\sum_{n=0}^{\infty} \frac{f_n(x)x^n}{n!} = c_1 e^{ax} + c_2 e^{bx}$$

where $f_n(x) = c_1 a^n + c_2 b^n$.

Proof: If we use Binet formula, we have

$$LHS = \sum_{n=0}^{\infty} \frac{(c_1 a^n + c_2 b^n)x^n}{n!}$$

$$= c_1 \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} + c_2 \sum_{n=0}^{\infty} \frac{(bx)^n}{n!}$$

We know that

$$e^{ax} = \sum_{n=1}^{\infty} \frac{(ax)^n}{n!},$$

$$e^{bx} = \sum_{n=1}^{\infty} \frac{(bx)^n}{n!},$$

so,

$$LHS = c_1 e^{ax} + c_2 e^{bx}$$

Theorem 2.5: (Catalan Identity) *Catalan's identity relates to sums of binomial coefficients and is commonly used in combinatorics. It expresses a relationship between two sums of binomial coefficients, leading to simplifications in various problems.*

For $n \geq r$, we have

$$\mathcal{F}_{n-r}(x)\mathcal{F}_{n+r}(x) - \mathcal{F}_n^2(x) = c_1 c_2 2^{n-r}(a^{2r} + b^{2r} - 2^{r+1})$$

Proof:

$$LHS = (c_1 a^{n-r} + c_2 b^{n-r})(c_1 a^{n+r} + c_2 b^{n+r}) - (c_1 a^n + c_2 b^n)^2$$

$$= 2c_1 c_2 a^n b^n - c_1 c_2 a^{n-r} b^{n+r} - c_1 c_2 a^{n+r} b^{n-r}$$

$$= c_1c_2 2^{n-r}(a^{2r} + b^{2r} - 2^{r+1})$$

So, the desired is obtained.

Theorem 2.6: (Cassini Identity)

$$\mathcal{F}_{n-1}(x)\mathcal{F}_{n+1}(x) - \mathcal{F}_n^2(x) = c_1c_2 2^{n-1}(9k^2 - 8)$$

Proof: By setting $r = 1$ in the Catalan identity, it reduces to the Cassini identity.

Theorem 2.7: (D'ocagne's Identity) *This identity relates to binomial coefficients and is used in number theory and combinatorics. It is an identity involving sums and products of binomial coefficients and can be used to derive other mathematical identities.*

Let $m, n \in \mathbb{Z}$.

$$\mathcal{F}_n(x)\mathcal{F}_{m+1}(x) - \mathcal{F}_m(x)\mathcal{F}_{n+1}(x) = -c_1c_2(a-b)(a^nb^m - a^mb^n)$$

Proof:

$$\begin{aligned} LHS &= (c_1a^n + c_2b^n)(c_1a^{m+1} + c_2b^{m+1}) \\ &\quad - (c_1a^m + c_2b^m)(c_1a^{n+1} + c_2b^{n+1}) \\ &= c_1c_2a^nb^{m+1} + c_1c_2a^{m+1}b^n - c_1c_2a^mb^{n+1} - c_1c_2a^{n+1}b^m \\ &= c_1c_2a^nb^m(b-a) + c_1c_2a^mb^n(a-b) \\ &= -c_1c_2(a-b)(a^nb^m - a^mb^n) \end{aligned}$$

Theorem 2.8: (Honsberger Identity) *Honsberger's identity is a formula involving sums of products of binomial coefficients and has applications in combinatorial mathematics. It is useful for simplifying complex binomial expressions and finding closed forms for sums.*

Let n and m be any integers.

$$\mathcal{F}_{m-1}(x)\mathcal{F}_n(x) + \mathcal{F}_m(x)\mathcal{F}_{n+1}(x) = {c_1}^2a^{m+n-1}(1+a^2) + {c_2}^2b^{m+n-1}(1+b^2) + 3c_1c_2(a^{m-1}b^n - a^nb^{m-1})$$

