

Solving a quadratic fractional problem via a new quadratic formulation

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Abstract

We present two different techniques for solving a quadratic fractional program. The first one uses Taylor's development which makes it possible to transform the quadratic fractional programming problem into an equivalent linear programming problem that will be solved via an interior point approach. The second one allows us to transform the quadratic fractional programming problem into an equivalent quadratic program that will be solved by an efficient algorithm for this class of problems. The established comparative numerical tests show that the algorithm works properly and confirms the effectiveness of our proposed techniques.

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1. Introduction

Quadratic fractional programs (QFP) are considered as a special type of mathematical programming, where the objective function is the ratio of two quadratic functions. This type of problems have been a subject of wide interest and have attracted considerable researchers since they have applications in various fields such as financial, corporate and production

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planning, health care, combinatorial optimization and operational research, engineering, data base,...etc.

Many methods to solve (QFP) were proposed, among them, the one proposed by Beck et al. [2] in 2006, who resolved a fractional quadratic problem with quadratic constraints and found a global optimal solution with applications to the regularized total least square. Khurana et al. [1] in 2011, have presented an algorithm for solving (QFP) by converting it into an equivalent (QFP) with less constraints. In 2013, S. Nadjmaddin et al. [16] have resolved (QFP) using Wolfe's method and a new modified simplex approach. In 2014, in their paper, Van Bong Nguyen et al. [24] have resolved (QFP) via an SDP approach and M. Rashidul Hasan et al. [22] have extended the usual simplex method for solving (QFP) problems.

The aim of this work concerns the resolution of (QFP) via two different formulations. The first one is based on the work of Sivri et al. [23]. It allows us to approximate (QFP) via Taylor's developpement by a sequence of (LP) which are easier to solve. In contrary to Sivri et al., we introduce the projective variant of interior points of Ye-lustig [14] to solve the sequence of (LP) obtained via this approximation. There are two types of methods to solve (LP) , simplicial methods and interior point methods, see for instance [4, 8, 15, 17]. Note that in their original paper, Sivri et al. used the Simplex approach [7]. The second formulation allows us to transform (QFP) in a suitable way to a quadratic problem (QP) . We dispose many variants for solving (QP) . Generally, the obtained (QP) is non-convex. In case where (QP) is a convex problem, we dispose many algorithms. See for instance [5, 6]. Likewise, if (QP) is non-convex, Pham Dinh Tao in 1985 was the first one to propose an efficient algorithm to solve such problems called DCA [18]. Since then, many researchers proposed different variants until these days [9, 10, 13, 19, 20, 21, 25].

The paper is composed of four sections organized as follows: Firstly, we describe in section 2, the mathematical formulation of (QFP) . Then, inspired by [3], we present the first formulation based on interior point methods where we convert (QFP) into a sequence of (LP) and we recall the algorithm of Ye-Lutig for (LP) . After that, we present our new quadratic formulation where we convert (QFP) into a (QP) . After that, we recall the general idea and the mains steps of the algorithm DCA. Section 3 is reserved to comparative numerical experiments by considering the two formulations presented previously. Finally, the last section draws overall conclusion.

2. Description and resolution of (QFP)

Consider the quadratic fractional program (QFP):

$$(QFP) \begin{cases} \min J(x) = \frac{\frac{1}{2}x^t Hx + c^t x + c_0}{\frac{1}{2}x^t Dx + d^t x + d_0} \\ P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}, \end{cases}$$

where H and D are two symmetric matrices of $\mathbb{R}^{n \times n}$; $c, d \in \mathbb{R}^n$; $b \in \mathbb{R}^m$; $c_0, d_0 \in \mathbb{R}$; $A \in \mathbb{R}^{m \times n}$ and $\frac{1}{2}x^t Dx + d^t x + d_0 > 0$ for all $x \in P$.

Throughout, the paper, we make the following assumptions :

- a1): The matrix A has full rank ($\text{rank}(A) = m < n$).
- a2): The set P is compact (closed and bounded).
- a3): (QFP) satisfies the interior point condition, i.e., it exists $x_0 > 0$ such that $Ax_0 = b$.

Remark 2.1: Since the fonction J is continuous on a compact set P , then according to Weirstass's theorem (QFP) admits at least one optimal solution.

In order to solve (QFP), we present two different manners which require the reformulation of (QFP) into a simple equivalent problem easier to solve and obtain its optimal solution. In fact, we present two formulations of (QFP), the first one concerns the formulation of Sivri et al. [23] given in 2018 where (QFP) is converted into a sequence of equivalent linear problems (LP), the second one is the principal object of our new quadratic formulation.

