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Non-instantaneous impulsive fractional integro-differential inclusions with state-dependent delay

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Abstract

This paper is mainly concerned with the existence of mild solutions of a class of fractional integro-differential inclusions with state-dependent delay subject to not instantaneous impulses. The main results are obtained by using the nonlinear alternative of Leray-Schauder for multivalued maps due to Kakutani. An example is provided to illustrate the theory.

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1. Introduction

In recent years, there has been considerable interest among physicists, mathematicians, and engineers in fractional differential equations. This area has seen significant advancements in both theory and applications, as evidenced by the extensive development highlighted in [5, 19, 30, 33, 35, 41-44]. The investigation into the existence of solutions for fractional differential inclusions, initially prompted by [21], has been pursued by numerous researchers across various problem types. Moreover, several qualitative results have been attained, as documented in [2-4, 16, 40].

Hernández and O'Regan recently introduced a novel class of non-instantaneous impulsive differential equations in their work [26]. In their framework, impulsive behaviors commence abruptly at a specific time and endure for a finite duration. This type of non-instantaneous impulsive system is particularly suitable for exploring the dynamics of evolving processes in pharmacotherapy. Additionally, the existence of solutions for non-instantaneous impulsive fractional/integer order differential equations is discussed by various authors in [1, 12-14, 22, 32, 36, 38]. For further insights into differential equations involving different types of delays, interested readers are referred to [6-11, 15, 25, 31].

In this contribution, we study the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delay and non instantaneous impulses described by the form

$$\begin{cases} {}^{C}D_{\xi}^{\kappa\wp(\xi)+\mathcal{G}\wp(\xi)\in} \int^{\xi} a(\xi,\vartheta)\Psi(\vartheta,\wp_{\varpi(\vartheta,\wp\vartheta)},\wp(\vartheta))d\vartheta, \xi\in(\vartheta_j,\xi_{j+1}], j=0,\dots,\beta, \\ \wp(\xi)=\chi_j(\xi,\wp_{\varpi(\xi,\wp\xi)},\wp(\xi)), \quad \xi\in(\xi_j,\vartheta_j], j=1,\dots,\beta, \\ \wp_0=\hbar\in\overline{\Theta} \end{cases} \tag{1}$$

Where ${}^{C}D_{\xi}^{\kappa}$ is the Caputo fractional derivative of order $0<\kappa<1, \mathcal{G}$ generates a compact and uniformly bounded semigroup $\{\mathcal{X}(\xi)\}_{\xi\geq 0}$ on Ξ, $\mathfrak{I}=[0,\omega]$, $\omega>0$, $\Psi:\mathfrak{I}\times\overline{\Theta}\times\Xi\to\mathcal{P}(\Xi)$ is a multivalued map ($\mathcal{P}(\Xi)$ is the family of all nonempty subsets of Ξ), and $\varpi:\mathfrak{I}\times\overline{\Theta}\to(-\infty,\omega]$ are appropriated functions, $a: D \to \mathbb{R}$ $(D = \{(\xi, \vartheta) \in \Im \times \Im : \xi \geq \vartheta\})$. Here $0 = \xi_0 = \vartheta_0 < \xi_1 \leq \vartheta_1 \leq \xi_2 < \cdots < \xi_{\beta-1} \leq \vartheta_\beta \leq \xi_\beta \leq \xi_{\beta+1} = \omega$ be pre-fixed numbers, and $\chi_j \in C((\xi_j, \vartheta_j] \times \overline{\Theta} \times \Xi, \Xi)$, for all $j = 1,2, \dots, \beta$, stand for the impulsive conditions. $(\Xi, \|\cdot\|)$ be a real separable Banach space. For any piecewise continuous function $\wp$ defined on $(-\infty, \omega]$ and any $\xi \in \Im$, we denote by $\wp_\xi$ the element of $\overline{\Theta}$ defined by

$$\wp_\xi(\theta) = \wp(\xi + \theta), \qquad \theta \in (-\infty, 0].$$

2. Preliminaries

Let $\nabla = C(\Im, \Xi)$ be the space of all Ξ-valued continuous functions on $\Im$. By $L(\Xi)$ we denotes the Banach space of all linear and bounded operators on Ξ.

Let $L^1(\Im, \Xi)$ be the space of Ξ −valued Bochner integrable functions on $\Im$ with the norm

$$\|\gamma\|_{L^1} = \int_0^\omega \|\gamma(\xi)\| d\xi.$$

$L^\infty(\Im, \mathbb{R})$ is the Banach space of essentially bounded functions, normed by

$$\|\gamma\|_{L^\infty} = \inf\{\eta > 0 : |\gamma(\xi)| \leq \eta, a.e. \xi \in \Im\}.$$

Denote by

$$\mathcal{P}_{cl}(\Xi) = \{\mathfrak{K} \in \mathcal{P}(\Xi) : \mathfrak{K} \text{ closed}\},$$
$$\mathcal{P}_b(\Xi) = \{\mathfrak{K} \in \mathcal{P}(\Xi) : \mathfrak{K} \text{ bounded}\},$$
$$\mathcal{P}_{cp}(\Xi) = \{\mathfrak{K} \in \mathcal{P}(\Xi) : \mathfrak{K} \text{ compact}\},$$

and

$$\mathcal{P}_{cp,c}(\Xi) = \{\mathfrak{K} \in \mathcal{P}(\Xi) : \mathfrak{K} \text{ compact, convex}\}.$$

A multivalued map $\Phi: \Xi \to \mathcal{P}(\Xi)$ is convex (closed) valued if $\Phi(\Xi)$ is convex (closed) for all $\wp \in \Xi$. Φ is bounded on bounded sets if $\Phi(B) = \cup_{\wp \in B} \Phi(\wp)$ is bounded in Ξ for all $B \in \mathcal{P}_b(\Xi)$ (i.e. $\sup_{\wp \in B}\{\sup\{\|\gamma\| : \gamma \in \Phi(\wp)\}\} < \infty$).

Φ is called upper semi-continuous (u.s.c.) on Ξ if for each $\wp_0 \in \Xi$ the set $\Phi(\wp_0)$ is a nonempty, closed subset of Ξ, and if for each open set $\mathcal{F}$ of Ξ containing $\Phi(\wp_0)$, there exists an open neighborhood $\Bbbk$ of $\wp_0$ such that $\Phi(\Bbbk) \subseteq \mathcal{F}$.

Φ is said to be completely continuous if $\Phi(B)$ is relatively compact for every $B \in \mathcal{P}_b(\Xi)$. If the multivalued map Φ is completely continuous with nonempty compact values, then Φ is u.s.c. if and only if Φ has a closed graph (i.e. $\wp_\alpha \to \wp_*, \gamma_\alpha \to \gamma_*, \gamma_\alpha \in \Phi(\wp_\alpha)$ imply $\gamma_* \in \Phi(\wp_*)$). For more details on multivalued maps see [18, 23, 29].

Definition 2.1: The multivalued map $\Psi: \Im \times \overline{\Theta} \times \Xi \to \mathcal{P}(\Xi)$ is Carathéodory if

- $\xi \mapsto \Psi(\xi, \wp, \gamma)$ is measurable for each $(\wp, \gamma) \in \overline{\Theta} \times \Xi$;
- $(\wp, \gamma) \mapsto \Psi(\xi, \wp, \gamma)$ is upper semicontinuous for almost all $\xi \in \Im$.