Proof:

$$\begin{aligned} LHS &= (c_1a^{m-1} + c_2b^{m-1})(c_1a^n + c_2b^n) \\ &\quad - (c_1a^m + c_2b^m)(c_1a^{n+1} + c_2b^{n+1}) \\ &= {c_1}^2a^{m+n-1} + c_1c_2a^{m-1}b^n + c_1c_2b^{m-1}a^n + {c_2}^2b^{m+n+1} \\ &\quad + c_1c_2a^mb^{n+1} + c_1c_2a^{n+1}b^m + {c_2}^2b^{m+n+1} \end{aligned}$$

$$= {c_1}^2a^{m+n-1}(1+a^2) + {c_2}^2b^{m+n-1}(1+b^2) + 3c_1c_2(a^{m-1}b^n - a^nb^{m-1})$$

Theorem 2.9: (Vajda Identity) *Vajda's identity is related to sequences and sums involving binomial coefficients, often used in number theory. It provides a powerful tool for transforming and simplifying expressions in combinatorial contexts.*

Let n and m be any integers.

$$\mathcal{F}_{n+r}(x)\mathcal{F}_{n+s}(x) + \mathcal{F}_n(x)\mathcal{F}_{n+r+s}(x) = 2^nc_1c_2(a^rb^s + a^sb^r - b^{r+s} - a^{r+s}$$

Proof:

$$LHS = (c_1a^{n+r} + c_2b^{n+r})(c_1a^{n+s} + c_2b^{n+s})$$
$$-(c_1a^n + c_2b^n)(c_1a^{n+r+s} + c_2b^{n+r+s})$$
$$= c_1c_2a^{n+r}b^{n+s} + c_1c_2a^{n+s}b^{n+r} - c_1c_2a^nb^{n+r+s} - c_1c_2a^{n+r+s}b^n$$
$$= 2^nc_1c_2(a^rb^s + a^sb^r - b^{r+s} - a^{r+s})$$

So, the proof is complete.

Theorem 2.10: *The connection between Fermat polynomials and Mersenne polynomials is expressed as follows:*

$$\mathcal{F}_n(x) = 3M_n(x) - 4M_{n-1}(x).$$

Proof: The proof can be easily seen by substituting the Binet formula for Mersenne polynomials.

Theorem 2.11: *Let $n \to \infty, b > a$. The ratio for the Fermat polynomials, with the larger term divided by the smaller, is expressed as follows:*

$$\lim_{n\to\infty} \frac{\mathcal{F}_{n+1}(x)}{\mathcal{F}_n(x)} = b.$$

Proof: The Fermat polynomials' Binet formula is derived as follows:

$$\lim_{n\to\infty} \frac{\mathcal{F}_{n+1}(x)}{\mathcal{F}_n(x)} = \frac{a^{n+1}c_1 + b^{n+1}c_2}{a^nc_1 + b^nc_2} = \frac{b^{n+1}\left(c_2 + \left(\frac{a}{b}\right)^{n+1} c_1\right)}{b^n\left(c_2 + \left(\frac{a}{b}\right)^{n} c_1\right)} = b.$$

Theorem 2.12: *The sum for the Fermat polynomials is given*

$$\sum_{s=0}^{n} \mathcal{F}_s(x) = \frac{2\mathcal{F}_n(x) - 3x\mathcal{F}_n(x) + 2\mathcal{F}_{n-1}(x) + 5 - 6x}{1 - 3x + 2}$$

Proof: We have

$$\mathcal{F}_2(x) = 3x\mathcal{F}_1(x) - 2\mathcal{F}_0(x)$$
$$\mathcal{F}_3(x) = 3x\mathcal{F}_2(x) - 2\mathcal{F}_1(x)$$
$$\vdots$$
$$\mathcal{F}_n(x) = 3x\mathcal{F}_{n-1}(x) - 2\mathcal{F}_{n-2}(x)$$

so, we have

$$-5 + \sum_{s=0}^{n} \mathcal{F}_s(x) = 3x\left[-2 - \mathcal{F}_n(x) + \sum_{s=0}^{n} \mathcal{F}_s(x)\right]$$
$$-2\left[-\mathcal{F}_n(x) - \mathcal{F}_{n-1}(x) + \sum_{s=0}^{n} \mathcal{F}_s(x)\right]$$
$$\sum_{s=0}^{n} \mathcal{F}_s(x)\,[1 - 3x + 2] = -6x - 3x\mathcal{F}_n(x) + 2\mathcal{F}_n(x) + 2\mathcal{F}_{n-1}(x) + 5$$

thus, we get

$$\sum_{s=0}^{n} \mathcal{F}_s(x) = \frac{2\mathcal{F}_n(x) - 3x\mathcal{F}_n(x) + 2\mathcal{F}_{n-1}(x) + 5 - 6x}{1 - 3x + 2}.$$

