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A note on Pell-Padovan Tetranacci numbers and their ordinary generating functions with special numbers and Chebyshev polynomials

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Abstract

In this paper, we use the symmetrizing operators $\delta^1_{a_1a_2}$, $\delta^2_{a_1a_2}$ and $\delta^3_{a_1a_2}$ for giving some ordinary generating functions related to the products of the Pell-Padovan Tetranacci numbers with *k*-Fibonacci, *k*-Pell, *k*-Jacobsthal, *k*-Mersenne numbers and Chebyshev polynomials.

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1. Introduction and preliminaries

Generating functions are one of the most surprising and useful inventions in discrete mathematics. There have been many studies of the sequences of numbers in the literature that are defined recursively. On these subjects there is a vast amount of literature. As examples, we mention [6], [8], [9], [15], [16], [17], [20] and [22].

In [21], the Pell-Padovan Tetranacci numbers PT_n is defined by the recursive relation

$$PT_{n+4} = PT_{n+2} + 2PT_{n+1} + PT_n, \tag{1.1}$$

with initial values $PT_0 = 0, PT_1 = 1, PT_2 = 1, PT_3 = 1$. Thus, 0, 1, 1, 1, 3, 4, 6, 11, 17, 27, 45, 72, 116 are the first few values of the Pell-Padovan Tetranacci sequence.

In ([12], [14], [18], [23]), the k -Fibonacci, k -Pell, k -Jacobsthal, k-Mersenne numbers are also given by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, F_{k,0} = 1, F_{k,1} = k,$$

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$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, P_{k,0} = 0, P_{k,1} = 1,$$
$$J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, J_{k,0} = 0, J_{k,1} = 1,$$

and

$$M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2}, M_{k,0} = 0, M_{k,1} = 1.$$

In ([11], [13]), the Chebyshev polynomials of the first, second, third and fourth kinds are defined, respectively, by

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), T_0(x) = 1, T_1(x) = x,$$
$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), U_0(x) = 1, U_1(x) = 2x,$$
$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), V_0(x) = 1, V_1(x) = 2x - 1,$$

and

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x), W_0(x) = 1, W_1(x) = 2x + 1.$$

Next, we recall some backgrounds about the symmetric functions. For details, see, [4], [19]

Definition 1.1: [2] Given two alphabets A and E, we have

$$\sum_{n=0}^{+\infty} S_n(A - E)z^n = \frac{\prod_{e\in E}(1-ze)}{\prod_{a\in A}(1-za)}, \tag{1.2}$$

where $S_n(A - E) = 0$ for $n < 0$.

Corollary 1.1: *By putting* $A = \{0\}$ *in equation* (1.2), *we obtain*

$$\prod_{e\in E}(1 - ze) = \sum_{n=0}^{+\infty} S_n(-E)z^n. \tag{1.3}$$

Furthermore, if either A or $E = \{0\}$, we have

$$\sum_{n=0}^{+\infty} S_n(A - E)z^n = \sum_{n=0}^{+\infty} S_n(A)z^n \times \sum_{n=0}^{+\infty} S_n(-E)z^n. \tag{1.4}$$

Thus, $S_n(A - E) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-E)$ (see [1])

Definition 1.2: [5] Given a positive number n and a set of variables $A_2 = \{a_1, a_2\}$, the n-th symmetric function $S_n(a_1 + a_2)$ is defined as

$$S_n(A_2) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$S_0(A_2) = S_0(a_1 + a_2) = 1,$$
$$S_1(A_2) = S_1(a_1 + a_2) = a_1 + a_2,$$
$$S_2(A_2) = S_2(a_1 + a_2) = a_1^2 + a_1a_2 + a_2^2,$$
$$\vdots$$

Definition 1.3: [5] Let $\delta_{a_1a_2}^k$ the symmetrizing operator defined by

$$\delta_{a_1a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \text{forall } k \in \mathbb{N}_0. \tag{1.5}$$

Remark 1.1: If $f(a_1) = a_1$, the operator (1.5) gives us

$$\delta_{a_1a_2}^k(a_1) = S_k(a_1 + a_2).$$

2. Principal Theorems

In this section, we use the symmetrizing operators $\delta^1_{a_1a_2}$, $\delta^2_{a_1a_2}$ and $\delta^3_{a_1a_2}$ to give some new theorems. The next theorem we start with is this one.