Definition 2.2: ([17]). Let $\kappa > 0$ and $f: \mathbb{R}_+ \to \Xi$ be in $L^1(\mathbb{R}_+, \Xi)$. Then the Riemann-Liouville integral is given by

$$I_\xi^\kappa f(\xi) = \frac{1}{\Gamma(\kappa)} \int_0^\xi \frac{f(\vartheta)}{(\xi-\vartheta)^{1-\kappa}} d\vartheta.$$

Definition 2.3: ([39]). The Caputo derivative of order κ for a function $f: [0, +\infty) \to \Xi$ can be written as

$$D_\xi^\kappa f(\xi) = \frac{1}{\Gamma(\alpha-\kappa)} \int_0^\xi \frac{f^{(\alpha)}(\vartheta)}{(\xi-\vartheta)^{\kappa+1-\alpha}} d\vartheta = I^{\alpha-\kappa} f^{(\alpha)}(\xi), \quad \xi > 0, \alpha - 1 \leq \kappa < \alpha.$$

If $0 \leq \kappa < 1$, then

$$D_\xi^\kappa f(\xi) = \frac{1}{\Gamma(1-\kappa)} \int_0^\xi \frac{f^{(1)}(\vartheta)}{(\xi-\vartheta)^\kappa} d\vartheta.$$

Let $\overline{\Theta}$ be a linear space of functions mapping $(-\infty, 0]$ into Ξ with a seminorm $\|\cdot\|_{\overline{\Theta}}$, and verifies the axioms [24]:

(A1) If $\wp: (-\infty, \omega\,] \to \Xi$ is continuous on $\mathfrak{I}$ and $\wp_0 \in \overline{\Theta}$, then $\wp_\xi \in \overline{\Theta}$ and $\wp_\xi$ is piecewise continuous in $\xi \in \mathfrak{I}$ and

$$\|\wp(\xi)\| \leq \Omega \|\wp_\xi\|_{\overline{\Theta}}, \tag{2}$$

where $\Omega \geq 0$ is a constant.

(A2) There exist a continuous function $\Omega_1(\xi) > 0$ and a locally bounded function $\Omega_2(\xi) \geq 0$ in $\xi \geq 0$ such that

$$\|\wp_\xi\|_{\overline{\Theta}} \leq \Omega_1(\xi) \sup_{\vartheta \in [0,\xi]} \|\wp(\vartheta)\| + \Omega_2(\xi) \|\wp_0\|_{\overline{\Theta}}, \qquad ()$$

for $\xi \in [0, \omega]$ and $\wp$ as in $(A1)$.

(A3) The space $\overline{\Theta}$ is complete.

Remark 2.4: Condition (2) in (A1) is equivalent to $\|\hbar(0)\| \leq \|P\hbar\|_{\overline{\Theta}}$, for all $\hbar \in \overline{\Theta}$.

Example 2.5: *The phase space* $\Omega_\varepsilon \times L^\rho(\chi, \Xi)$.

Let $\varepsilon \geq 0, 1 \leq \rho < \infty$, and let $\chi: (-\infty, -\varepsilon) \to \mathbb{R}$ be a nonnegative measurable function which verifies the conditions $(\chi - 5), (\chi - 6)$ in [28]. Thus, χ is locally integrable and there exists a nonnegative, locally bounded function Λ on $(-\infty, 0]$, such that $\chi(x + \theta) \leq \Lambda(x)\chi(\theta)$, for all $x \leq 0$ and $\theta \in (-\infty, -\varepsilon) \backslash N_x$, where $N_x \subseteq (-\infty, -\varepsilon)$ is a set with Lebesgue measure zero.

The space $\Omega_\varepsilon \times L^\rho(\chi, \Xi)$ consists of all classes of functions $\sigma: (-\infty, 0] \to \Xi$, such that σ is continuous on $[-\varepsilon, 0]$, Lebesgue-measurable, and $\chi\|\sigma\|^\rho$ on $(-\infty, -\varepsilon)$. The seminorm in $\|.\|_{\overline{\Theta}}$ is defined by

$$\|\sigma\|_{\overline{\Theta}} = \sup_{\theta \in [-\varepsilon, 0]} \|\sigma(\theta)\| + \left(\int_{-\infty}^{-\varepsilon} \chi(\theta) \|\sigma(\theta)\|^\rho d\theta\right)^{\frac{1}{\rho}}.$$

The space $\overline{\Theta} = \Omega_\varepsilon \times L^\rho(\chi, \Xi)$ satisfies axioms **(A1)**, **(A2)**, **(A3)**. And, for $\varepsilon = 0$ and $\rho = 2$, this space coincides with $\Omega_0 \times L^2(\chi, \Xi), H = 1, M(\xi) =$

$\Lambda(-\xi)^{\frac{1}{2}}, K(\xi) = 1 + \left(\int_{-\varepsilon}^{0} \chi(\varsigma)d\varsigma\right)^{\frac{1}{2}}$, for $\xi \geq 0$ (see [28], Theorem 1.3.8 for details).

Let $S_{\Psi,\wp}$ be a set defined by

$$S_{\Psi,\wp} = \{v \in L^1(\Im,\Xi): v(\xi) \in \Psi(\xi, \wp_{\varpi(\xi,\wp_\xi)}, \wp(\xi))\ a.e. \xi \in \Im\}.$$

Lemma 2.6: ([34]). *Let Ξ be a Banach space. Let $\Psi: \Im \times \overline{\Theta} \times \Xi \to \mathcal{P}_{cp,c}(\Xi)$ be an L^1-Carathéodory multivalued map and let $\Bbbk$ be a linear continuous mapping from $L^1(\Im,\Xi)$ to ∇, then the operator*

$$\Bbbk \circ S_\Psi: \nabla \to \mathcal{P}_{cp,c}(\nabla),$$
$$\wp \mapsto (\Bbbk \circ S_\Psi)(\wp) := \Bbbk(S_{\Psi,\wp})$$

is a closed graph operator in $\nabla \times \nabla$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps. By $\overline{\mathcal{F}}$ and $\partial\mathcal{F}$ we denote the closure of $\mathcal{F}$ and the boundary of $\mathcal{F}$ respectively.

Lemma 2.7: *(Nonlinear alternative of Leray-Schauder type [20]). Let Ξ be a Banach space, D a closed convex subset of Ξ, $\mathcal{F}$ an open subset of D and $0 \in \mathcal{F}$. Suppose that $N: \overline{\mathcal{F}} \to \mathcal{P}_{cp,c}(D)$ is an upper semi-continuous compact map. Then either,*

(i) *N has a fixed point in $\overline{\mathcal{F}}$, or*

(ii) *there is a point $\wp \in \partial\mathcal{F}$ and $\daleth \in (0,1)$ with $\wp \in N(\wp)$.*

Consider the space

$$PC(\Im,\Xi) = \{\gamma: \Im \to \Xi: \gamma \in C([0,\xi_1] \cup (\xi_k, \vartheta_k] \cup (\vartheta_k, \xi_{k+1}], \Xi), k = 1, \dots, m,$$
$$\text{and there exist } \gamma(\xi_k^-), \gamma(\xi_k^+), \gamma(\vartheta_k^-) \text{ and } \gamma(\vartheta_k^+), k = 1, \dots, m, \text{with}$$
$$\gamma(\xi_k^-) = \gamma(\xi_k) \text{ and } \gamma(\vartheta_k^-) = \gamma(\vartheta_k)\}.$$

Obviously, $PC(\Im,\Xi)$ is a Banach space with the norm

$$\|\wp\|_{PC} = \sup_{\xi\in\Im}\|\wp(\xi)\|_\Xi.$$

3. Mains Results

In this section, we state and prove the existence results for the system (1). Now we define the mild solution for our problem.