Definition 2.13: Fermat polynomials matrix $\mathcal{f}_n(x)$ is defined by

$$\not{f}_n(x) = \begin{pmatrix} 9x-4 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \mathcal{F}_{n+2}(x) & -2\mathcal{F}_{n+1}(x) \\ \mathcal{F}_{n+1}(x) & -2\mathcal{F}_n(x) \end{pmatrix}, n \geq 0.$$

For example,

For $n = 2$,

$$\begin{aligned} \not{f}_2(x) &= \begin{pmatrix} 9x-4 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 9x^2-2 & -6x \\ 3x & -2 \end{pmatrix} \\ &= \begin{pmatrix} 81x^3 - 36x^2 - 36x + 8 & -54x^2 + 24x + 12 \\ 27x^2 - 12x - 6 & -18x + 8 \end{pmatrix}. \end{aligned}$$

Proposition 2.14: For $n \geq 0$, we have

$$\det(\not{f}_n(x)) = 2^n(-36x + 34).$$

Proof: We can prove this by using the method of induction on n, such that:

For $n = 1$, we have

$$\left| \begin{pmatrix} 9x-4 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix}^1 \right| = 2(-36x + 34).$$

For $n = k$, assume that the equation holds. That is;

$$\left| \begin{pmatrix} 9x-4 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix}^k \right| = 2^k(-36x + 34).$$

Then $n = k + 1$, we have

$$\begin{aligned} \left| \begin{pmatrix} 9x-4 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix}^1 \right| &= 2.2^k(-36x + 34) \\ &= 2^{k+1}(-36x + 34). \end{aligned}$$

as desired.

Proposition 2.15: For $n \geq 0$ we have

$$\det(\not{f}_n^2(x)) + \det(\not{f}_{n+1}^2(x)) = 5.2^{2n}(-36x + 34)^2.$$

Proof: We get

$$\begin{aligned} \det(\not{f}_n(x)) \det(\not{f}_n(x)) &+ \det(\not{f}_{n+1}(x)) \det(\not{f}_{n+1}(x)) \\ &= 2^{2n}(-36x + 34)^2 + 2^{2n+2}(-36x + 34)^2 \\ &= 2^{2n}(-36x + 34)^2(1 + 4) = 5.2^{2n}(-36x + 34)^2. \end{aligned}$$

Proposition 2.16: For $n, m \geq 0$ we have

$$\begin{aligned} \det(\not{f}_{m+1}(x)) \det(\not{f}_{n+1}(x)) &+ \det(\not{f}_m(x)) \det(\not{f}_n(x)) \\ &= 5.2^{m+n}(-36x + 34)^2. \end{aligned}$$

Proof: The proof is clear from the definition the determinant.

Proposition 2.17: For $n \geq 0$ we have

$$\det(\not{f}_n^n(x)) + \det(\not{f}_{n+1}^n(x)) = 2^{n^2}(-36x + 34)^n(M_n)$$

$$\det\left(f_n^n(x)\right)+\det\left(f_{n+1}^n(x)\right)=2^{n^2}(-36x+34)^n(M_n)$$

where (M_n), n[th] Mersenne number.

Proof: From the definition of Fermat polynomials determinant proof is clear.

3. Hankel Transform

Fermat integer sequence is $\{2,3,5,9,17,33,65,129,257,\ldots\}$. The Hankel matrix of F Fermat sequences is infinite matrix as

$$F=\begin{bmatrix} b_1 & b_2 & \ldots \\ b_2 & b_3 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

with elements b_i which are Fermat numbers. The Hankel matrix of F_n of order n is the upper-left $n\times n$ submatrix of F, and f_n, the Hankel determinant of F_n , is the determinant of the coresponding Hankel matrix, $f_n=\det(F_n)$. Cvetkovic, Rajkovic, and Ivkovic [12] calculated the Catalan numbers, the Hankel transform of Fibonacci numbers. Inspired that study we present first four Hankel matrices of Fermat sequences as follow:

$$F_1=[2], F_2=\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, F_3=\begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix}, F_4=\begin{bmatrix} 2 & 3 & 5 & 9 \\ 3 & 5 & 9 & 17 \\ 5 & 9 & 17 & 33 \\ 9 & 17 & 33 & 65 \end{bmatrix},$$