Theorem 2.1: *Let* $E_4 = \{e_1, e_2, e_3, e_4\}$ *and* $A_2 = \{a_1, a_2\}$ *be two alphabets, then we have*

$$\sum_{n=0}^{+\infty} S_n(E_4)S_n(A_2)z^n = \frac{\begin{bmatrix} 1-a_1a_2S_2(-E_4)z^2-a_1a_2(a_1+a_2)S_3(-E_4)z^3 \\ -a_1a_2((a_1+a_2)^2-a_1a_2)S_4(-E_4)z^4 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \quad (2.1)$$

Proof: Let $f(a_1) = \sum_{n=0}^{+\infty} S_n(E_4)a_1^n z^n$, then the left hand side of formula (2.1) can be written as

$$\begin{aligned} \delta^1_{a_1a_2} f(a_1) &= \delta^1_{a_1a_2}\left(\sum_{n=0}^{+\infty} S_n(E_4)a_1^n z^n\right) \\ &= \frac{a_1\sum_{n=0}^{+\infty} S_n(E_4)a_1^n z^n - a_2\sum_{n=0}^{+\infty} S_n(E_4)a_2^n z^n}{a_1-a_2} \\ &= \sum_{n=0}^{+\infty} S_n(E_4)\left(\frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2}\right)z^n \\ &= \sum_{n=0}^{+\infty} S_n(E_4)S_n(A_2)z^n, \end{aligned}$$

and the right hand side of this formula can be written as

$$\begin{aligned} \delta^1_{a_1a_2}\left(\frac{1}{\prod_{e\in E_4}(1-a_1ez)}\right) &= \frac{a_1\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n - a_2\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n}{(a_1-a_2)\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)} \\ &= \frac{\sum_{n=0}^{+\infty} S_n(-E_4)\frac{a_1a_2^n-a_2a_1^n}{a_1-a_2}z^n}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)} \\ &= \frac{1-a_1a_2\sum_{n=1}^{+\infty} S_n(-E_4)\frac{a_1^{n-1}-a_2^{n-1}}{a_1-a_2}z^n}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)} \\ &= \frac{1-a_1a_2\sum_{n=2}^{+\infty} S_n(-E_4)S_{n-2}(A)z^n}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)} \\ &= \frac{\begin{bmatrix} 1-a_1a_2S_2(-E_4)z^2-a_1a_2(a_1+a_2)S_3(-E_4)z^3 \\ -a_1a_2((a_1+a_2)^2-a_1a_2)S_4(-E_4)z^4 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \end{aligned}$$

This completes the proof .

From (2.1), we deduce the following corollary.

Corollary 2.1: *Given two alphabets* $E_4 = \{e_1, e_2, e_3, e_4\}$ *and* $A_2 = \{a_1, a_2\}$, *we have*

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(A_2)z^n = \frac{\begin{bmatrix} z-a_1a_2S_2(-E_4)z^3-a_1a_2(a_1+a_2)S_3(-E_4)z^4 \\ -a_1a_2((a_1+a_2)^2-a_1a_2)S_4(-E_4)z^5 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \quad (2.2)$$

Theorem 2.2: *Given two alphabets* $E_4 = \{e_1, e_2, e_3, e_4\}$ *and* $A_2 = \{a_1, a_2\}$, *then*

$$\sum_{n=0}^{+\infty} S_n(E_4)S_{n+1}(A_2)z^n = \frac{\begin{bmatrix} a_1+a_2+a_1a_2S_1(-E_4)z-a_1^2a_2^2S_3(-E_4)z^3 \\ -a_1^2a_2^2(a_1+a_2)S_4(-E_4)z^4 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \quad (2.3)$$

Proof: Suffices to apply the operator $\delta_{a_1a_2}^2$ to the identity

$$\sum_{n=0}^{+\infty} S_n(E_4)a_1^nz^n = \frac{1}{\prod_{e\in E_4}(1-a_1ez)},$$

then

$$\delta_{a_1a_2}^2\left(\sum_{n=0}^{+\infty} S_n(E_4)a_1^nz^n\right) = \delta_{a_1a_2}^2\left(\frac{1}{\prod_{e\in E_4}(1-a_1ez)}\right)$$

$$\frac{a_1^2\sum_{n=0}^{+\infty} S_n(E_4)a_1^nz^n - a_2^2\sum_{n=0}^{+\infty} S_n(E_4)a_2^nz^n}{a_1-a_2} = \frac{a_1^2\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n - a_2^2\sum_{n=0}^{+\infty} S_n(-E_4)a_1^nz^n}{(a_1-a_2)\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}$$

$$\sum_{n=0}^{+\infty} S_n(E_4)\left(\frac{a_1^{n+2}-a_2^{n+2}}{a_1-a_2}\right)z^n = \frac{\sum_{n=0}^{+\infty} S_n(-E_4)\frac{a_1^2a_2^n-a_2^2a_1^n}{a_1-a_2}z^n}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}$$

$$\sum_{n=0}^{+\infty} S_n(E_4)S_{n+1}(A)z^n = \frac{\begin{bmatrix} a_1+a_2+a_1a_2S_1(-E)z-a_1^2a_2^2S_3(-E)z^3 \\ -a_1^2a_2^2(a_1+a_2)S_4(-E)z^4 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}.$$

This completes the proof.