Definition 3.1: A function $\wp: (-\infty,\omega] \to \Xi$ is said to be a mild solution of the equation (1) if $\wp_0 = \hbar$ on $(-\infty,\omega], \wp|_{[0,\omega]} \in PC([0,\omega],\Xi)$ and there exists $v(\cdot) \in L^1(\Im,\Xi)$, such that $v(\xi) \in \Psi(\xi, \wp_{\varpi(\xi,\wp_\xi)}, \wp(\xi)) a.e\ \xi \in [0,\omega]$, and $\wp$ satisfies

$$\wp(\xi) = \begin{cases} Q(\xi)\hbar(0) + \int_0^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\varsigma)d\varsigma d\vartheta, & \xi \in [0,\xi_1], \\ \chi_j\left(\xi, \wp_{\varpi(\xi,\wp_\xi)}, \wp(\xi)\right), & \xi \in (\xi_j, \vartheta_j], j = 1,2,\dots,\beta, \\ Q(\xi-\vartheta_j)\chi_j\left(\vartheta_j, \wp_{\varpi\left(\vartheta_j,\wp_{\vartheta_j}\right)}, \wp(\vartheta_j)\right) & \\ + \int_{\vartheta_j}^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\varsigma)d\varsigma d\vartheta, & \xi \in (\vartheta_j, \xi_{j+1}], \end{cases} \tag{4}$$

where

$$Q(\xi) = \int_0^\infty x_\kappa(\delta)\mathcal{X}(\xi^\kappa\delta)d\delta, \quad Y(\xi) = \kappa\int_0^\infty \delta\xi^{\kappa-1}x_\kappa(\delta)\mathcal{X}(\xi^\kappa\delta)d\delta,$$

and x_κ is a probability density function defined on $(0,\infty)$ such that

$$x_\kappa(\delta) = \frac{1}{\kappa}\delta^{-1-(\frac{1}{\kappa})}\chi_\kappa(\delta^{-\frac{1}{\kappa}}) \geq 0,$$

where

$$\chi_\kappa(\delta) = \frac{1}{\pi}\sum_{k=1}^\infty (-1)^{k-1}\delta^{-\kappa k-1}\frac{\Gamma(k\kappa+1)}{k!}\sin(k\pi\kappa), \quad \delta \in (0,\infty).$$

Remark 3.2: Note that $\{\mathcal{X}(\xi)\}_{\xi\geq 0}$ is a uniformly bounded i.e., there exists a constant $M > 0$ such that $\|\mathcal{X}(\xi)\| \leq M$ for all $\xi \geq 0$.

Remark 3.3: ([37]). We have by a direct calculation that

$$\|Y(\xi)\| \leq \Omega_{\kappa,M}\xi^{\kappa-1}, \xi > 0, \tag{5}$$

where $\Omega_{\kappa,M} = \frac{\kappa M}{\Gamma(1+\kappa)}$.

Set

$$Y(\varpi^-) = \{\varpi(\vartheta,\sigma) : (\vartheta,\sigma) \in \mathfrak{J} \times \overline{\Theta}, \varpi(\vartheta,\sigma) \leq 0\}.$$

Assume that $\varpi : \mathfrak{J} \times \overline{\Theta} \to (-\infty,\omega]$ is continuous. And, we introduce following hypothesis:

(H_σ) The function $\xi \to \sigma_\xi$ is continuous from $Y(\varpi^-)$ into $\overline{\Theta}$ and there exists a continuous and bounded function $L^\hbar : Y(\varpi^-) \to (0,\infty)$ such that

$$\|\hbar_\xi\|_{\overline{\Theta}} \leq L^\hbar(\xi)\|\hbar\|_{\overline{\Theta}} \text{ for every } \xi \in Y(\varpi^-).$$

Remark 3.4: The condition (H_σ), is frequently verified by the continuous and bounded functions. For more details, see [28].

Remark 3.5: In the rest of this section, Ω_1^* and Ω_2^* are the constants

$$\Omega_1^* = \sup_{\vartheta\in\mathfrak{J}} \Omega_1(\vartheta) \text{and } \Omega_2^* = \sup_{\vartheta\in\mathfrak{J}} \Omega_2(\vartheta).$$

Lemma 3.6: ([27]). *If* $\wp : \mathbb{R} \to \Xi$ *is a function such that* $\wp_0 = \hbar$, *then*

$$\|\wp_\vartheta\|_{\overline{\Theta}} \leq (\Omega_2^* + L^\hbar)\|\hbar\|_{\overline{\Theta}} + \Omega_1^*\sup\{|\wp(\theta)|; \theta \in [0, max\{0,\vartheta\}]\}, \vartheta \in Y(\varpi^-) \cup \mathfrak{J},$$

where $L^\hbar = sup_{\xi\in Y(\varpi^-)}L^\hbar(\xi)$.

Let us introduce the following hypotheses:

(H1) The multivalued map $\Psi\colon \mathfrak{J} \times \overline{\Theta} \times \Xi \to \mathcal{P}_{cp,c}(\Xi)$ is Carathéodory.

(H2) There exist a function $\mu \in L^1(\mathfrak{J}, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi\colon \mathbb{R}^+ \to (0, +\infty)$ such that

$$\|f(\xi, \wp, \gamma)\| \leq \mu(\xi)\psi\big(\|\wp\|_{\overline{\Theta}} + \mathrm{P}\gamma\mathrm{P}_{\Xi}\big), (\xi, \wp, \gamma) \in \mathfrak{J} \times \overline{\Theta} \times \Xi.$$

(H3) The functions $\chi_j\colon (\xi_j, \vartheta_j] \times \overline{\Theta} \times \Xi \to \Xi, j = 1, \dots, \beta,$ and there exist constants $d_j > 0, j = 1, \dots, \beta,$ such that

$$\left\|\chi_j(\xi, \wp, \gamma)\right\| \leq d_j\big(\|\wp\|_{\overline{\Theta}} + \|\gamma\|_{\Xi}\big).$$

(H4) For each $\xi \in \mathfrak{J}$, $a(\xi, \vartheta)$ is measurable on $[0, \xi]$ and $a(\xi) = esssup\{|a(\xi, \vartheta)|, 0 \leq \vartheta \leq \xi\}$ is bounded on $\mathfrak{J}$. The map $\xi \to a_\xi$ is continuous from $\mathfrak{J}$ to $L^\infty(\mathfrak{J}, \mathbb{R})$, here, $a_\xi(\vartheta) = a(\xi, \vartheta)$.

(H5) There is $M^* > 0$ with

$$\frac{M^*}{(\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + \frac{Md_j(\Omega_2^* + L^{\hbar})(\Omega_1^* + 1)\|\hbar\|_{\overline{\Theta}}}{1 - Md_j(\Omega_1^* + 1)} + \frac{a\Omega_{\kappa,M}\frac{\omega^\kappa}{\kappa}(\Omega_1^* + 1)\|\mu\|_{L^1}}{1 - Md_j(\Omega_1^* + 1)}\psi(M^*)} > 1, \tag{6}$$

where $a = \sup_{\xi \in \mathfrak{J}} a(\xi)$.

Theorem 3.7: *Assume that* $(H_\sigma), (H1) - (H4)$ *hold. If* $Md_j(\Omega_1^* + 1) < 1$ *then the problem (1) has a mild solution on* $(-\infty, \omega]$.