First four terms of Hankel transform of Fermat integer sequences are $f_1=2, f_2=1, f_3=0, f_4=0$. It can be calculated that other terms of this sequence equal 0. Ehrenbourg [13] calculated the Hankel determinant of exponential polynomials. Peart and Woan [14] have studied generating functions via Hankel and Stieltjes matrices. Using these results in [12] and [13] we calculated the Hankel determinant of Fermat numbers and its polynomials.

3.1 *Hankel Transformation of Fermat Polynomials*

Fermat polynomial sequence is $\{2,3,9x-4,27x^2-12x-6,81x^3-36x^2-36x+8,\ldots\}$. The Hankel matrix of F Fermat polynomial sequences is infinite matrix as

$$F(x)=\begin{bmatrix} b_1(x) & b_2(x) & \ldots \\ b_2(x) & b_3(x) & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

with elements $b_i(x)$ which are Fermat polynomials. The Hankel matrix of $F_n(x)$ of order n is the upper-left $n\times n$ submatix of $F(x)$, and $f_n(x)$, the Hankel determinant of F_n , is the determinant of the coresponding Hankel matrix, $f_n(x)=\det(F_n(x))$. First three Hankel matrices of Fermat polynomials are,

$$F_1(x)=[2], F_2(x)=\begin{bmatrix} 2 & 3 \\ 3 & 9x-4 \end{bmatrix},$$

$$F_n(x)=\begin{bmatrix}\mathcal{F}_0(x) & \mathcal{F}_1(x) & \dots & \mathcal{F}_{n-1}(x)\\ \mathcal{F}_1(x) & \mathcal{F}_2(x) & \dots & \mathcal{F}_n(x)\\ \dots & \dots & \dots & \dots\\ \mathcal{F}_{n-1}(x) & \mathcal{F}_n(x) & \dots & \mathcal{F}_{2n-2}(x)\end{bmatrix}.$$

First three terms of Hankel transform of Fermat sequences are $f_1(x)=2, f_2(x)=18x-17, f_3(x)=0$. It can be calculated that other terms of this sequence equal 0.

4. Catalan and binomial transformation

In [5], Barry introduced the Catalan transform. Furthermore, Özkan and Taştan [15] studied the Catalan transform applied to k-Jacobsthal sequence. The Catalan numbers $C(n)$ are defined by:

$$C_n=\frac{1}{n+1}\binom{2n}{n},$$

or

$$C_n=\frac{(2n)!}{(n+1)!n!}$$

We know the following relation

$$\frac{C_{n+1}}{C_n}=\frac{2(2n+1)}{n+2}$$

The ordinary generating function (o.g.f.) $c(x)$ as follows:

$$c(x)=\frac{1-\sqrt{1-4x}}{2x}.$$

$$c(x)=1+x+2x^2+5x^3+14x^4+\cdots$$

Now, we can give give the Catalan transform of Fermat sequence $(\mathcal{F}_n)$ as

$$C\mathcal{F}_n=\sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}\mathcal{F}_i$$

for $n\geq 1$ with $C\mathcal{F}_0=2$.

First few terms of $C\mathcal{F}_n$ as follow:

$$C\mathcal{F}_1=3,\ C\mathcal{F}_2=8,\ C\mathcal{F}_3=25,\ C\mathcal{F}_4=84$$

The sequence obtained here $\{3,8,25,84,\ldots\}$: indexed in OEIS as A038665. This numbers also is the convolution of A007054 with A000984 [16].

We give the Binomial transform of Fermat sequences $(\mathcal{F}_n)$ as

$$B\mathcal{F}_n=\sum_{j=0}^{n}\binom{n}{j}\mathcal{F}_j.$$

for $n\geq 0$ with $B\mathcal{F}_0=2, B\mathcal{F}_1=5$.