From (2.3), we deduce the following corollaries

Corollary 2.2: *Given two alphabets* $E_4 = \{e_1, e_2, e_3, e_4\}$ *and* $A_2 = \{a_1, a_2\}$, *then*

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(A_2)z^n = \frac{\begin{bmatrix} (a_1+a_2)z+a_1a_2S_1(-E_4)z^2-a_1^2a_2^2S_3(-E_4)z^4 \\ -a_1^2a_2^2(a_1+a_2)S_4(-E_4)z^5 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \quad (2.4)$$

Corollary 2.3: *Given two alphabets* $E_4 = \{e_1, e_2, e_3, e_4\}$ *and* $A_2 = \{a_1, a_2\}$, *then*

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(A_2)z^n = \frac{\begin{bmatrix} (a_1+a_2)z^2+a_1a_2S_1(-E_4)z^3-a_1^2a_2^2S_3(-E_4)z^5 \\ -a_1^2a_2^2(a_1+a_2)S_4(-E_4)z^6 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \quad (2.5)$$

Theorem 2.3: *Given two alphabets* $E_4 = \{e_1, e_2, e_3, e_4\}$ *and* $A_2 = \{a_1, a_2\}$, *then*

$$\sum_{n=0}^{+\infty} S_n(E_4)S_{n+2}(A_2)z^n = \frac{\begin{bmatrix} ((a_1+a_2)^2-a_1a_2)+a_1a_2(a_1+a_2)S_1(-E_4)z \\ +a_1^2a_2^2S_2(-E_4)z^2-a_1^3a_2^3S_4(-E_4)z^4 \end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \quad (2.6)$$

Proof: First step, the action of the operator $\delta_{a_1a_2}^3$ on $\sum_{n=0}^{+\infty} S_n(E_4)a_1^nz^n$, we get

$$\begin{aligned}\delta_{a_1a_2}^3\left(\sum_{n=0}^{+\infty} S_n(E_4)a_1^nz^n\right) &= \frac{a_1^3\sum_{n=0}^{+\infty} S_n(E_4)a_1^nz^n - a_2^3\sum_{n=0}^{+\infty} S_n(E_4)a_2^nz^n}{a_1-a_2} \\ &= \sum_{n=0}^{+\infty} S_n(E_4)\left(\frac{a_1^{n+3}-a_2^{n+3}}{a_1-a_2}\right)z^n \\ &= \sum_{n=0}^{+\infty} S_n(E_4)S_{n+2}(A_2)z^n,\end{aligned}$$

Second step, by action of the operator $\delta^3_{a_1a_2}$ on $\frac{1}{\prod_{e\in E_4}(1-a_1ez)}$, we get

$$\delta^3_{a_1a_2}\left(\frac{1}{\prod_{e\in E_4}(1-a_1ez)}\right) = \frac{a_1^3\sum_{n=0}^{+\infty}S_n(-E_4)a_2^nz^n - a_2^3\sum_{n=0}^{+\infty}S_n(-E_4)a_1^nz^n}{(a_1-a_2)\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}$$

$$= \frac{\sum_{n=0}^{+\infty}S_n(-E_4)\frac{a_1^3a_2^n-a_2^3a_1^n}{a_1-a_2}z^n}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}$$

$$= \frac{\begin{bmatrix}((a_1+a_2)^2-a_1a_2)+a_1a_2(a_1+a_2)S_1(-E_4)z\\ +a_1^2a_2^2S_2(-E_4)z^2-a_1^3a_2^3S_4(-E_4)z^4\end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}.$$

This completes the proof.

From (2.6) we deduce the following corollary.

Corollary 2.4: *Given two alphabets* $E_4=\{e_1,e_2,e_3,e_4\}$ *and* $A_2=\{a_1,a_2\}$, *then*

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_n(A_2)z^n = \frac{\begin{bmatrix}((a_1+a_2)^2-a_1a_2)z^2+a_1a_2(a_1+a_2)S_1(-E_4)z^3\\ +a_1^2a_2^2S_2(-E_4)z^4-a_1^3a_2^3S_4(-E_4)z^6\end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1-a_2ez)}. \quad (2.7)$$

If $A_2=\{1,0\}$ in the formulas (2.4) and (2.5), then we deduce the following result.