Proof: Let $\mathfrak{K} = \{\wp \in PC(\Xi)\colon \wp(0) = \hbar(0) = 0\}$ endowed with the uniform convergence topology and consider the multivalued operator $P\colon \mathfrak{K} \to \mathcal{P}(\mathfrak{K})$ defined by $P(\wp) = \{\varphi \in \mathfrak{K}\}$ with

$$\varphi(\xi) = \begin{cases} Q(\xi)\hbar(0) + \int_0^\xi \int_0^\vartheta Y(\xi - \vartheta)a(\vartheta, \varsigma)v(\varsigma)d\varsigma d\vartheta, & \xi \in [0, \xi_1], \\ \chi_j(\xi, \overline{\wp}_{\varpi(\xi, \overline{\wp}_\xi)}, \overline{\wp}(\xi)), & \xi \in (\xi_j, \vartheta_j], j = 1,2, \dots, \beta, \\ Q(\xi - \vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j, \overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j)) & \\ + \int_{\vartheta_j}^\xi \int_0^\vartheta Y(\xi - \vartheta)a(\vartheta, \varsigma)v(\varsigma)d\varsigma d\vartheta, & \xi \in (\vartheta_j, \xi_{j+1}], \end{cases}$$

where $\overline{\wp}\colon (-\infty, \omega] \to \Xi$ is such that $\wp_0 = \hbar$ and $\overline{\wp} = \wp$ on $\mathfrak{J}$. Let $\overline{\hbar}\colon (-\infty, \omega] \to \Xi$ be the extension of $\hbar$ to $(-\infty, \omega]$ such that $\overline{\hbar}(\theta) = \hbar(0) = 0$ on $\mathfrak{J}$.

We will show that P satisfies the assumptions of Leray-Schauder nonlinear alternative (Lemma 2.7). The proof consists of several steps.

Step 1: $P(\wp)$ is convex for each $\wp \in \mathfrak{K}$.

Indeed, if φ_1 and φ_2 belong to P, then there exist $v_1, v_2 \in S_{\Psi, \overline{\wp}_{\varpi(\vartheta, \overline{\wp}_\vartheta)}}$ such that, for $\xi \in \mathfrak{J}$, we have

$$\varphi_j(\xi)=\begin{cases} Q(\xi)\hbar(0)+\int_0^\xi\int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v_j(\varsigma)d\varsigma d\vartheta, & \xi\in[0,\xi_1],\\ \chi_j(\xi,\overline{\wp}_{\varpi(\xi,\overline{\wp}_\xi)},\overline{\wp}(\xi)), & \xi\in(\xi_j,\vartheta_j], j=1,2,\dots,\beta,\\ Q(\xi-\vartheta_j)\chi_j(\vartheta_j,\overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})},\overline{\wp}(\vartheta_j)) & \\ +\int_{\vartheta_j}^\xi\int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v_j(\varsigma)d\varsigma d\vartheta, & \xi\in(\vartheta_j,\xi_{j+1}].\end{cases}$$

Let $\eta\in[0,1]$. Then, for each $\xi\in[0,\xi_1]$, we have

$$\eta\varphi_1(\xi)+(1-\eta)\varphi_2(\xi)=Q(\xi)\hbar(0)+\int_0^\xi\int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)[dv_1(\varsigma)+(1-\eta)v_2(\varsigma)]d\varsigma d\vartheta.$$

Similarly, for any $\xi\in(\vartheta_j,\xi_{j+1}], j=1,2,\dots,\beta$ we have

$$\eta\varphi_1(\xi)+(1-\eta)\varphi_2(\xi)=Q(\xi-\vartheta_j)\chi_j(\vartheta_j,\overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})},\overline{\wp}(\vartheta_j))+\int_{\vartheta_j}^\xi\int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)[dv_1(\varsigma)+(1-\eta)v_2(\varsigma)]d\varsigma d\vartheta.$$

Since $S_{\Psi,\overline{\wp}_{\varpi(\vartheta,\overline{\wp}_\vartheta)}}$ is convex (because Ψ has convex values), we have

$$\eta\varphi_1+(1-\eta)\varphi_2\in P(\wp).$$

Step 2: P maps bounded sets into bounded sets in $\mathfrak{K}$.

Indeed, it is enough to show that for any $\varepsilon>0$, there exists a positive constant ℓ such that for each $\wp\in B_\varepsilon=\{\wp\in\mathfrak{K}:\|\wp\|_{PC}\le\varepsilon\}$, we have $\|P(\wp)\|_{PC}\le\ell$. Let $\varphi\in P(z)$ and there exists $v\in S_{\Psi,\overline{\wp}_{\varpi(\vartheta,\overline{\wp}_\vartheta)}}$, then if $\xi\in[0,\xi_1]$, we get

$$\begin{aligned}\|\varphi(\xi)\| &\le\|Q(\xi)\hbar(0)\|+\int_0^\xi\int_0^\vartheta\|Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\varsigma)\|d\varsigma d\vartheta\\ &\le M\Omega\|\hbar\|_{\overline{\Theta}}+a\Omega_{\kappa,M}\int_0^\xi\int_0^\vartheta(\xi-\vartheta)^{\kappa-1}\mu(\varsigma)\psi(\left\|\overline{\wp}_{\varpi(\varsigma,\overline{\wp}_\varsigma)}\right\|_{\overline{\Theta}}+\|\overline{\wp}\|)d\varsigma d\vartheta\\ &\le M\Omega\|\hbar\|_{\overline{\Theta}}+a\Omega_{\kappa,M}\int_0^\xi\int_0^\vartheta(\xi-\vartheta)^{\kappa-1}\mu(\varsigma)\\ &\times\psi\big((\Omega_2^*+L^\hbar)\|\hbar\|_{\overline{\Theta}}+\Omega_1^*\varepsilon+\varepsilon\big)d\varsigma d\vartheta\\ &\le M\Omega\|\hbar\|_{\overline{\Theta}}+a\Omega_{\kappa,M}\|\mu\|_{L^1}\frac{\omega^\kappa}{\kappa}\psi\big((\Omega_2^*+L^\hbar)\|\hbar\|_{\overline{\Theta}}+(\Omega_1^*+1)\varepsilon\big).\end{aligned}$$

If $\xi\in[\xi_j,\vartheta_j), j=1,2,\dots,\beta,$ we have

$$\begin{aligned}\|\varphi(\xi)\| &\le\left\|\chi_j(\xi,\overline{\wp}_{\varpi(\xi,\overline{\wp}_\xi)},\overline{\wp}(\xi))\right\|\\ &\le d_j\left(\left\|\overline{\wp}_{\varpi(\xi,\overline{\wp}_\xi)}\right\|_{\overline{\Theta}}+\|\overline{\wp}\|\right)\\ &\le d_j\big((\Omega_2^*+L^\hbar)\|\hbar\|_{\overline{\Theta}}+(\Omega_1^*+1)\varepsilon\big).\end{aligned}$$

If $\xi\in(\vartheta_j,\xi_{j+1}], j=1,2,\dots,\beta,$ we obtain

$$\|\varphi(\xi)\|\le\left\|Q(\xi-\vartheta_j)\chi_j(\vartheta_j,\overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})},\overline{\wp}(\vartheta_j))\right\|+\int_{\vartheta_j}^\xi\int_0^\vartheta\|Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\varsigma)\|d\varsigma d\vartheta,$$

$$\leq Md_j\big((\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big)$$
$$+a\Omega_{\kappa,M}\|\mu\|_{L^1}\frac{\omega^\kappa}{\kappa}\psi\big((\Omega_2^* + L^{\hbar})\|\hbar\|_B + (\Omega_1^* + 1)\varepsilon\big)$$
$$\leq \ell.$$

Step 3: P maps bounded sets into equicontinuous sets of $\mathfrak{K}$.