First few terms of $B\mathcal{F}_n$ as follow:

$$B\mathcal{F}_2=\sum_{j=0}^{2}\binom{2}{j}\mathcal{F}_j=13,$$

$$B\mathcal{F}_3=\sum_{j=0}^{3}\binom{3}{j}\mathcal{F}_j=35,$$

$$B\mathcal{F}_4 = \sum_{j=0}^{4} \binom{4}{j} \mathcal{F}_j = 97,$$
$$B\mathcal{F}_5 = \sum_{j=0}^{5} \binom{5}{j} \mathcal{F}_j = 275.$$

The Binomial transform of Fermat sequences may be represented in the form of a matrix product as shown below:

$$\begin{bmatrix} B\mathcal{F}_0 \\ B\mathcal{F}_1 \\ B\mathcal{F}_2 \\ B\mathcal{F}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \begin{bmatrix} \mathcal{F}_0 \\ \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 5 \\ 13 \\ 35 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ \vdots \end{bmatrix}.$$

The Catalan transform of $(\mathcal{F}_n)_{n\geq 0}$ is introduced to be the sequence whose o.g.f is $A\big(xc(x)\big)$.

Binomial transform [14] allows the sequence a_n to be transformed into b_n using the following transformation:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_n.$$

The matrix representation of coefficients for his transformation is as follows:

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Additionally, the Binomial transform for this transform can be obtained as follows:

$$BC(a_n) = \sum_{j=1}^{n} \sum_{i=1}^{j} \frac{i}{2j-i} \binom{n}{j} \binom{2j-i}{j-i} a_n.$$

4.1. *The Catalan transform of Fermat polynomials*

We can define the Catalan transform for $(\mathcal{F}_n)$ (x) as

$$C\mathcal{F}_n(x) = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} \mathcal{F}_i(x)$$

for $n \geq 0$ with $C\mathcal{F}_0(x) = 2, C\mathcal{F}_1(x) = 3.$

First few terms of $C\mathcal{F}_n(x)$ as follow:

$$C\mathcal{F}_2(x) = \sum_{i=0}^{n} \frac{i}{4-i}\binom{4-i}{2-i}\mathcal{F}_i(x) = 9x - 1$$

$$C\mathcal{F}_3(x) = \sum_{i=0}^{n} \frac{i}{6-i}\binom{6-i}{3-i}\mathcal{F}_i(x) = 27x^2 + 6x - 8$$

$$C\mathcal{F}_4(x) = \sum_{i=0}^{n} \frac{i}{8-i}\binom{8-i}{4-i}\mathcal{F}_i(x) = 81x^3 + 45x^2 - 27x - 15$$

Giving $x = 1$ we find the Catalan transform for Fermat sequence. Deveci and Shannon [17] investigated k-Fibonacci numbers including complex-type the Catalan transform. For Fermat polynomials, this transformation is as follows:

$$\begin{bmatrix} C\mathcal{F}_1(x) \\ C\mathcal{F}_2(x) \\ C\mathcal{F}_3(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \end{bmatrix} . \begin{bmatrix} \mathcal{F}_1(x) \\ \mathcal{F}_2(x) \\ \mathcal{F}_3(x) \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 9x + 2 \\ 27x^2 + 14x - 4 \\ 81x^3 + 126x^2 + 36x - 29 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & \dots \\ 5 & 5 & 3 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} . \begin{bmatrix} 3 \\ 9x - 1 \\ 27x^2 + 6x - 8 \\ 81x^3 + 45x^2 - 27x - 15 \\ \vdots \end{bmatrix}$$

In [18] authors explored the binomial transformation. In this context, we define the binomial transform for $(\mathcal{F}_n)\ (x)$ as follows:

$$B\mathcal{F}_n(x) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{F}_j(x).$$

for $n \geq 0$ with $B\mathcal{F}_0(x) = 2, B\mathcal{F}_1(x) = 5.$

First few terms of $B\mathcal{F}_n(x)$ as follow:

$$B\mathcal{F}_2(x) = \sum_{j=0}^{2} \binom{2}{j} \mathcal{F}_j(x) = 9x + 4,$$

$$B\mathcal{F}_3(x) = \sum_{j=0}^{3} \binom{3}{j} \mathcal{F}_j(x) = 27x^2 + 15x - 7,$$

$$B\mathcal{F}_4(x) = \sum_{j=0}^{4} \binom{4}{j} \mathcal{F}_j(x) = 81x^3 + 72x^2 - 30x - 26,$$

$$B\mathcal{F}_5(x) = \sum_{j=0}^{5} \binom{5}{j} \mathcal{F}_j(x) = 243x^4 + 297x^3 - 72x^2 - 162x - 31.$$

