Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 47 (2026), No. 1, pp. 119–131
DOI : 10.47974/JIOS-1468

Numerical solution of some facility location problems

Stefan M. Stefanov
Department of Mathematics
South-West University "Neofit Rilski"
2700 Blagoevgrad
Bulgaria

Abstract

Some facility location problems are considered in the paper, in particular, box constrained facility location problems. The optimal solution is described by a characterization theorem, and an efficient convergent polynomial algorithm is proposed.

Subject Classification (2020): *90B05, 90C25.*

Keywords: *Convex separable programming, Facility location problem, Characterization theorem.*

1 Introduction

Consider the following box constrained facility location problem.
(C)

$$\min\left\{c(x) \equiv c(x_1,\dots,x_n) = \sum_{j=1}^{n}\left(\frac{s_j a_j}{x_j} + \frac{p_j x_j}{2}\right)\right\} \tag{1}$$

subject to

$$\sum_{j=1}^{n} d_j x_j \le \beta \tag{2}$$

$$x_j \ge 0, \quad j = 1,\dots,n. \tag{3}$$

In problem (C), for the item j, where $j = 1,\dots,n$, s_j is the cost of setup, a_j is the demand rate, x_j is quantity of the order, p_j is the cost for holding a unit of the item, d_j is the storage area per facility unit, and β is the capacity of the store (the maximum storage area).

E-mail: **stefm@swu.bg**

The objective function describes the total cost and it must be minimized.

The unconstrained minimum of function $c(\mathrm{x})$, defined by (1), is

$$x_j^* = \sqrt{\frac{2s_j a_j}{p_j}}, \quad \forall j. \tag{4}$$

Taking the positive values of the square roots in (4), then x_j^*'s fulfil constraints (3) of problem (C). If x_j^*'s fulfil constraint (2) as well, then x_j^*'s constitute the optimal solution of (C).

Otherwise, if $\sum_{j=1}^{n} d_j x_j^* > \beta$, constraint (2) has to be fulfilled as equality

$$\sum_{j=1}^{n} d_j x_j^* = \beta.$$

The function of Lagrange for the considered problem (C) (1)-(3) is

$$L(x,\gamma) = \sum_{j=1}^{n} \left(\frac{s_j a_j}{x_j} + \frac{p_j x_j}{2} \right) + \gamma \left(\sum_{j=1}^{n} d_j x_j - \beta \right),$$

with $\gamma \geq 0$ the dual variable, corresponding to the single inequality constraint (2).

Because $L(x,\gamma)$ is convex regarding γ and x, the optimal γ^* and x^* fulfil the optimality conditions

$$\frac{\partial L}{\partial x_j} \equiv -\frac{s_j a_j}{x_j^2} + \frac{p_j}{2} + \gamma d_j = 0, \quad \forall j, \tag{5}$$

$$\frac{\partial L}{\partial \gamma} \equiv \sum_{j=1}^{n} d_j x_j - \beta = 0. \tag{6}$$

Conditions (5) imply

$$x_j^* = \sqrt{\frac{2s_j a_j}{p_j + 2\gamma d_j}}, \quad \forall j. \tag{7}$$

If $\gamma = 0$, that is, if there is no limit on the storage, expressions (4) and (7) coincide, and (4) is known as the *Wilson's economic lot size formula*.

Consider problem (CBV), which is a version of problem (C) (1)-(3), such that instead of nonnegativity constraints (3), box constraints are included.

(CBV)

$$\min\left\{c(\mathrm{x}) \equiv c(x_1,\ldots,x_n) \equiv \sum_{j=1}^{n} c_j(x_j) = \sum_{j=1}^{n}\left(\frac{s_j a_j}{x_j}+\frac{p_j x_j}{2}\right)\right\} \tag{8}$$

subject to

$$\sum_{j=1}^{n} d_j x_j \le \beta \tag{9}$$

$$l_j \le x_j \le u_j, \quad \forall j, \tag{10}$$

where $l_j \ge 0,\ j=1,\ldots,n,$ in accordance with economic interpretation of x_j's; and the values of $s_j,\ a_j,\ p_j,$ and d_j are also assumed to be nonnegative (even positive) for each $j=1,\ldots,n,$ in accordance with their economic meaning.

Note that if $l_j = 0$ and $u_j = +\infty \quad \forall j=1,\ldots,n,$ constraints (10) imply constraints (3).