Let $0 \leq \varsigma_2 \leq \varsigma_1 \leq \xi_1$, and let $\varphi \in P(\wp)$ for $\wp \in \mathfrak{K}$, then we have
$$\|\varphi(\varsigma_1) - \varphi(\varsigma_2)\| \leq I_1 + I_2 + I_3,$$
where
$$I_1 = \|Q(\varsigma_1) - Q(\varsigma_2)\|\ \|\hbar(0)\|$$
$$I_2 = \left\|\int_0^{\varsigma_2}\int_0^{\vartheta} [Y(\varsigma_1 - \vartheta) - Y(\varsigma_2 - \vartheta)]a(\vartheta,\varsigma)v(\varsigma)d\varsigma d\vartheta\right\|$$
$$I_3 = \left\|\int_{\varsigma_2}^{\varsigma_1}\int_0^{\vartheta} Y(\varsigma_1 - \vartheta)a(\vartheta,\varsigma)v(\varsigma)d\varsigma d\vartheta\right\|.$$

I_1 tends to zero as $\varsigma_2 \to \varsigma_1$, since $\mathcal{X}(\xi)$ is uniformly continuous operator.

For I_2, using (5) and (H2), we have

$$I_2 \leq a\psi\big((\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big)\|\mu\|_{L^1}\int_0^{\varsigma_2} [Y(\varsigma_1 - \vartheta) - Y(\varsigma_2 - \vartheta)]d\vartheta$$
$$\leq a\psi\big((\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big)\|\mu\|_{L^1}$$
$$\times \int_0^{\varsigma_2}\Big[\kappa\int_0^{\infty}\delta(\varsigma_1 - \vartheta)^{\kappa-1}x_\kappa(\delta)\mathcal{X}((\varsigma_1 - \vartheta)^\kappa\delta)d\delta$$
$$-\kappa\int_0^{\infty}\delta(\varsigma_2 - \vartheta)^{\kappa-1}x_\kappa(\delta)\mathcal{X}((\varsigma_2 - \vartheta)^\kappa\delta)d\delta\Big]d\vartheta$$
$$\leq a\psi\big((\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big)\|\mu\|_{L^1}$$
$$\times\Big[\kappa\int_0^{\varsigma_2}\int_0^{\infty}\delta\|[(\varsigma_1 - \vartheta)^{\kappa-1} - (\varsigma_2 - \vartheta)^{\kappa-1}]x_\kappa(\delta)\mathcal{X}((\varsigma_1 - \vartheta)^\kappa\delta)$$
$$+\kappa\int_0^{\varsigma_2}\int_0^{\infty}\delta(\varsigma_2 - \vartheta)^{\kappa-1}x_\kappa(\delta)\mathrm{P}\mathcal{X}((\varsigma_1 - \vartheta)^\kappa\delta) - \mathcal{X}((\varsigma_2 - \vartheta)^\kappa\delta)\|\Big]$$
$$\leq a\psi\big((\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big)\|\mu\|_{L^1}$$
$$\times [\Omega_{\kappa,M}\int_0^{\varsigma_2}|(\varsigma_1 - \vartheta)^{\kappa-1} - (\varsigma_2 - \vartheta)^{\kappa-1}|d\vartheta$$
$$+\kappa\int_0^{\varsigma_2}\int_0^{\infty}\delta(\varsigma_2 - \vartheta)^{\kappa-1}x_\kappa(\delta)\|\mathcal{X}((\varsigma_1 - \vartheta)^\kappa\delta) - \mathcal{X}((\varsigma_2 - \vartheta)^\kappa\delta)\|d\delta d\vartheta].$$

Thus, the first term on the right-hand side of the above inequality tends to zero as $\varsigma_2 \to \varsigma_1$. From the continuity of $\mathcal{X}(\xi)$ in the uniform operator topology for $\xi > 0$, The second term on the right-hand side of the above inequality tends to zero as $\varsigma_2 \to \varsigma_1$.

In view of (H2), we have
$$I_3 \leq a\,\Omega_{\kappa,M}\int_{\varsigma_2}^{\varsigma_1}\int_0^{\vartheta}(\varsigma_1 - \vartheta)^{\kappa-1}\|v(\varsigma)\|d\varsigma d\vartheta$$
$$\leq a\,\Omega_{\kappa,M}\psi\big((\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big)\|\mu\|_{L^1}\int_{\varsigma_2}^{\varsigma_1}(\varsigma_1 - \vartheta)^{\kappa-1}d\vartheta.$$

As $\varsigma_2 \to \varsigma_1$, I_3 tends to zero.

For any $\xi \in [\xi_j, \vartheta_j), j = 1,2,\dots,\beta, \xi_j \leq \varsigma_2 \leq \varsigma_1 \leq \vartheta_j$, we have

$$\|\varphi(\varsigma_1) - \varphi(\varsigma_2)\| = \left\|\chi_j(\varsigma_1, \overline{\wp}_{\varpi(\varsigma_1,\overline{\wp}_{\varsigma_1})}, \overline{\wp}(\varsigma_1)) - \chi_j(\varsigma_2, \overline{\wp}_{\varpi(\varsigma_2,\overline{\wp}_{\varsigma_2})}, \overline{\wp}(\varsigma_2))\right\|$$
$$\to 0 \text{ as } \varsigma_2 \to \varsigma_1.$$

In the same way, for any $\xi \in (\vartheta_j, \xi_{j+1}], j = 1,2, \dots, \beta, \vartheta_j \leq \varsigma_2 \leq \varsigma_1 \leq \xi_{j+1}$, then we get

$$\|\varphi(\varsigma_1) - \varphi(\varsigma_2)\| \leq \left\|Q(\varsigma_1 - \vartheta_j) - Q(\varsigma_2 - \vartheta_j)\right\| \left\|\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j, \overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j))\right\| + I_1 + I_2 + I_3.$$

Since $\mathcal{X}(\xi)$ is uniformly continuous operator, so

$$\lim_{\varsigma_2 \to \varsigma_1} \left\|Q(\varsigma_1 - \vartheta_j) - Q(\varsigma_2 - \vartheta_j)\right\| = 0, j = 1, \dots, \beta.$$

Consequently

$$\lim_{\varsigma_2 \to \varsigma_1} \|\varphi(\varsigma_1) - \varphi(\varsigma_2)\| = 0.$$

Thus, $P(B_\varepsilon)$ is equicontinuous.

Step 4: $(PB_\varepsilon)(\xi)$ is relatively compact for each $\xi \in \Im$, where

$$(PB_\varepsilon)(\xi) = \{\varphi(\xi) : \varphi \in P(B_\varepsilon)\}.$$

Let $0 < \xi \leq \xi_1$ be fixed and let ε be a real number satisfying $0 < \varepsilon < \xi$. For arbitrary $\nu > 0$, we define

$$\begin{aligned}
\varphi_{\varepsilon,\nu}(\xi) &= Q(\xi)\hbar(0) + \kappa \int_0^{\xi-\varepsilon} (\xi - \vartheta)^{\kappa-1} \int_\nu^\infty \delta x_\kappa(\delta) \mathcal{X}((\xi - \vartheta)^\kappa \delta) \\
&\quad \times \int_0^\vartheta a(\vartheta, \varsigma) v(\varsigma) d\varsigma d\delta d\vartheta \\
&= \kappa \mathcal{X}(\varepsilon^\kappa \nu) \int_0^{\xi-\varepsilon} (\xi - \vartheta)^{\kappa-1} \int_\nu^\infty \delta x_\kappa(\delta) \mathcal{X}((\xi - \vartheta)^\kappa \delta - \varepsilon^\kappa \nu) \\
&\quad \times \int_0^\vartheta a(\vartheta, \varsigma) v(\varsigma) d\varsigma d\delta d\vartheta,
\end{aligned}$$

where $v \in S_{\Psi, \overline{\wp}_{\varpi(\vartheta, \overline{\wp}_\vartheta)}}$. Using the compactness of $\mathcal{X}(\xi)$ for $\xi > 0$, we deduce that the set

$$H_{\varepsilon,\nu} = \{\varphi_{\varepsilon,\nu}(\xi) : \varphi \in P(B_\varepsilon)\}$$