Giving $x = 1$ we find the Binomial transform of Fermat sequence.

$$\begin{bmatrix} B\mathcal{F}_0(x) \\ B\mathcal{F}_1(x) \\ B\mathcal{F}_2(x) \\ B\mathcal{F}_3(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \begin{bmatrix} \mathcal{F}_0(x) \\ \mathcal{F}_1(x) \\ \mathcal{F}_2(x) \\ \mathcal{F}_3(x) \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 5 \\ 9x+4 \\ 27x^2+15x-7 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \begin{bmatrix} 2 \\ 3 \\ 9x-4 \\ 27x^2-12x-6 \\ \vdots \end{bmatrix}$$

We have the binomial transforms of the Catalan-transformed Fermat polynomials, utilizing the procedure described above if we apply the binomial transform to the this transform.

First few terms of $BC\mathcal{F}_n(x)$ as follow:

$$BC\mathcal{F}_1(x) = 1.\mathcal{F}_1(x),$$
$$BC\mathcal{F}_2(x) = 4.\mathcal{F}_2(x),$$
$$BC\mathcal{F}_3(x) = 14.\mathcal{F}_3(x).$$

The sequence of coefficients $\{1,4,14,50,\dots\}$ represents the self-convolution of the sequence produced by the Binomial transform of the Catalan numbers Ba_n. This transformation, known as the Ballot transform [5], is denoted as:

$$Bal = Cat \circ Bin\,.$$

4.2. *The Generating Functions*

The generating function for the Catalan Fermat polynomials is expressed as $C\mathcal{F}_n\big(x * c(x)\big) = \mathcal{J}(x)$ where $u(x)$ is the generating functions of Fermat polynomials and $c(x)$ is the generating functions of Catalan sequences. From this, we obtain

$$\mathcal{J}(x) = C\mathcal{F}_n\big(x * c(x)\big) = \frac{1+3\sqrt{1-4x}}{1-4x-\sqrt{1-4x}}$$

4.3. *Hankel Transform of the Catalan Fermat Polynomials*

Now we apply the Hankel transformation onto the Catalan Fermat polynomials. Using this transformation, we obtain the terms as follows:

$HC\mathcal{F}_1(x) = det[C\mathcal{F}_1(x)] = det[3] = 3,$

$HC\mathcal{F}_2(x) = det\begin{bmatrix} 3 & 9x+2 \\ 9x+2 & 27x^2+14x-4 \end{bmatrix} = 6x-16,$

$HC\mathcal{F}_3(x)$

$= det\begin{bmatrix} 3 & 9x+2 & 27x^2+14x-4 \\ 9x+2 & 27x^2+14x-4 & 81x^3+126x^2+36x-29 \\ 27x^2+14x-4 & 81x^3+126x^2+36x-29 & 243x^4+216x^3-63x^2-78x-24 \end{bmatrix}$

$= -13284x^4 - 15074x^3 + 6600x^2 + 6584x - 1611$

5. Conclusion

Fermat polynomials $\mathcal{F}_n(x)$ were introduced alongside a range of fundamental identities. Key results include generating functions for $\mathcal{F}_n(x)$ and significant relationships such as those of Cassini, Catalan, Vajda, Honsberger, and D'Ocagne. Matrix representations of these polynomials were explored, with particular attention to the constant determinants of these matrices. Additionally, special transformations like the Hankel transform were applied to both the Fermat sequence and polynomials, yielding further insights.

This study provides a comprehensive exploration of Fermat numbers and their associated polynomials, offering significant insights into their identities, properties, and matrix representations. For future research, it would be valuable to extend the analysis of Fermat polynomials to higher-order recurrence relations, potentially uncovering new patterns and identities. Additionally, investigating the connections between Fermat polynomials and other well-known sequences, such as Fibonacci or Lucas numbers, could lead to further generalizations and applications. A deeper exploration of the computational aspects, including efficient algorithms for generating and working with these polynomials, would also be beneficial. Furthermore, applying the results of this study to areas such as cryptography, number theory, and combinatorial mathematics could open new avenues for practical applications and theoretical advancements.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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