Corollary 2.5: *Let* $E_4=\{e_1,e_2,e_3,e_4\}$, *we get*

$$\sum_{n=0}^{+\infty}(S_{n-1}(E_4)+S_{n-2}(E_4))z^n = \frac{z+z^2}{1+S_1(-E_4)z+S_2(-E_4)z^2+S_3(-E_4)z^3+S_4(-E_4)z^4}. \quad (2.8)$$

Assuming $S_1(-E_4)=0, S_2(-E_4)=S_4(-E_4)=-1, S_3(-E_4)=-2$ in (2.8), we obtain the generating function of the Pell-Padovan Tetranacci numbers given by [21], such that $S_{n-1}(E_4)+S_{n-2}(E_4)=PT_n$.

3. On the ordinary generating functions

The ordinary generating functions of the Pell-Padovan Tetranacci products with certain numbers (k-Fibonacci, k-Pell, k-Jacobsthal, k-Mersenne) are now derived in this section using corollaries 2, 3, 4 and 5.

- For the case $A_2=\{a_1,-a_2\}$ with replacing a_2 by $(-a_2)$ in (2.2), (2.4), (2.5) and (2.7), we have

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{\begin{bmatrix}z+a_1a_2S_2(-E_4)z^3+a_1a_2(a_1-a_2)S_3(-E_4)z^4\\ +a_1a_2((a_1-a_2)^2+a_1a_2)S_4(-E_4)z^5\end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1+a_2ez)}, \quad (3.1)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(a_1+[-a_2])z^n = \frac{\begin{bmatrix}(a_1-a_2)z-a_1a_2S_1(-E_4)z^2-a_1^2a_2^2S_3(-E_4)z^4\\ -a_1^2a_2^2(a_1-a_2)S_4(-E_4)z^5\end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1+a_2ez)}, \quad (3.2)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{\begin{bmatrix}(a_1-a_2)z^2-a_1a_2S_1(-E_4)z^3-a_1^2a_2^2S_3(-E_4)z^5\\ -a_1^2a_2^2(a_1-a_2)S_4(-E_4)z^6\end{bmatrix}}{\prod_{e\in E}(1-a_1ez)\prod_{e\in E}(1+a_2ez)}, \quad (3.3)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_n(a_1+[-a_2])z^n = \frac{\begin{bmatrix}((a_1-a_2)^2+a_1a_2)z^2-a_1a_2(a_1-a_2)S_1(-E)z^3\\ +a_1^2a_2^2S_2(-E)z^4+a_1^3a_2^3S_4(-E)z^6\end{bmatrix}}{\prod_{e\in E_4}(1-a_1ez)\prod_{e\in E_4}(1+a_2ez)}, \quad (3.4)$$

respectively.

This case is composed of four parts.

Firstly, the substitutions $S_1(-E_4)=0, S_2(-E_4)=S_4(-E_4)=-1, S_3(-E_4)=-2, a_1-a_2=k$ and $a_1a_2=1$ in (3.2) and (3.4), we obtain

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(a_1+[-a_2])z^n = \frac{kz+2z^4+kz^5}{D_1(z)}, \quad (3.5)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_n(a_1+[-a_2])z^n = \frac{(k^2+1)z^2-z^4-z^6}{D_1(z)}, \quad (3.6)$$

with $D_1(z)=1-(k^2+2)z^2-(2k^3+6k)z^3-(k^4+4k^2+1)z^4+2kz^5+(k^2-2)z^6-2kz^7+z^8$, respectively, we find the following theorem.

Theorem 3.1: *For* $n\in\mathbb{N}$, *we have*

$$\sum_{n=0}^{+\infty} PT_nF_{k,n}z^n = \frac{kz+(k^2+1)z^2+z^4+kz^5-z^6}{D_1(z)},$$

with $D_1(z)=1-(k^2+2)z^2-(2k^3+6k)z^3-(k^4+4k^2+1)z^4+2kz^5+(k^2-2)z^6-2kz^7+z^8$.