Since $c_j'(x_j) = -\frac{s_j a_j}{x_j^2}+\frac{p_j}{2},$ and therefore $c_j''(x_j) = \frac{2 s_j a_j}{x_j^3} > 0$ by assumption, then functions $c_j(x_j)$ are strictly convex, and therefore, $c(\mathrm{x})$ is also a strictly convex function.

Problems (C), (CBV) and related problems arise in facility location and other settings. Related topics are considered, e.g., in reference titles [1-33].

2. Characterization theorem for problem (CBV)

Assume that

(A.1) $l_j \le u_j$ for each $j=1,\ldots,n.$ If $l_k = u_k$ for an index $k,$ where $1 \le k \le n,$ then $x_k := l_k = u_k$ is known a priori.

(A.2)

$$\sum_{j=1}^{n} d_j l_j \le \beta, \tag{11}$$

otherwise the constraints (9) and (10) are incompatible and therefore the feasible region X (9)-(10) would be empty. We also assume that $\beta \le \sum_{j=1}^{n} d_j u_j$ in certain cases, which will be described below.

The function of Lagrange for (CBV) is as follows

$$L(\mathrm{x},\mathrm{v},\mathrm{w},\gamma) = \sum_{j=1}^{n}\left(\frac{s_j a_j}{x_j}+\frac{p_j x_j}{2}\right) + \gamma\Big(\sum_{j=1}^{n} d_j x_j - \beta\Big) + \sum_{j=1}^{n} v_j(l_j - x_j) + \sum_{j=1}^{n} w_j(x_j - u_j), \tag{12}$$

where $\gamma \in \mathbb{R}^1_+$, $\mathrm{v} = (v_1, \dots, v_n) \in \mathbb{R}^n_+$, $\mathrm{w} = (w_1, \dots, w_n) \in \mathbb{R}^n_+$.

The KKT conditions for (CBV) are as follows

$$-\frac{s_j a_j}{(x_j^*)^2} + \frac{p_j}{2} + \gamma d_j - v_j + w_j = 0, \quad \forall j, \tag{13}$$

$$v_j(l_j - x_j^*) = 0, \quad \forall j, \tag{14}$$

$$w_j(x_j^* - u_j) = 0, \quad \forall j, \tag{15}$$

$$\gamma\left(\sum_{j=1}^n d_j x_j^* - \beta\right) = 0, \quad \gamma \in \mathbb{R}^1_+, \tag{16}$$

$$\sum_{j=1}^n d_j x_j^* \le \beta, \tag{17}$$

$$l_j \le x_j^* \le u_j, \quad \forall j, \tag{18}$$

$$v_j \in \mathbb{R}^1_+, w_j \in \mathbb{R}^1_+, \quad \forall j, \tag{19}$$

where γ, v_j, and w_j, are the dual variables (Lagrange multipliers) for the constraints (9), $l_j \le x_j$, and $x_j \le u_j$ for each $j = 1, \dots, n$, respectively. If $l_j = -\infty$ or $u_j = +\infty$ for an index j, $1 \le j \le n$, then the KKT condition (14) (condition (15), respectively) as well as the dual variable v_j (dual variable w_j, respectively) are not taken into account.

Theorem 1: (Characterization theorem for problem (CBV))

Denote $J = \{1, \dots, n\}$. *A point* $x^* = (x_j^*)_{j \in J} \in X$ *(9)-(10) is the optimal solution of problem* (CBV) *iff there is a scalar* $\gamma \in \mathbb{R}^1_+$, *such that*

$$x_j^* = l_j, \quad j \in J_l^\gamma \stackrel{\text{def}}{=} \left\{ j \in J : \gamma \ge \frac{s_j a_j}{d_j l_j^2} - \frac{p_j}{2d_j} \right\}, \tag{20}$$

$$x_j^* = u_j, \quad j \in J_u^\gamma \stackrel{\text{def}}{=} \left\{ j \in J : \gamma \le \frac{s_j a_j}{d_j u_j^2} - \frac{p_j}{2d_j} \right\}, \tag{21}$$

$$x_j^* = \sqrt{\frac{2 s_j a_j}{p_j + 2\gamma d_j}}, \quad j \in J^\gamma \stackrel{\text{def}}{=} \left\{ j \in J : \frac{s_j a_j}{d_j u_j^2} - \frac{p_j}{2d_j} < \gamma < \frac{s_j a_j}{d_j l_j^2} - \frac{p_j}{2d_j} \right\}. \tag{22}$$

Note that expressions of x_j^* in (22) and in (7) coincide.