2.1 Formulation of Sivri (FS)

This formulation is based essentially on Taylor's development of the objective function, i.e., insted of solving (QFP), we resolve a sequence of equivalent linear programs.

2.1.1 Taylor's development

Consider $G : \mathbb{R}^n \rightarrow \mathbb{R}$ a differentiable function of class C^1 .

The first two terms of Taylor's development of the function G at the point $a = (a_1, a_2, \dots, a_n)$ are:

$$G(a) + \nabla^t G(a)(x - a),$$

which linearize the function G in n variables, i.e.,:

$$\begin{aligned}
G(x) &\simeq G(a) + \nabla^t G(a)(x - a) \\
&\simeq G(a) + \nabla^t G(a)x - \nabla^t G(a)a \\
&\simeq \nabla^t G(a)x + (G(a) - \nabla^t G(a)a)
\end{aligned} \tag{1}$$

where $\nabla^t G(x) = \left(\frac{\partial G(x)}{\partial x_1}, \frac{\partial G(x)}{\partial x_2}, \dots, \frac{\partial G(x)}{\partial x_n} \right)$.

In the following part, we give a detailed description of the main steps of this formulation.

2.1.2 Description of the algorithm formulated by Sivri (FS)

- **Step 0:** Reconsider the precedent quadratic fractional program:

$$(QFP) \begin{cases} \min J(x) = \frac{\frac{1}{2}x^t Hx + c^t x + c_0}{\frac{1}{2}x^t Dx + d^t x + d_0} \\ P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}, \end{cases}$$

- **Step 1:** We start from a known strictly feasible solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^t$, which satisfies the condition **(a3)**.
- **Step 2:** We linearize the objective function J of (QFP) via Taylor's development at the point $\tilde{x}$ using (1). We obtain the following linear program:

$$(LP_{\tilde{x}}) \begin{cases} \min [\nabla^t J(\tilde{x})x + (J(\tilde{x}) - \nabla^t J(\tilde{x})\tilde{x})] \\ P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}, \end{cases}$$

where

$$\nabla J(x) = \frac{\left(\frac{1}{2}x^t Dx + d^t x + d_0 \right) (Hx + c) - \left(\frac{1}{2}x^t Hx + c^t x + c_0 \right) (Dx + d)}{\left(\frac{1}{2}x^t Dx + d^t x + d_0 \right)^2}$$

- **Step 3:** We resolve $(LP_{\tilde{x}})$ to get its optimal solution noted $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)^t$.
- **Step 4 :** We develop again the objective function J of (QFP) via Taylor's development this time at the point $\bar{x}$ (optimal solution of $(LP_{\tilde{x}})$) using (1). We obtain the following linear program:

$$(LP_{\bar{x}}) \begin{cases} \min[\nabla^t J(\bar{x})x + (J(\bar{x}) - \nabla^t J(\bar{x})\bar{x})] \\ P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}, \end{cases}$$

- **Step 5:** We resolve $(LP_{\bar{x}})$ to get its optimal solution noted $\underline{x} = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n)^t$.
- **Step 6:** If the two successive solutions $\bar{x}$ and $\underline{x}$ coincid, then it's the optimal solution of (QFP). Else, assign $\underline{x}$ to $\bar{x}$ and return to Step 4.

Remark 2.2: In the original paper of Sivri et al. [23], the authors resolve the obtained linear programs $(LP_{\bar{x}}, LP_{\underline{x}})$ by the simplex method. In this paper, we resolve the previous sequence of linear programs via the interior point method of Ye-Lustig's approach [14] which is described below.

2.1.3 Algorithm of Ye-Lustig

Let the linear program below:

$$(LP_Y) \begin{cases} \min c_1^t x \\ A_1 x = b_1 \\ x \geq 0, \end{cases}$$

where $A_1 \in \mathbb{R}^{m \times n}$ with $\text{rank}(A_1) = m < n$, $b_1 \in \mathbb{R}^m$ and $c_1 \in \mathbb{R}^n$.

Next, we give the description of the algorithm of Ye-Lustig to solve (LP_Y) . For more details, see [14, 15].

Prototype of the algorithm of Ye-Lustig

Begin

Initialization

x_0 a strictly feasible solution for (LP_Y) (i.e., x_0 satisfies the condition (a3)), $k=0$ and ε be a given precision.