Proof: By [10]), We have $F_{k,n}=S_n(a_1+[-a_2])$. Then,

$$\begin{aligned}\sum_{n=0}^{+\infty} PT_nF_{k,n}z^n &= \sum_{n=0}^{+\infty}(S_{n-1}(E_4)+S_{n-2}(E_4))S_n(a_1+[-a_2])z^n\\ &= \sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(a_1+[-a_2])z^n+\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_n(a_1+[-a_2])z^n,\end{aligned}$$

using the formulas (3.5) and (3.6), we have

$$\begin{aligned}\sum_{n=0}^{+\infty} PT_nF_{k,n}z^n &= \frac{kz+2z^4+kz^5}{D_1(z)}+\frac{(k^2+1)z^2-z^4-z^6}{D_1(z)}\\ &= \frac{kz+(k^2+1)z^2+z^4+kz^5-z^6}{D_1(z)}.\end{aligned}$$

This completes the proof.

Secondly, the substitutions $S_1(-E_4)=0, S_2(-E_4)=S_4(-E_4)=-1, S_3(-E_4)=-2, a_1-a_2=2$ and $a_1a_2=k$ in (3.1) and (3.3), we get

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{z-kz^3-4kz^4-(k^2+4k)z^5}{D_2(z)}, \quad (3.7)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{2z^2+2k^2z^5+2k^2z^6}{D_2(z)}, \quad (3.8)$$

with $D_2(z)=1-(2k+4)z^2-(12k+16)z^3-(k^2+16k+16)z^4+4k^2z^5-(2k^3-16k^2)z^6-4k^3z^7+k^4z^8$, respectively, we have the following result.

Theorem 3.2: *For* $n\in\mathbb{N}$, *we have*

$$\sum_{n=0}^{+\infty} PT_n P_{k,n} z^n = \frac{z+2z^2-kz^3-4kz^4+(k^2-4k)z^5+2k^2z^6}{D_2(z)},$$

with $D_2(z) = 1-(2k+4)z^2-(12k+16)z^3-(k^2+16k+16)z^4+4k^2z^5+(2k^3-16k^2)z^6-4k^3z^7+k^4z^8$.

Proof: By [10], we have $P_{k,n} = S_{n-1}(a_1+[-a_2])$. Then,

$$\sum_{n=0}^{+\infty} PT_n P_{k,n} z^n = \sum_{n=0}^{+\infty} (S_{n-1}(E_4)+S_{n-2}(E_4))S_{n-1}(a_1+[-a_2])z^n$$
$$= \sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(a_1+[-a_2])z^n + \sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(a_1+[-a_2])z^n,$$

using (3.7) and (3.8), the above equation becomes

$$\sum_{n=0}^{+\infty} PT_n P_{k,n} z^n = \frac{z-kz^3-4kz^4-(k^2+4k)z^5}{D_2(z)} + \frac{2z^2+2k^2z^5+2k^2z^6}{D_2(z)}$$
$$= \frac{z+2z^2-kz^3-4kz^4+(k^2-4k)z^5+2k^2z^6}{D_2(z)}.$$

Hence, we obtain the desired result.

Thirdly, the substitutions $S_1(-E_4)=0, S_2(-E_4)=S_4(-E_4)=-1, S_3(-E_4)=-2, a_1-a_2=k$ and $a_1a_2=2$ in (3.1) and (3.3), we give

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{z-2z^3-4kz^4-(2k^2+4)z^5}{D_3(z)}, \tag{3.9}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{kz^2+8z^5+4kz^6}{D_3(z)}, \tag{3.10}$$

where $D_3(z) = 1-(k^2+4)z^2-(2k^3+12k)z^3-(k^4+8k^2+4)z^4+8kz^5+(4k^2-16)z^6-16kz^7+16z^8$, respectively, we find the following result.

Theorem 3.3: *For* $n \in \mathbb{N}$, *we have*

$$\sum_{n=0}^{+\infty} PT_n J_{k,n} z^n = \frac{z+kz^2-2z^3-4kz^4-(2k^2-4)z^5+4kz^6}{D_3(z)},$$

where $D_3(z) = 1-(k^2+4)z^2-(2k^3+12k)z^3-(k^4+8k^2+4)z^4+8kz^5-(4k^2-16)z^6-16kz^7+16z^8$.

Proof: By referred to [10], we have $J_{k,n} = S_{n-1}(a_1+[-a_2])$. Then,

$$\sum_{n=0}^{+\infty} PT_n J_{k,n} z^n = \sum_{n=0}^{+\infty} (S_{n-1}(E_4)+S_{n-2}(E_4))S_{n-1}(a_1+[-a_2])z^n$$
$$= \sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(a_1+[-a_2])z^n + \sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(a_1+[-a_2])z^n$$
$$= \frac{z-2z^3-4kz^4-(2k^2+4)z^5}{D_3(z)} + \frac{kz^2+8z^5+4kz^6}{D_3(z)},$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} PT_n J_{k,n} z^n = \frac{z+kz^2-2z^3-4kz^4-(2k^2-4)z^5+4kz^6}{D_3(z)}.$$

This completes the proof.