Proof: *Necessity.* Suppose that $x^* = (x_j^*)_{j\in J}$ is the optimal solution to (CBV). Then $\exists$ scalars (dual variables) γ, v_j, w_j, $j \in J$, fulfilling the KKT optimality conditions (13)-(19).

(1) Let $\gamma > 0$. Then system (13)-(19) of the KKT conditions is of the form

$$(13),(14),(15),(18),(19) \quad \text{and} \quad \sum_{j\in J} d_j x_j^* = \beta.$$

(a) If $x_j^* = l_j$, then (19) implies $v_j \ge 0$ and (15) implies $w_j = 0$. Then (13) implies

$$-\frac{s_j a_j}{(x_j^*)^2} + \frac{p_j}{2} = v_j - \gamma d_j \ge -\gamma d_j.$$

By assumption $d_j > 0$, hence

$$\gamma \ge \frac{s_j a_j}{d_j (x_j^*)^2} - \frac{p_j}{2d_j} \equiv \frac{s_j a_j}{d_j l_j^2} - \frac{p_j}{2d_j}.$$

(b) If $x_j^* = u_j$, then (14) implies $v_j = 0$ and (19) implies $v_j \ge 0$. Then from (13) it follows that

$$-\frac{s_j a_j}{(x_j^*)^2} + \frac{p_j}{2} = -w_j - \gamma d_j \le -\gamma d_j.$$

Hence,

$$\gamma \le \frac{s_j a_j}{d_j (x_j^*)^2} - \frac{p_j}{2d_j} \equiv \frac{s_j a_j}{d_j u_j^2} - \frac{p_j}{2d_j}.$$

(c) If $l_j < x_j^* < u_j$, then (14) and (15) imply $v_j = w_j = 0$. Then from (13) it follows that

$$\frac{s_j a_j}{(x_j^*)^2} - \frac{p_j}{2} = \gamma d_j.$$

Hence,

$$\gamma = \frac{s_j a_j}{d_j (x_j^*)^2} - \frac{p_j}{2d_j} \quad \text{and} \quad x_j^* = \sqrt{\frac{2 s_j a_j}{p_j + 2\gamma d_j}}.$$

By the assumption, $d_j > 0 \;\; \forall \;\; j \in J$, $\gamma > 0$, $l_j < x_j^* < u_j$, therefore

$$\gamma = \frac{s_j a_j}{d_j (x_j^*)^2} - \frac{p_j}{2d_j} < \frac{s_j a_j}{d_j l_j^2} - \frac{p_j}{2d_j}, \quad \gamma = \frac{s_j a_j}{d_j (x_j^*)^2} - \frac{p_j}{2d_j} > \frac{s_j a_j}{d_j u_j^2} - \frac{p_j}{2d_j},$$

hence,

$$\frac{s_j a_j}{d_j u_j^2} - \frac{p_j}{2d_j} < \gamma < \frac{s_j a_j}{d_j l_j^2} - \frac{p_j}{2d_j}.$$

(2) Let $\gamma = 0$. Then system (13)-(19) of the KKT optimality conditions is of the form

$$-\frac{s_j a_j}{(x_j^*)^2} + \frac{p_j}{2} - v_j + w_j = 0, j \in J, \quad \text{and} \quad (14),(15),(17),(18),(19).$$

(a) If $x_j^* = l_j$, then (19) implies $v_j \geq 0$ and (15) implies $w_j = 0$. Then from (13) it follows that

$$-\frac{s_j a_j}{l_j^2} + \frac{p_j}{2} \equiv -\frac{s_j a_j}{(x_j^*)^2} + \frac{p_j}{2} = v_j \geq 0.$$

Multiply both sides of the above inequality by $-\frac{1}{d_j}$ (< 0 by the assumption), hence

$$\frac{s_j a_j}{d_j l_j^2} - \frac{p_j}{2d_j} \leq 0 \equiv \gamma.$$

(b) If $x_j^* = u_j$, then (14) implies $v_j = 0$ and (19) implies $w_j \geq 0$. Then from (13) it follows that

$$-\frac{s_j a_j}{u_j^2} + \frac{p_j}{2} \equiv -\frac{s_j a_j}{(x_j^*)^2} + \frac{p_j}{2} = -w_j \leq 0.$$