- **While** $\frac{\|P_k\|}{|c_1^t x_0|} > \varepsilon$ **do**

1. Construct the matrix

$$\bar{B}_k = [\bar{A}_k, -b_1], \text{ where } \bar{A}_k = A_1 D_k \text{ and } D_k = \text{diag}(x_k)$$

2. Calculate the projection P_k

$$P_k = [I - \bar{B}_k^t (\bar{B}_k \bar{B}_k^t)^{-1} \bar{B}_k] \begin{pmatrix} D_k c_1 \\ -c_1^t x_k \end{pmatrix} \text{ and take } d_k = \frac{P_k}{\|P_k\|}$$

3. Calculate the current iterate

$$\bar{y}^{(k+1)} = \frac{e_{n+1}}{n+1} - \alpha_k r d_k, \quad r = \left[\sqrt{n(n+1)} \right]^{-1},$$

with $0 < \alpha_k < 1$ and $e_{n+1} = (1, 1, \dots, 1)^t \in \mathbb{R}^{n+1}$.

4. Calculate

$$x_{k+1} = \frac{D_k \bar{y}^{(k+1)}[n]}{\bar{y}_{n+1}^{(k+1)}}, \quad k = k + 1.$$

- **End while**

End

In the following part, we present our new quadratic formulation allowing us to transform (QFP) into an equivalent quadratic program (QP). The obtained quadratic program (QP) will be processed and solved by the DCA algorithm [18] (difference of two convex functions algorithm)

2.2 New quadratic formulation (FB)

Reconsider again the precedent problem (QFP):

$$(QFP) \begin{cases} \min J(x) = \frac{\frac{1}{2}x^t H x + c^t x + c_0}{\frac{1}{2}x^t D x + d^t x + d_0} \\ P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\} \end{cases}$$

Lemma 2.1: *The quadratic fractional programming problem (QFP) can be written in the following equivalent quadratic programming problem (QP):*

$$(QP) \begin{cases} \min \frac{1}{2} x^t Q x + q^t x + q_0 \\ P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}, \end{cases}$$

where

$$\begin{aligned} Q &= H - z^k D \in \mathbb{R}^{n \times n}, \\ q^t &= c^t - z^k d^t \in \mathbb{R}^n, \\ q_0 &= c_0 - z^k d_0 \in \mathbb{R}, \end{aligned}$$

with z^k is an upper bound of the optimal value of the objective function of (QFP), we take

$$z^k = J(x^k),$$

such that x^k is a strictly feasible solution of (QFP) i.e., x_0 satisfies the condition (a3).

Proof: If z^* is an optimal value of (QFP), then:

$$(QFP) \left\{ \begin{aligned} &\min \left[\frac{\frac{1}{2}x^t Hx + c^t x + c_0}{\frac{1}{2}x^t Dx + d^t x + d_0} \right] = z^* \\ &P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}, \end{aligned} \right.$$

is equivalent to:

$$\left\{ \begin{aligned} &\min \left[\frac{1}{2}x^t Hx + c^t x + c_0 - z^* \left(\frac{1}{2}x^t Dx + d^t x + d_0 \right) \right] = 0 \\ &P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}. \end{aligned} \right.$$

This give us:

$$\left\{ \begin{aligned} &\min \left[\frac{1}{2}x^t (H - z^* D)x + (c^t - z^* d^t)x + (c_0 - z^* d_0) \right] = 0 \\ &P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\}. \end{aligned} \right.$$

Since z^* is generally unknown, we approximate it by an upper bound z^k :

$$z^* \leq z^k = J(x^k).$$

Then, we set:

$$\begin{aligned} Q &= H - z^* D \simeq H - z^k D, \\ q^t &= c^t - z^* d^t \simeq c^t - z^k d^t, \\ q_0 &= c_0 - z^* d_0 \simeq c_0 - z^k d_0, \end{aligned}$$

with:

$$z^k = \frac{\frac{1}{2}(x^k)^t Hx^k + c^t x^k + c_0}{\frac{1}{2}(x^k)^t Dx^k + d^t x^k + d_0},$$

and x^k satisfies the assumption **a3**.

Therefore, (QFP) becomes equivalent to the following quadratic program:

$$(QFP) \Leftrightarrow (QP) \begin{cases} \min \frac{1}{2} x^t Q x + q^t x + q_0 \\ P = \{x \in \mathbb{R}^n / Ax = b; x \geq 0\} \end{cases}$$

Remark 2.3: In order to solve the obtained equivalent problem (QP) (usually non-convexe), we use an algorithm called **DCA** (difference of two convex functions algorithm), proposed for the first time by Pham Dinh Tao [18] in 1985.