Finaly, the substitutions $S_1(-E_4)=0, S_2(-E_4)=S_4(-E_4)=-1, S_3(-E_4)=-2, a_1-a_2=3k$ and $a_1a_2=-2$ in (3.1) and (3.3), we obtain

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{z+2z^3+12kz^4+(18k^2-4)z^5}{D_4(z)},$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(a_1+[-a_2])z^n = \frac{3kz^2+8z^5+12kz^6}{D_4(z)},$$

with $D_4(z) = 1-(9k^2-4)z^2-(54k^3-36k)z^3-(81k^4-72k^2+4)z^4+24kz^5+(36k^2+16)z^6+48kz^7+16z^8$, respectively, we have the following result.

Theorem 3.4: *For* $n \in \mathbb{N}$, *we have*

$$\sum_{n=0}^{+\infty} PT_n M_{k,n} z^n = \frac{z+3kz^2+2z^3+12kz^4+(18k^2+4)z^5+12kz^6}{D_4(z)},$$

with $D_4(z) = 1-(9k^2-4)z^2-(54k^3-36k)z^3-(81k^4-72k^2+4)z^4+24kz^5+(36k^2+16)z^6+48kz^7+16z^8$.

Proof: We have $M_{k,n} = S_{n-1}(a_1+[-a_2])$, (see [10]). Then

$$\begin{aligned}
&\sum_{n=0}^{+\infty} PT_n M_{k,n} z^n \\
&= \sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])z^n + \sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])z^n \\
&= \frac{z+2z^3+12kz^4+(18k^2-4)z^5}{D_4(z)} + \frac{3kz^2+8z^5+12kz^6}{D_4(z)} \\
&= \frac{z+3kz^2+2z^3+12kz^4+(18k^2+4)z^5+12kz^6}{D_4(z)}.
\end{aligned}$$

The proof is completed.

- If we set $k = 1$ in theorems [4,5,6,7], we obtain the following corollary.

Corollary 3.1: *The ordinary generating functions of the product of Pell-Padovan Tetranacci numbers with Fibonacci numbers, Pell numbers, Jacobsthal numbers and Mersenne numbers are expressed, respectively, by*

$$\sum_{n=0}^{+\infty} PT_n F_n z^n = \frac{z+2z^2+z^4+z^5-z^6}{1-3z^2-8z^3-6z^4+2z^5-z^6-2z^7+z^8},$$

$$\sum_{n=0}^{+\infty} PT_n P_n z^n = \frac{z+2z^2-z^3-4z^4-3z^5+2z^6}{1-6z^2-28z^3-33z^4+4z^5+14z^6-4z^7+z^8},$$

$$\sum_{n=0}^{+\infty} PT_n J_n z^n = \frac{z+z^2-2z^3-4z^4+2z^5+4z^6}{1-5z^2-14z^3-13z^4+8z^5-12z^6-16z^7+16z^8},$$

and

$$\sum_{n=0}^{+\infty} PT_n M_n z^n = \frac{z+3z^2+2z^3+12z^4+22z^5+12z^6}{1-5z^2-18z^3-13z^4+24z^5+52z^6+48z^7+16z^8}.$$

The ordinary generating functions of the products of the first, second, third and fourth kinds of Chebyshev polynomials and Pell-Padovan Tetranacci numbers are now derived in this part.

- For the case $A_2 = \{2a_1, -2a_2\}$ with changing a_1 by $(2a_1)$ and a_2 by $(-2a_2)$ in (2.2), (2.4), (2.5) and (2.7), we get