Multiply the above inequality by $-\frac{1}{d_j} < 0$, hence

$$\frac{s_j a_j}{d_j u_j^2} - \frac{p_j}{2d_j} \geq 0 \equiv \gamma.$$

(c) If $l_j < x_j^* < u_j$, then (14) and (15) imply $v_j = w_j = 0$. Then (13) implies

$$\frac{s_j a_j}{(x_j^*)^2} - \frac{p_j}{2} = 0.$$

By the assumption, $l_j < x_j^*$, $x_j^* < u_j$, $j \in J$. Then

$$0 = \frac{s_j a_j}{(x_j^*)^2} - \frac{p_j}{2} < \frac{s_j a_j}{l_j^2} - \frac{p_j}{2}, \quad 0 = \frac{s_j a_j}{(x_j^*)^2} - \frac{p_j}{2} > \frac{s_j a_j}{u_j^2} - \frac{p_j}{2}.$$

Multiply the above two inequalities by $\frac{1}{d_j} > 0$ and take into consideration that $\gamma = 0$ in the considered case (2). Then in the subcase (c) we get

$$\frac{s_j a_j}{d_j u_j^2} - \frac{p_j}{2d_j} < 0 \equiv \gamma < \frac{s_j a_j}{d_j l_j^2} - \frac{p_j}{2d_j}.$$

Introduce the index sets J_l^γ, J_u^γ, and J^γ, defined by (20), (21) and (22), respectively. Evidently $J_l^\gamma \cup J_u^\gamma \cup J^\gamma = J$. Using these three index sets, subcases (a), (b), and (c) of cases (1)-(2) are described by (20), (21), and (22), respectively.

Sufficiency. Let $x^* \in X$, and let entries of the vector (point) x^* fulfil (20), (21), and (22) with $\gamma \in \mathbb{R}_+^1$.

(1) Let $\gamma > 0$. Set:

$$\gamma = \frac{s_j a_j}{d_j (x_j^*)^2} - \frac{p_j}{2d_j} (> 0), \quad v_j = w_j = 0, \quad j \in J^\gamma;$$

$$v_j = -\frac{s_j a_j}{l_j^2} + \frac{p_j}{2} + \gamma d_j \quad (\geq 0), \quad w_j = 0, \quad j \in J_l^\gamma;$$

$$v_j = 0, \quad w_j = \frac{s_j a_j}{u_j^2} - \frac{p_j}{2} - \gamma d_j \quad (\geq 0), \quad j \in J_u^\gamma.$$

Obviously the KKT optimality conditions (13), (14), (15), (16), (19) are fulfilled, and the KKT conditions (17) and (18) are fulfilled as well according to the assumption $x^* \in X$ of Theorem 1.

(2) Let $\gamma = 0$. Therefore (22) with $\gamma = 0$ implies $p_j (x_j^*)^2 - 2 s_j a_j = 0$. Set:

$$\gamma = \frac{s_j a_j}{d_j (x_j^*)^2} - \frac{p_j}{2d_j} \quad (= 0), \quad v_j = w_j = 0, \quad j \in J^{\gamma=0};$$

$$v_j = -\frac{s_j a_j}{l_j^2} + \frac{p_j}{2} + \gamma d_j = -\frac{s_j a_j}{l_j^2} + \frac{p_j}{2} \quad (\geq 0), \quad w_j = 0, \quad j \in J_l^{\gamma=0};$$

$$v_j = 0, \quad w_j = \frac{s_j a_j}{u_j^2} - \frac{p_j}{2} - \gamma d_j = \frac{s_j a_j}{u_j^2} - \frac{p_j}{2} \quad (\geq 0), \quad j \in J_u^{\gamma=0}.$$

Evidently the KKT optimality conditions (13), (14), (15), (19) are fulfilled, the KKT conditions (17) and (18) are fulfilled as well according to the assumption of Theorem 1 that $x^* \in X$, and the KKT condition (16) evidently is fulfilled with $\gamma = 0$.

From cases (1) and (2) it follows that the KKT sufficient optimality conditions (13)-(19) for (CBV) are fulfilled. Therefore x^* is an optimal solution to (CBV). This optimal solution is unique due to the strict convexity of function $c(x)$.