We present in the following part a reminder of this algorithm.

2.2.1 General principle of DCA algorithm

A difference of two convex functions program (DC) is an optimization problem defined as:

$$(P_{DC}) : \alpha = \inf \{f(x) = g_c(x) - h_c(x), x \in \mathbb{R}^n\},$$

with g_c and h_c are two convex functions. $g_c - h_c$ is the DC decomposition of the function f . (P_{DC}) is an unconstrained optimization problem.

A constrained DC program of the form

$$\inf \{f(x) = g_c(x) - h_c(x), x \in C\},$$

where C is a bounded convex set, can be written in the (P_{DC}) form by adding the indicator function of C to the function g_c , i.e.,

$$(P_{DC}) : \alpha = \inf \{F(x) = \varphi(x) - h_c(x), x \in \mathbb{R}^n\},$$

where

$$\varphi(x) = g_c(x) + \chi_C(x)$$

and

$$\chi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{otherwise.} \end{cases}$$

2.2.2 Choice of decomposition

For a general quadratic program (not necessarily convex), many DC decompositions were proposed in [11, 12], that's why we should choose the best decomposition which accelerates the algorithm's convergence.

Consider the quadratic program below:

$$\begin{cases} \min f(x) = \frac{1}{2} x^t Q x + q^t x \\ x \in C, \end{cases} \quad (2)$$

where Q is a symmetric matrix of $\mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$.

In this paper, we pose

$$f(x) = g_c(x) - h_c(x),$$

where

$$g_c(x) = \chi_c(x) + \frac{1}{2} x^t (Q + \rho I_n) x + q^t x \text{ and } h_c(x) = \frac{1}{2} \rho \|x\|^2. \quad (3)$$

With $\rho \geq \rho_- = \max(0, -\lambda_-)$ where λ_- is the smallest eigenvalue of the matrix Q . It is clear that both g_c and h_c are convex.

Then, (2) is a **DC** program of the form

$$\min \{f(x) = g_c(x) - h_c(x), x \in \mathbb{R}^n\}. \quad (4)$$

2.2.3 Description of DCA algorithm with the last decomposition

Begin

- **Initialization**

Let $x^0 \in C$ be a starting point, ε a given precision and $k = 0$.

- **While** the stopping criterion of optimality is not satisfied do

- **Step 1:** Compute $y^k = \rho x^k$.
- **Step 2:** Compute x^{k+1} which gives an optimal solution of the following quadratic convex problem

$$\min \left[\frac{1}{2} x^t (Q + \rho I_n) x + x^t (q - y^k), x \in C \right].$$

- **Step 3:** Set $k = k + 1$.

End.

Remark 2.4: The stopping criterion used is either $\|x^{k+1} - x^k\|$ or $|(g_c - h_c)(x^k + 1) - (g_c - h_c)(x^k)| \leq \varepsilon$.

3. Numerical tests

We establish a comparative numerical study between the formulation given by Sivri (FS) and our new quadratic formulation (FB) to solve (QFP). In Sivri's formulation, we consider Ye-Lustig's approach conversely to their original paper where Sivri et al. used the simplex approach. Besides, in our new quadratic formulation, we use DCA algorithm based on the choice of the DC decomposition described in (3)

We take $\varepsilon = 10^{-6}$ and we denote by:

- **k** : The iteration number required to get an optimal solution.
- **T(s)** : The execution time expressed in seconds.

Example 3.1: Let:

$$(QFP1) \begin{cases} \max \left(\frac{x_2^2 + x_1 x_2 + x_1 + 4x_2 + 3}{x_1^2 + x_1 x_2 + 6x_1 + 4x_2 + 8} \right) \\ -x_1 + x_2 \leq 1 \\ x_1 + 2x_2 \leq 7, \\ x_1, x_2 \geq 0. \end{cases}$$

Optimal solution and optimal value obtained respectively via Ye-Lustig's algorithm with the formulation FS:

$$\begin{aligned} x^* &= (1.6667, 2.6667)^t \\ z^* &= 0.7492 \end{aligned}$$

Optimal solution and optimal value obtained respectively via DCA algorithm with the formulation FB:

$$\begin{aligned} x^* &= (1.6667, 2.6667)^t \\ z^* &= 0.7492 \end{aligned}$$

Example 3.2: Let:

$$(QFP2) \begin{cases} \max \left(\frac{2x_1^2 + 2x_2^2 + 4x_1x_2 + 46x_1 + 46x_2 + 204}{-4x_1^2 - x_2^2 - 4x_1x_2 + 8x_1 - 4x_2 + 165} \right) \\ x_1 + 2x_2 \leq 2 \\ 3x_1 + x_2 \leq 4 \\ -5x_1 + 3x_2 \leq 0 \\ -x_1 + x_2 \leq 0 \\ x_1, x_2 \geq 0. \end{cases}$$