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n = \frac{\begin{bmatrix} z+4a_1a_2S_2(-E_4)z^3+8a_1a_2(a_1-a_2)S_3(-E_4)z^4 \\ +16a_1a_2\left((a_1-a_2)^2+a_1a_2\right)S_4(-E_4)z^5 \end{bmatrix}}{\prod_{e\in E_4}(1-2a_1ez)\prod_{e\in E_4}(1+2a_2ez)}, \tag{3.11}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(2a_1+[-2a_2])z^n = \frac{\begin{bmatrix} 2(a_1-a_2)z-4a_1a_2S_1(-E_4)z^2-16a_1^2a_2^2S_3(-E_4)z^4 \\ -32a_1^2a_2^2(a_1-a_2)S_4(-E_4)z^5 \end{bmatrix}}{\prod_{e\in E_4}(1-2a_1ez)\prod_{e\in E_4}(1+2a_2ez)}, \tag{3.12}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_n(2a_1+[-2a_2])z^n = \frac{\begin{bmatrix} \left(4(a_1-a_2)^2+4a_1a_2\right)z^2-8a_1a_2(a_1-a_2)S_1(-E)z^3 \\ +16a_1^2a_2^2S_2(-E)z^4-64a_1^3a_2^3S_4(-E)z^6 \end{bmatrix}}{\prod_{e\in E_4}(1-2a_1ez)\prod_{e\in E_4}(1+2a_2ez)}, \tag{3.13}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n = \frac{\begin{bmatrix} 2(a_1-a_2)z^2-4a_1a_2S_1(-E_4)z^3-16a_1^2a_2^2S_3(-E_4)z^5 \\ -32a_1^2a_2^2(a_1-a_2)S_4(-E_4)z^6 \end{bmatrix}}{\prod_{e\in E_4}(1-2a_1ez)\prod_{e\in E_4}(1+2a_2ez)}, \tag{3.14}$$

respectively.

By making the following restrictions $S_1(-E_4)=0, S_2(-E_4)=S_4(-E_4)=-1$, $S_3(-E_4)=-2$, $a_1-a_2=x$ and $4a_1a_2=-1$ in (3.11), (3.12), (3.13) and (3.14), we, respectively, obtain

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n = \frac{z+z^3+4xz^4+(x^2-1)z^5}{D_5(z)}, \tag{3.15}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(2a_1+[-2a_2])z^n = \frac{2xz+2z^4+2xz^5}{D_5(z)}, \tag{3.16}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_n(2a_1+[-2a_2])z^n = \frac{(4x^2-1)z^2-z^4-z^6}{D_5(z)}, \tag{3.17}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n = \frac{2xz^2+2z^5+2xz^6}{D_5(z)}, \tag{3.18}$$

where $D_5(z)=1-(4x^2-2)z^2-(16x^3-12x)z^3-(16x^4+16x^2+1)z^4+4xz^5+(4x^2+2)z^6+4xz^7+z^8$.

Thus, we have the following theorems.

Theorem 3.5: *For $n\geq 0$ be an integer. Then we have*

$$\sum_{n=0}^{+\infty} PT_nU_n(x)z^n = \frac{2xz+(4x^2-1)z^2+z^4+2xz^5-z^6}{D_5(z)},$$

with $D_5(z)=1-(4x^2-2)z^2-(16x^3-12x)z^3-(16x^4+16x^2+1)z^4+4xz^5+(4x^2+2)z^6+4xz^7+z^8$.

Proof: We know that $U_n(x)=S_n(2a_1+[-2a_2])$, (see [7]). Then,

$$\sum_{n=0}^{+\infty} PT_nU_n(x)z^n = \sum_{n=0}^{+\infty}\left(S_{n-1}(E_4)+S_{n-2}(E_4)\right)S_n(2a_1+[-2a_2])z^n$$
$$= \sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(2a_1+[-2a_2])z^n+\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_n(2a_1+[-2a_2])z^n,$$

from (3.16) and (3.17), we easily obtain

$$\begin{aligned}\sum_{n=0}^{+\infty} PT_nU_n(x)z^n &= \frac{2xz+2z^4+2xz^5}{D_5(z)}+\frac{(4x^2-1)z^2-z^4-z^6}{D_5(z)}\\ &= \frac{2xz+(4x^2-1)z^2+z^4+2xz^5-z^6}{D_5(z)}.\end{aligned}$$

The proof is finished.

Theorem 3.6: *For* $n \geq 0$ *be an integer. Then we have*

$$\sum_{n=0}^{+\infty} PT_nT_n(x)z^n = \frac{xz+(2x^2-1)z^2-xz^3-(4x^2-1)z^4-(x^3-x)z^5-(2x^2+1)z^6}{D_5(z)},$$

with $D_5(z) = 1-(4x^2-2)z^2-(16x^3-12x)z^3-(16x^4+16x^2+1)z^4+4xz^5+(4x^2+2)z^6+4xz^7+z^8$.