3. The Algorithm

The KKT optimality condition (16), with the use of expressions (20), (21), and (22) of the statement of characterization Theorem 1, can be presented as follows.

$$\gamma\left(\sum_{j\in J_l^\gamma} d_j l_j + \sum_{j\in J_u^\gamma} d_j u_j + \sum_{j\in J^\gamma} d_j \sqrt{\frac{2s_j a_j}{p_j+2\gamma d_j}} - \beta\right) = 0, \quad \gamma \in \mathbb{R}^1_+. \tag{23}$$

Define

$$\delta(\gamma) \stackrel{\text{def}}{=} \sum_{j\in J_l^\gamma} d_j l_j + \sum_{j\in J_u^\gamma} d_j u_j + \sum_{j\in J^\gamma} d_j \sqrt{\frac{2s_j a_j}{p_j+2\gamma d_j}} - \beta. \tag{24}$$

Introduce the following notation at iteration k of the proposed algorithm: $\gamma^{(k)}$ is value of the dual variable γ for the constraint (9), $\beta^{(k)}$ is the right-hand side of (9), $J^{(k)}$, $J_l^{\gamma(k)}$, $J_u^{\gamma(k)}$, and $J^{\gamma(k)}$, respectively, are the current index sets J, J_l^γ, J_u^γ, and J^γ.

The next algorithm for (CBV) is based on the characterization Theorem 1.

Algorithm 1

0. (Initialization) $n^{(0)} := n$, $k := 0$, $J := \{1,\dots,n\}$,
   and $J^{(0)} := J$, $J_l^\gamma := \varnothing$, $J_u^\gamma := \varnothing$, $\beta^{(0)} := \beta$.
   If $\sum_{j\in J} d_j l_j \le \beta$, to step 1, otherwise go to step 9.
1. Construct the initial index sets J_l^0, J_u^0, J^0 with $\gamma = 0$. Calculate the value

   $$\delta(0) := \sum_{j\in J_l^0} d_j l_j + \sum_{j\in J_u^0} d_j u_j + \sum_{j\in J^0} d_j \sqrt{\frac{2s_j a_j}{p_j}} - \beta,$$

   using expression (24) with $\gamma = 0$.
   If $\delta(0) \le 0$, then $\gamma := 0$, to step 8
   otherwise if $\delta(0) > 0$:
   if $\beta \le \sum_{j\in J} d_j u_j$, then to step 2

else if $\beta > \sum_{j\in J} d_j u_j$, then to step 9.

2. Set $J^{\gamma(k)} := J^{(k)}$. Determine $\gamma^{(k)}$ via the closed form expression for γ. Go to step 3.
3. Construct the current index sets $J_l^{\gamma(k)}$, $J_u^{\gamma(k)}$, $J^{\gamma(k)}$ via (20), (21), (22) (with $j \in J^{(k)}$) and determine $|J_l^{\gamma(k)}|$, $|J_u^{\gamma(k)}|$, and $|J^{\gamma(k)}|$. Go to step 4.
4. Calculate the value

$$\delta(\gamma^{(k)}) := \sum_{j\in J_l^{\gamma(k)}} d_j l_j + \sum_{j\in J_u^{\gamma(k)}} d_j u_j + \sum_{j\in J^{\gamma(k)}} d_j \sqrt{\frac{2s_j a_j}{p_j + 2\gamma^{(k)} d_j}} - \beta^{(k)}.$$

   Go to step 5.
5. If $\delta(\gamma^{(k)}) = 0$ or $J^{\gamma(k)} = \varnothing$, then:

$$\gamma := \gamma^{(k)},\ \ J_l^{\gamma} := J_l^{\gamma} \cup J_l^{\gamma(k)},\ \ J_u^{\gamma} := J_u^{\gamma} \cup J_u^{\gamma(k)},\ \ J^{\gamma} := J^{\gamma(k)},\ \text{to step 8}$$