is relatively compact. Moreover,

$$\begin{aligned}
&\left\|\varphi(\xi) - \varphi_{\varepsilon,\nu}(\wp)(\xi)\right\| \\
&\leq \int_0^\xi \int_0^\vartheta \|Y(\xi - \vartheta) a(\vartheta, \varsigma) v(\varsigma)\| d\varsigma d\vartheta \\
&\leq \kappa \int_0^{\xi-\varepsilon} (\xi - \vartheta)^{\kappa-1} \int_0^\nu \delta x_\kappa(\delta) \|\mathcal{X}((\xi - \vartheta)^\kappa \delta)\| \int_0^\vartheta \|a(\vartheta, \varsigma)\| \, \|v(\varsigma)\| d\varsigma d\delta d\vartheta \\
&+ \kappa \int_{\xi-\varepsilon}^\xi (\xi - \vartheta)^{\kappa-1} \int_0^\infty \delta x_\kappa(\delta) \|\mathcal{X}((\xi - \vartheta)^\kappa \delta)\| \int_0^\vartheta \|a(\vartheta, \varsigma)\| \, \|v(\varsigma)\| d\varsigma d\delta d\vartheta \\
&\leq \omega^\kappa M a \psi\big((\Omega_2^* + L^\hbar)\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big) \|\mu\|_{L^1} \int_0^\nu \delta x_\kappa(\delta) d\delta \\
&+ \frac{\varepsilon^\kappa M a}{\Gamma(1+\kappa)} \psi\big((\Omega_2^* + L^\hbar)\|\hbar\|_{\overline{\Theta}} + (\Omega_1^* + 1)\varepsilon\big) \|\mu\|_{L^1}.
\end{aligned}$$

Let $\vartheta_j < \xi \leq \xi_{j+1}$ be fixed and let ε be a real number satisfying $0 < \varepsilon < \xi$. For arbitrary $\nu > 0$, we define

$$\varphi_{\varepsilon,\nu}(\wp)(\xi) = Q(\xi - \vartheta_j) \chi_j(\xi, \overline{\wp}_{\varpi(\xi, \overline{\wp}_\xi)}, \overline{\wp}(\xi))$$

$$+\kappa \int_0^{\xi-\varepsilon} (\xi-\vartheta)^{\kappa-1} \int_\nu^\infty \delta x_\kappa(\delta)\mathcal{X}((\xi-\vartheta)^\kappa\delta)$$
$$\times \int_0^\vartheta a(\vartheta,\varsigma)v(\varsigma)d\varsigma d\delta d\vartheta$$
$$= Q(\xi-\vartheta_j)\chi_j(\xi,\overline{\wp}_{\varpi(\xi,\overline{\wp}_\xi)},\overline{\wp}(\xi)) + \kappa\mathcal{X}(\varepsilon^\kappa\nu)\int_0^{\xi-\varepsilon} (\xi-\vartheta)^{\kappa-1}\int_\nu^\infty \delta x_\kappa(\delta)$$
$$\times\, \mathcal{X}((\xi-\vartheta)^\kappa\delta-\varepsilon^\kappa\nu)\int_0^\vartheta a(\vartheta,\varsigma)v(\varsigma)d\varsigma d\delta d\vartheta,$$

where $v \in S_{\Psi,\overline{\wp}_{\varpi(\vartheta,\overline{\wp}_\vartheta)}}$. Since $\mathcal{X}(\xi)$ is a compact operator, the set

$$H_{\varepsilon,\nu} = \{\varphi_{\varepsilon,\nu}(\xi) \colon \varphi \in P(B_\varepsilon)\}$$

is relatively compact. Moreover,

$$\|\varphi(\xi)-\varphi_{\varepsilon,\nu}(\wp)(\xi)\|$$
$$\le \int_{\vartheta_j}^\xi \int_0^\vartheta \|Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\varsigma)\|d\varsigma d\vartheta$$
$$\le \kappa\int_{\vartheta_j}^{\xi-\varepsilon}(\xi-\vartheta)^{\kappa-1}\int_0^\nu \delta x_\kappa(\delta)\|\mathcal{X}((\xi-\vartheta)^\kappa\delta)\|\int_0^\vartheta \|a(\vartheta,\varsigma)\|\,\|v(\varsigma)\|d\varsigma d\delta d\vartheta$$
$$+\kappa\int_{\xi-\varepsilon}^{\xi}(\xi-\vartheta)^{\kappa-1}\int_0^\infty \delta x_\kappa(\delta)\|\mathcal{X}((\xi-\vartheta)^\kappa\delta)\|\int_0^\vartheta \|a(\vartheta,\varsigma)\|\|v(\varsigma)\|d\varsigma d\delta d\vartheta$$
$$\le \omega^\kappa Ma\psi\big((\Omega_2^*+L^\hbar)\|\hbar\|_{\overline{\Theta}}+(\Omega_1^*+1)\varepsilon\big)\|\mu\|_{L^1}\int_0^\nu \delta x_\kappa(\delta)d\delta$$
$$+\frac{\varepsilon^\kappa Ma}{\Gamma(1+\kappa)}\psi\big((\Omega_2^*+L^\hbar)\|\hbar\|_{\overline{\Theta}}+(\Omega_1^*+1)\varepsilon\big)\|\mu\|_{L^1},$$

which implies that $(PB_\varepsilon)(\xi)$ is relatively compact. Therefore it follows by the Ascoli-Arzelá theorem that $P\colon \mathfrak{K}\to\mathcal{P}(\mathfrak{K})$ is completely continuous.

Step 5: P has a closed graph.

Let $\wp_\alpha \to \wp_*, \varphi_\alpha \in P(\wp_\alpha)$, and $\varphi_\alpha \to \varphi_*$. We shall show that $\varphi_* \in P(\wp_*)$. $\varphi_\alpha \in P(\wp_\alpha)$ means that there exists $v_\alpha \in S_{\Psi,\overline{\wp}_\alpha \varpi(\vartheta,\overline{\wp}_{ns})}$ such that, for each $\xi \in [0,\xi_1]$,

$$\varphi_\alpha(\xi) = Q(\xi)\hbar(0) + \int_0^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v_\alpha(\varsigma)d\varsigma d\vartheta.$$

We must prove that there exists $v_* \in S_{\Psi,\overline{\wp}_*\varpi(\vartheta,\overline{\wp}_{*\vartheta})}$ such that, for each $\xi \in [0,\xi_1]$,

$$\varphi_*(\xi) = Q(\xi)\hbar(0) + \int_0^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v_*(\varsigma)d\varsigma d\vartheta.$$

Consider the linear and continuous operator $\Upsilon\colon L^1([0,\xi_1],\Xi)\to C([0,\xi_1],\Xi)$ defined by

$$(\Upsilon v)(\xi) = \int_0^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\vartheta)d\varsigma d\vartheta.$$

We have

$$\|(\varphi_\alpha(\xi)-Q(\xi)\hbar(0))-(\varphi_*(\xi)-Q(\xi)\hbar(0))\| = \|\varphi_\alpha(\xi)-\varphi_*(\xi)\| \to 0,$$
$$\text{as } \alpha\to\infty.$$

From Lemma 2.6 it follows that $\Upsilon o S_\Psi$ is a closed graph operator and from the definition of Υ one has

$$\varphi_\alpha(\xi) - Q(\xi)\hbar(0) \in \Upsilon(S_{\Psi,\overline{\wp}_\alpha\varpi(\vartheta,\overline{\wp}_{ns})}).$$