Proof. We know that $T_n(x) = S_n(2a_1+[-2a_2]) - xS_{n-1}(2a_1+[-2a_2])$, (see [7]). Then

$$\begin{aligned}&\sum_{n=0}^{+\infty} PT_nT_n(x)z^n\\ &= \sum_{n=0}^{+\infty} (S_{n-1}(E_4)+S_{n-2}(E_4))S_n(2a_1+[-2a_2])z^n\\ &\quad -x\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\\ &\quad -x\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\\ &= \sum_{n=0}^{+\infty} PT_nU_n(x)z^n - x\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\\ &\quad -x\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\\ &= \frac{2xz+(4x^2-1)z^2+z^4+2xz^5-z^6}{D_5(z)} - \frac{xz+xz^3+4x^2z^4+(x^3-x)z^5}{D_5(z)} - \frac{2x^2z^2+2xz^5+2x^2-z^6}{D_5(z)}\\ &= \frac{xz+(2x^2-1)z^2-xz^3-(4x^2-1)z^4-(x^3-x)z^5-(2x^2+1)z^6}{D_5(z)}.\end{aligned}$$

The proof is finished.

Theorem 3.7: *For* $n \geq 0$ *be an integer. Then we have*

$$\sum_{n=0}^{+\infty} PT_nV_n(x)z^n = \frac{(2x-1)z+(4x^2-2x-1)z^2-z^3-(4x-1)z^4-(x^2-2x+1)z^5-(2x+1)z^6}{D_5(z)},$$

with $D_5(z) = 1-(4x^2-2)z^2-(16x^3-12x)z^3-(16x^4-16x^2+1)z^4+4xz^5+(4x^2+2)z^6+4xz^7+z^8$.

Proof: Recall that, we have $V_n(x) = S_n(2a_1+[-2a_2]) - S_{n-1}(2a_1+[-2a_2])$, (see [3]). We get

$$\begin{aligned}&\sum_{n=0}^{+\infty} PT_nV_n(x)z^n\\ &= \sum_{n=0}^{+\infty} (S_{n-1}(E_4)+S_{n-2}(E_4))S_n(2a_1+[-2a_2])z^n\\ &\quad -\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\\ &\quad -\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\\ &= \sum_{n=0}^{+\infty} PT_nU_n(x)z^n - \sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\\ &\quad -\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n\end{aligned}$$

$$= \frac{2xz+(4x^2-1)z^2+z^4+2xz^5-z^6}{D_5(z)} - \frac{z+z^3+4xz^4+(x^2-1)z^5}{D_5(z)} - \frac{2xz^2+2z^5+2xz^6}{D_5(z)}$$
$$= \frac{(2x-1)z+(4x^2-2x-1)z^2-z^3-(4x-1)z^4-(x^2-2x+1)z^5-(2x+1)z^6}{D_5(z)}.$$

The proof is finished.

Theorem 3.8: *For* $n \geq 0$ *be an integer. Then we have*

$$\sum_{n=0}^{+\infty} PT_n W_n(x) z^n = \frac{(2x+1)z+(4x^2+2x-1)z^2+z^3+(4x+1)z^4+(x^2+2x+1)z^5+(2x-1)z^6}{D_5(z)},$$

with $D_5(z) = 1-(4x^2-2)z^2-(16x^3-12x)z^3-(16x^4+16x^2+1)z^4+4xz^5+(4x^2+2)z^6+4xz^7+z^8$.

Proof: Recall that, we have $W_n(x) = S_n(2a_1+[-2a_2]) + S_{n-1}(2a_1+[-2a_2])$, (see [3]). We get

$$\sum_{n=0}^{+\infty} PT_n W_n(x) z^n$$
$$= \sum_{n=0}^{+\infty} (S_{n-1}(E_4)+S_{n-2}(E_4))S_n(2a_1+[-2a_2])z^n$$
$$+\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n$$
$$+\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n$$
$$= \sum_{n=0}^{+\infty} PT_n U_n(x) z^n + \sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-1}(2a_1+[-2a_2])z^n$$
$$+\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(2a_1+[-2a_2])z^n$$
$$= \frac{2xz+(4x^2-1)z^2+z^4+2xz^5-z^6}{D_5(z)} + \frac{z+z^3+4xz^4+(x^2-1)z^5}{D_5(z)} + \frac{2xz^2+2z^5+2xz^6}{D_5(z)}$$
$$= \frac{(2x+1)z+(4x^2+2x-1)z^2+z^3+(4x+1)z^4+(x^2+2x+1)z^5+(2x-1)z^6}{D_5(z)}.$$

The proof is finished.

4. Conclusion

In this study, ordinary generating functions for the products of Pell-Padovan Tetranacci numbers with certain numbers and Chebyshev polynomials were developed through the application of theorems (2,1), (2,2) and (2,3). Symmetric functions and their products of numbers and polynomials provide the basis of derived theorems and corollaries.

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