   otherwise if $\delta(\gamma^{(k)}) > 0$, then go to step 6
   otherwise if $\delta(\gamma^{(k)}) < 0$, then go to step 7.
6. $x_j^* := l_j$, $j \in J_l^{\gamma(k)}$; $J^{(k+1)} := J^{(k)} \setminus J_l^{\gamma(k)}$; $\beta^{(k+1)} := \beta^{(k)} - \sum_{j\in J_l^{\gamma(k)}} d_j l_j$; $n^{(k+1)} := n^{(k)} - |J_l^{\gamma(k)}|$; $J_l^{\gamma} := J_l^{\gamma} \cup J_l^{\gamma(k)}$; $k := k+1$. Go to step 2.
7. $x_j^* := u_j$, $j \in J_u^{\gamma(k)}$; $J^{(k+1)} := J^{(k)} \setminus J_u^{\gamma(k)}$; $\beta^{(k+1)} := \beta^{(k)} - \sum_{j\in J_u^{\gamma(k)}} d_j u_j$; $n^{(k+1)} := n^{(k)} - |J_u^{\gamma(k)}|$; $J_u^{\gamma} := J_u^{\gamma} \cup J_u^{\gamma(k)}$; $k := k+1$. Go to step 2.
8. $x_j^* := l_j$ for $j \in J_l^{\gamma}$, $x_j^* := u_j$ for $j \in J_u^{\gamma}$, $x_j^* := \sqrt{\frac{2s_j a_j}{p_j + 2\gamma d_j}}$ for $j \in J^{\gamma}$. Go to step 10.
9. Problem (CBV) does not have an optimal solution since the feasible region X is empty or because there is no $\gamma^* > 0$ which fulfils Theorem 1.
10. End.

Theorem 2: *(Convergence of the proposed Algorithm 1)*
If $\{\gamma^{(k)}\}$ is obtained by Algorithm 1, then:

(i) $\delta(\gamma^{(k)}) > 0$ *implies* $\gamma^{(k)} \le \gamma^{(k+1)}$;

(ii) $\delta(\gamma^{(k)}) < 0$ *implies* $\gamma^{(k)} \ge \gamma^{(k+1)}$.

The relations (i) and (ii) of Theorem 2, combined with the expressions (20), (21), and (22) of the formulation of Theorem 1, state that the calculation of the optimal value of γ and construction of the optimal index sets J_l^γ, J_u^γ, and J^γ, are in accordance with the statement of the characterization Theorem 1.

Analysis of the proposed Algorithm 1 shows that the value of at least one variable of problem (CBV) is calculated at each iteration of the algorithm by solving a problem of the type (CBV) with *less* variables at Steps 2-7. Hence, Algorithm 1 is convergent for at most $n = |J|$ iterations. Furthermore, since each step of Algorithm 1 takes time, bounded from above by $O(n)$, then Algorithm 1 is an algorithm of polynomial complexity with $O(n^2)$ run time.

4. Conclusions

Some location problems are considered in this paper, in particular, box constrained facility location problems. The optimal solution is described by a characterization theorem, and an efficient convergent polynomial algorithm is suggested.

Open problems for future research are to study other related problems and to develop polynomial convergent algorithms for the corresponding optimization problems.

References

[1] G. R. Bitran and A. C. Hax, "Disaggregation and resource allocation using convex knapsack problems with bounded variables," *Management Science*, vol. 27, no. 4, pp. 431-441 (1981).

[2] P. Brucker, "An O(n) algorithm for quadratic knapsack problems," *Operations Research Letters*, vol. 3, no. 3, pp. 163-166 (1984).

[3] J.-P. Dussault, J. A. Ferland, and B. Lemaire, "Convex quadratic programming with one constraint and bounded variables," *Mathematical Programming*, vol. 36, no. 1, pp. 90-104 (1986).

[4] R. Helgason, J. Kennington, and H. Lall, "A polynomially bounded algorithm for a singly constrained quadratic program," *Mathematical Programming*, vol. 18, no. 3, pp. 338-343 (1980).

[5] A. Kozma, C. Conte, and M. Diehl, "Benchmarking large-scale distributed convex quadratic programming algorithms," *Optimization Methods and Software*, vol. 30, pp. 191-214 (2015).

[6] J. J. Moré and G. Toraldo, "Algorithms for bound constrained quadratic programming problems," *Numerische Mathematik*, vol. 55, no. 4, pp. 377-400 (1989).

[7] P. M. Pardalos and N. Kovoor, "An algorithm for a singly constrained class of quadratic programs subject to upper and lower bounds," *Mathematical Programming*, vol. 46, no. 3, pp. 321-328 (1990).

[8] S. M. Stefanov, A Lagrangian Dual Method for Solving Variational Inequalities, Kluwer Series in Mathematical Programming and Operations Research, Working Paper WP-KSMPOR-99-11, February, 10 pp (1999).

[9] S. M. Stefanov, On the Solution of Variational Inequality Problems by Using Cutting Plane Methods, Kluwer Series in Mathematical Programming and Operations Research, Working Paper WP-KSMPOR-99-12, February, 9 pp (1999).