As $z_\alpha \to z_*$ and $\varphi_\alpha \to \varphi_*$, there is a $v_* \in S_{\Psi,\overline{\wp}_*\varpi(\vartheta,\overline{\wp}_{*\vartheta})}$ such that

$$\varphi_*(\xi) - Q(\xi)\hbar(0) = \int_0^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\vartheta)d\varsigma d\vartheta.$$

Similarly, for any $\xi \in (\vartheta_j, \xi_{j+1}], j = 1,2,\dots,\beta$, we have

$$\varphi_\alpha(\xi) = Q(\xi-\vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j)) + \int_{\vartheta_j}^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v_\alpha(\varsigma)d\varsigma d\vartheta.$$

We shall prove that there exists $v_* \in S_{\Psi,\overline{\wp}_*\varpi(\vartheta,\overline{\wp}_{*\vartheta})}$ such that, for each $\xi \in (\vartheta_j, \xi_{j+1}]$,

$$\varphi_*(\xi) = Q(\xi-\vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j)) + \int_{\vartheta_j}^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v_*(\varsigma)d\varsigma d\vartheta.$$

Clearly, we have

$$\begin{aligned}&\|(\varphi_\alpha(\xi) - Q(\xi-\vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j))) - \\ &\qquad (\varphi_*(\xi) - Q(\xi-\vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j)))\| \\ &= \|\varphi_\alpha(\xi) - \varphi_*(\xi)\| \to 0, \text{as } \alpha \to \infty.\end{aligned}$$

Consider the linear and continuous operator $\Upsilon: L^1((\vartheta_j, \xi_{j+1}], \Xi) \to C((\vartheta_j, \xi_{j+1}], \Xi)$ defined by

$$(\Upsilon v)(\xi) = \int_{\vartheta_j}^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\vartheta)d\varsigma d\vartheta.$$

In view of Lemma 2.6, we deduce that $\Upsilon o S_\Psi$ is a closed graph operator. Also, from the definition of Υ, we have that, for every $\xi \in (\vartheta_j, \xi_{j+1}], j = 1,2,\dots,\beta$,

$$\varphi_\alpha(\xi) - Q(\xi-\vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j)) \in \Upsilon(S_{\Psi,\overline{\wp}_\alpha\varpi(\vartheta,\overline{\wp}_{ns})}).$$

Since $\wp_\alpha \to \wp_*$, for some $v_* \in S_{\Psi,\overline{\wp}_*\varpi(\vartheta,\overline{\wp}_{*\vartheta})}$ it follows from Lemma 2.6 that, for every $\xi \in (\vartheta_j, \xi_{j+1}]$, we have

$$\varphi_*(\xi) = Q(\xi-\vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j)) + \int_{\vartheta_j}^\xi \int_0^\vartheta Y(\xi-\vartheta)a(\vartheta,\varsigma)v_*(\varsigma)d\varsigma d\vartheta.$$

Hence the multivalued operator P is upper semi-continuous.

Step 6: We show that there exists an open set $\hat{o} \subseteq \mathfrak{K}$ with $\wp \notin \daleth P(\wp)$ for any $\daleth \in (0,1)$ and $\wp \in \partial\mathcal{F}$.

Suppose on the contrary that there exist $\daleth \in (0,1)$ and $\wp \in \daleth P(\wp)$. Then there exists $v \in S_{\Psi,\overline{\wp}\varpi(\vartheta,\overline{\wp}_\vartheta)}$ such that, for $\xi \in [0,\xi_1]$, we have

$$\begin{aligned}\|\wp(\xi)\| &\le M\Omega\|\hbar\|_{\overline{\Theta}} + \Omega_{\kappa,M}\int_0^\xi (\xi-\vartheta)^{\kappa-1}\int_0^\vartheta \|a(\vartheta,\varsigma)\|\|v(\varsigma)\|d\varsigma d\vartheta \\ &\le M\Omega\|\hbar\|_{\overline{\Theta}} + a\Omega_{\kappa,M}\int_0^\xi (\xi-\vartheta)^{\kappa-1}\int_0^\vartheta \|v(\varsigma)\|d\varsigma d\vartheta \\ &\le M\Omega\|\hbar\|_{\overline{\Theta}} \\ &\quad + a\Omega_{\kappa,M}\int_0^\xi (\xi-\vartheta)^{\kappa-1}\int_0^\vartheta \mu(\varsigma)\psi\big((\Omega_2^* + L^\hbar)\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\varsigma)\|\big)d\varsigma d\vartheta \\ &\le M\Omega\|\hbar\|_{\overline{\Theta}} + a\Omega_{\kappa,M}\frac{\omega^\kappa}{\kappa}\int_0^\xi \mu(\vartheta)\psi\big((\Omega_2^* + L^\hbar)\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\vartheta)\|\big)d\vartheta.\end{aligned}$$

For any $\xi \in [\xi_j, \vartheta_j), j = 1,2, \dots, \beta, \xi_j \le \varsigma_2 \le \varsigma_1 \le \vartheta_j$, we have

$$\begin{aligned}\|\wp(\xi)\| &\le \left\|\chi_j(\xi, \overline{\wp}_{\varpi(\xi,\overline{\wp}_\xi)}, \overline{\wp}(\xi))\right\| \\ &\le d_j\left(\left\|\overline{\wp}_{\varpi(\xi,\overline{\wp}_\xi)}\right\|_{\overline{\Theta}} + \|\overline{\wp}(\xi)\|\right) \\ &\le d_j\big((\Omega_2^* + L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\xi)\|\big).\end{aligned}$$

Then

$$\|\wp(\xi)\| \le \frac{d_j(\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}}}{1-d_j(\Omega_1^*+1)}.$$

In the same way, for any $\xi \in (\vartheta_j, \xi_{j+1}], j = 1,2, \dots, \beta$, we have

$$\begin{aligned}\|\wp(\xi)\| &\le \left\|Q(\xi-\vartheta_j)\chi_j(\vartheta_j, \overline{\wp}_{\varpi(\vartheta_j,\overline{\wp}_{\vartheta_j})}, \overline{\wp}(\vartheta_j))\right\| \\ &\quad + \int_{\vartheta_j}^{\xi}\int_0^{\vartheta} \|Y(\xi-\vartheta)a(\vartheta,\varsigma)v(\varsigma)\|d\varsigma d\vartheta \\ &\le Md_j\big((\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\xi)\|\big)\end{aligned}$$

$$\begin{aligned}&+a\Omega_{\kappa,M}\int_{\vartheta_j}^{\xi}\int_0^{\vartheta}(\xi-\vartheta)^{\kappa-1}\mu(\varsigma)\psi\big((\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\xi)\|\big)d\varsigma d\vartheta \\ &\le Md_j\big((\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\xi)\|\big) \\ &+a\Omega_{\kappa,M}\frac{\omega^{\kappa}}{\kappa}\int_{\vartheta_j}^{\xi}\mu(\vartheta)\psi\big((\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\xi)\|\big)d\vartheta.\end{aligned}$$

Then

$$\begin{aligned}\|\wp(\xi)\| &\le \frac{Md_j(\Omega_2^*+L^{\hbar})\mathrm{P}\hbar\mathrm{P}_{\overline{\Theta}}}{1-Md_j(\Omega_1^*+1)} \\ &\quad + \frac{a\Omega_{\kappa,M}\frac{\omega^{\kappa}}{\kappa}}{1-Md_j(\Omega_1^*+1)}\int_{\vartheta_j}^{\xi}\mu(\vartheta)\psi\big((\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}} + (\Omega_1^*+1)\|\wp(\vartheta)\|\big)d\vartheta.\end{aligned}$$