Optimal solution and optimal value obtained respectively via Ye-Lustig's algorithm with the formulation FS:

$$\begin{aligned} x^* &= (1.2, 0.4)^t \\ z^* &= 1.6729 \end{aligned}$$

Optimal solution and optimal value obtained respectively via DCA algorithm with the formulation FB:

$$\begin{aligned} x^* &= (1.2, 0.4)^t \\ z^* &= 1.6729 \end{aligned}$$

Example 3.3: Let:

$$(QFP3) \begin{cases} \max \left(\frac{x_1^2 + x_1x_2 + 3x_1 + 2x_2 + 2}{2x_2^2 + x_1x_2 + 3x_1 + 9x_2 + 9} \right) \\ 4x_1 + 2x_2 \leq 8 \\ x_1 + 2x_2 \leq 6 \\ x_1, x_2 \geq 0. \end{cases}$$

Optimal solution and optimal value obtained respectively via Ye-Lustig's algorithm with the formulation FS:

$$\begin{aligned} x^* &= (2, 0)^t \\ z^* &= 0.8 \end{aligned}$$

Optimal solution and optimal value obtained respectively via DCA algorithm with the formulation FB:

$$\begin{aligned} x^* &= (2, 0)^t \\ z^* &= 0.8 \end{aligned}$$

Example 3.4: Let:

$$(QFP4) \begin{cases} \max \left(\frac{4x_1^2 + 8x_2^2 + 12x_1x_2 + 4x_1 + 4x_2}{4x_1 + 8x_2 + 4} \right) \\ -2x_1 + x_2 \leq 3 \\ 4x_1 + 2x_2 \leq 8 \\ x_1, x_2 \geq 0. \end{cases}$$

Optimal solution and optimal value obtained respectively via Ye-Lustig's algorithm with the formulation FS:

$$\begin{aligned} x^* &= (0.25, 3.5)^t \\ z^* &= 3.75 \end{aligned}$$

Optimal solution and optimal value obtained respectively via DCA algorithm with the formulation FB:

$$\begin{aligned} x^* &= (0.25, 3.5)^t \\ z^* &= 3.75 \end{aligned}$$

Example 3.5: Let:

$$(QFP5) \begin{cases} \max \left(\frac{8x_1^2 + 18x_2^2 + 24x_1x_2 - 2}{6x_1 + 9x_2 + 3} \right) \\ x_1 + 3x_2 \leq 5 \\ 2x_1 + x_2 \leq 2 \\ x_1, x_2 \geq 0. \end{cases}$$

Optimal solution and optimal value obtained respectively via Ye-Lustig's algorithm with the formulation FS:

$$\begin{aligned} x^* &= (0.2, 1.6)^t \\ z^* &= 2.8 \end{aligned}$$

Optimal solution and optimal value obtained respectively via DCA algorithm with the formulation FB:

$$\begin{aligned} x^* &= (0.2, 1.6)^t \\ z^* &= 2.8 \end{aligned}$$

In this table, we summarize the results obtained via the formulations FS and FB.

Examples	FS(Ye-Lustig)		FB(DCA)	
	k	T(s)	k	T(s)
QFP 1	1	0.004829	2	0.005409
QFP 2	1	0.005111	2	0.004243
QFP 3	1	0.005017	2	0.006852
QFP 4	1	0.005161	2	0.005666
QFP 5	1	0.005607	2	0.005603

Comments

For different dimensions, of the tested examples, we conclude that the numerical tests carried out give the optimal solution of (QFP) in a short time and a small number of iterations via the two formulations FS and FB.

4. Conclusion

- We exposed two formulations for solving a quadratic fractional programming problems (QFP). At first, we convert (QFP) into a sequence of equivalent linear problems (LP) then solve them using interior point methods. The second formulation consists at converting (QFP) into a quadratic programming problem (QP) and solve it using DCA algorithm.
- We developed an efficient algorithm for solving such problems and tested a number of numerical examples to illustrate the effectiveness of the two formulations. The two formulations can solve any type of (QFP) using the obtained algorithms, where we get the optimal solution of (QFP) very quickly with few number of iterations in a short time.

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