[10] S. M. Stefanov, "On the implementation of stochastic quasigradient methods to some facility location problems," *Yugoslav Journal of Operations Research*, vol. 10, no. 2, pp. 235-256 (2000).

[11] S. M. Stefanov, Convex Separable Programming: Theory and Methods, Kluwer Academic Publishers, Dordrecht-Boston-London (2000).

[12] S. M. Stefanov, "Convex separable minimization subject to bounded variables," *Computational Optimization and Applications: An International Journal*, vol. 18, no. 1, pp. 27-48 (2001).

[13] S. M. Stefanov, "Polynomial algorithms for projecting a point onto a region defined by a linear constraint and box constraints in $\mathbf{R}^n$," *Journal of Applied Mathematics*, vol. 2004, no. 5, pp. 409-431 (2004).

[14] S. M. Stefanov, "Convex quadratic minimization subject to a linear constraint and box constraints," *Applied Mathematics Research Express AMRX*, vol. 2004, no. 1, pp. 17-42 (2004).

[15] S. M. Stefanov, "An efficient method for minimizing a convex separable logarithmic function subject to a convex inequality constraint or linear equality constraint," *Journal of Applied Mathematics and Decision Sciences* vol. 2006, Article ID 89307 (2006).

[16] S. M. Stefanov, "Minimization of a convex linear-fractional separable function subject to a convex inequality constraint or linear equality constraint and bounds on the variables," *Applied Mathematics Research eXpress*, vol. 2006, no. 4, Article ID 36581 (2006).

[18] S. M. Stefanov, "Solution of some convex separable resource allocation and production planning problems with bounds on the vari-

ables," *Journal of Interdisciplinary Mathematics*, vol. 13, no. 5, pp. 541-569 (2010).

[19] S. M. Stefanov, "Well-posedness and primal-dual analysis of some convex separable optimization problems," *Advances in Operations Research*, vol. 2013, pp. 10 pages, Article ID 279030 (2013).

[20] S. M. Stefanov, "On the solution of multidimensional convex separable continuous knapsack problem with bounded variables," *European Journal of Operational Research*, vol. 247, no. 2, pp. 366-369 (2015).

[21] S. M. Stefanov, *Separable Programming: Theory and Methods*, 4th rev. ed., Springer Science+Business Media, B.V., Dordrecht (2016).

[22] S. M. Stefanov, "Strictly convex separable optimization with linear equality constraints and bounded variables," *Journal of Statistics and Management Systems*, vol. 21, no. 2, pp. 261-272 (2018).

[23] S. M. Stefanov, "Characterization of the optimal solution of the convex generalized nonlinear transportation problem," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 5, pp. 745-756 (2019).

[24] S. M. Stefanov, "Characterization of the optimal solution of the convex separable continuous knapsack problem and related problems," *Journal of Information and Optimization Sciences*, vol. 42, no. 1, pp. 1-16 (2021).

[25] S. M. Stefanov, "On the numerical solution of separable stochastic inventory control problems," *Journal of Information and Optimization Sciences*, vol. 42, no. 3, pp. 533-561 (2021).

[26] S. M. Stefanov, *Separable Optimization: Theory and Methods*, Springer Optimization and Its Applications, vol. 177, Springer, Cham (2021).

[27] S. M. Stefanov, "Separable programming: a dynamic programming approach," In: *Separable Optimization*, Springer Optimization and Its Applications, vol. 177, Springer, Cham, pp. 85-129 (2021).

[28] S. M. Stefanov, "Numerical solution of some systems of nonlinear algebraic equations," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 6, pp. 1545-1564 (2021).

[29] S. M. Stefanov, "Numerical solution of systems of nonlinear equations defined by convex functions," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 4, pp. 951-962 (2022).

[30] S. M. Stefanov, "On some properties of linear functionals," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 8, pp. 2491-2502 (2022).

[31] S. M. Stefanov, "On the solution of quadratic programming problems," *Journal of Information and Optimization Sciences*, vol. 44, no. 2, pp. 243-253 (2023).

[32] S. M. Stefanov, "Continuous linear knapsack problems revisited," *Journal of Information and Optimization Sciences*, vol. 44, no. 5, pp. 909-922 (2023).

[33] D. G. Tian, "An exterior point polynomial-time algorithm for convex quadratic programming," *Computational Optimization and Applications*, vol. 61, pp. 51-78 (2015).

Received June, 2023