Let $m(\xi) = \sup\{\|\wp(\vartheta)\| : 0 \le \vartheta \le \xi\}, \xi \in \mathfrak{I}$. Then we have

$$\begin{aligned}m(\xi) &\le \frac{Md_j(\Omega_2^*+L^{\hbar})\mathrm{P}\hbar\mathrm{P}_{\overline{\Theta}}}{1-Md_j(\Omega_1^*+1)} \\ &\quad + \frac{a\Omega_{\kappa,M}\frac{\omega^{\kappa}}{\kappa}}{1-Md_j(\Omega_1^*+1)}\int_{\vartheta_j}^{\xi}\mu(\vartheta)\psi\big((\Omega_2^*+L^{\hbar})\mathrm{P}\hbar\mathrm{P}_{\overline{\Theta}} + (\Omega_1^*+1)m(\vartheta)\big)d\vartheta.\end{aligned}$$

If $x(\xi) = (\Omega_2^*+L^{\hbar})\mathrm{P}\hbar\mathrm{P}_{\overline{\Theta}} + (\Omega_1^*+1)m(\xi)$ then we have,

$$\begin{aligned}x(\xi) &\le (\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}} + \frac{Md_j(\Omega_2^*+L^{\hbar})(\Omega_1^*+1)\|\hbar\|_{\overline{\Theta}}}{1-Md_j(\Omega_1^*+1)} \\ &\quad + \frac{a\Omega_{\kappa,M}\frac{\omega^{\kappa}}{\kappa}(\Omega_1^*+1)}{1-Md_j(\Omega_1^*+1)}\int_{\vartheta_j}^{\xi}\mu(\vartheta)\psi(x(\vartheta))d\vartheta \\ &\le (\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}} + \frac{Md_j(\Omega_2^*+L^{\hbar})(\Omega_1^*+1)\|\hbar\|_{\overline{\Theta}}}{1-Md_j(\Omega_1^*+1)} \\ &\quad + \frac{a\Omega_{\kappa,M}\frac{\omega^{\kappa}}{\kappa}(\Omega_1^*+1)\|\mu\|_{L^1}}{1-Md_j(\Omega_1^*+1)}\psi(\|x\|).\end{aligned}$$

Then

$$\frac{\|x\|}{(\Omega_2^*+L^{\hbar})\|\hbar\|_{\overline{\Theta}}+\frac{Md_j(\Omega_2^*+L^{\hbar})(\Omega_1^*+1)\|\hbar\|_{\overline{\Theta}}}{1-Md_j(\Omega_1^*+1)}+\frac{a\Omega_{\kappa,M}\frac{\omega^{\kappa}}{\kappa}(\Omega_1^*+1)\|\mu\|_{L^1}}{1-Md_j(\Omega_1^*+1)}\psi(\|x\|)} \leq 1.$$

In view of (H5), there exists M^* such that $\wp \neq M^*$. Let us set $\mathcal{F} = \{\wp \in \mathfrak{K}: \|\wp\|_{PC} < M^*\}$. Note that the operator $P: \overline{\mathcal{F}} \to \mathcal{P}(\mathfrak{K})$ is a compact multivalued map, u.s.c. with convex closed values. From the choice of $\mathcal{F}$, there is no $\wp \in \partial\mathcal{F}$ such that $\wp \in \daleth P(\wp)$ for some $\daleth \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.7), we deduce that P has a fixed point $\wp \in \overline{\mathcal{F}}$ which is a solution of the problem (1). This completes the proof.

4. An Example

We consider the following fractional integro-differential inclusions with state-dependent delay and the non-instantaneous impulsive:

$$\begin{cases} \frac{\partial_{\xi}^{\kappa}}{\partial \xi^{\kappa}} v(\xi,\chi) + \frac{\partial^2}{\partial \chi^2} v(\xi,\chi) \in \int_0^{\xi} (\xi-\vartheta)^2 v(\vartheta, v(\vartheta-\delta(v(\vartheta,0)),\chi)) \\ \qquad + \int_0^{\xi} (\xi-\vartheta)^2 \cos|v(\vartheta,\chi)| d\vartheta, & (\xi,\chi) \in \mathbb{N} \in \cup_{j=1}^{\alpha} [\vartheta_j, \xi_{j+1}] \times [0,\pi], \\ v(\xi,0) = v(\xi,\pi) = 0, & \xi \in [0,\omega], \\ v(\varsigma,\chi) = v_0(\theta,\chi), & \theta \in (-\infty,0], \chi \in [0,\pi] \\ v(\xi,\chi) = G_j(\xi, v(\xi-\delta(v(\xi,0)),\chi), \chi(\xi)), & (\xi,\chi) \in (\xi_j,\vartheta_j] \times [0,\pi], j = 1,2,\ldots,\beta, \end{cases} \tag{7}$$

where $0 < \kappa < 1, 0 = \xi_0 = \vartheta_0 < \xi_1 \leq \vartheta_1 \leq \xi_2 < \cdots < \xi_{\beta-1} \leq \vartheta_\beta \leq \xi_\beta \leq \xi_{\beta+1} = \omega$ are prefixed real numbers, $\delta \in C(\mathbb{R},[0,\infty))$ and the functions $a_1: \mathbb{R} \to \mathbb{R}, \varpi_j: [0,+\infty) \to [0,+\infty), j = 1,2$ are continuous functions.

Let $\Xi = L^2([0,\pi])$ and define the operator $\mathcal{G}: D(\mathcal{G}) \subset \Xi \to \Xi$ by $\mathcal{G}\rho = \rho''$ with domain

$D(\mathcal{G}) = \{\rho \in E: \rho, \rho'\ are absolutely continuous, \rho'' \in E, \rho(0) = \rho(\pi) = 0\}$.

Then

$$\mathcal{G}\rho = \sum_{\alpha=1}^{\infty} \alpha^2 (\rho,\rho_\alpha)\rho_\alpha, \quad \rho \in D(\mathcal{G}),$$

where $\rho_\alpha(\wp) = \sqrt{\frac{2}{\pi}} \sin(\alpha\wp), \alpha \in \mathbb{N}$ is the orthogonal set of eigenvectors of $\mathcal{G}$. It is well known that $\mathcal{G}$ is the infinitesimal generator of an analytic semigroup $\{\mathcal{X}(\xi)\}_{\xi\geq 0}$ in Ξ and is given by

$$\mathcal{X}(\xi)\rho = \sum_{\alpha=1}^{\infty} e^{-\alpha^2\xi}(\rho,\rho_\alpha)\rho_\alpha, \quad \forall \rho \in \Xi, \text{ and every } \xi > 0.$$

From these expressions, it follows that $\{\mathcal{X}(\xi)\}_{\xi\geq 0}$ is a uniformly bounded compact semigroup on Ξ. For the phase space, we choose $\overline{\Theta} = \Omega_0 \times L^2(\chi,\Xi)$, see Example 2.5 for details.

Set

$$
\begin{aligned}
\wp(\xi)(\chi) &= v(\xi,\chi),\\
\hbar(\theta)(\chi) &= v_0(\theta,\chi),\\
a(\xi,\vartheta) &= (\xi-\vartheta)^2,\\
f(\xi,\sigma,\wp(\xi))(\chi) &= v(\xi,v(\xi-\delta(v(\xi,0)),\chi))+\cos|\wp(\xi)(\chi)|,\\
\chi_j(\xi,\sigma,\wp(\xi))(\chi) &= G_j(\xi,v(\xi-\delta(v(\xi,0)),\chi),\chi(\xi))\\
\varpi(\xi,\sigma) &= \xi-\delta(\hbar(0,0)).
\end{aligned}
$$

Let $\sigma \in \overline{\Theta}$ be such that (H_σ) holds, and let $\xi \to \sigma_\xi$ be continuous on $Y(\varpi^-)$. Then by Theorem 3.7, there exists a mild solution of (7).

Declarations

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