

Laplacian energy of power fuzzy graphs with application in cryptography

S. Shiny Paulin [†]

T. Bharathi ^{*}

Department of Mathematics

Loyola College

University of Madras

Chennai 600034

Tamil Nadu

India

Abstract

The primary aim of this study is to present and expand on the idea of Laplacian energy within the context of power fuzzy graphs. The aim is to establish the Laplacian energy sequence for the power fuzzy graph with increasing power. The Laplacian spectrum and matrix are defined using the degree and adjacency matrices. The Laplacian energy of the power fuzzy graph is computed using eigenvalues of the corresponding Laplacian matrix and the size of the power fuzzy graph. The formal definition of the Laplacian matrix for power fuzzy graphs is defined, and the Laplacian energy sequence as power increases is computed. This concept is explored as a unique extension of fuzzy graphs and power graphs. Applying size, Laplacian matrix, and energy of regular power fuzzy graph theory to cryptography is a new method in this study.

Subject Classification: 03B52, 05C72, 94D05, 94A60.

Keywords: Power fuzzy graph, Regular PFG, Laplacian energy, Cryptography.

1. Introduction

Lotfi Zadeh in 1965 [1] introduced fuzzy sets and then Azriel Rosenfeld in 1975 [2] developed the fuzzy graph theory. An extension of the fuzzy graph

[†] ORCID-ID: 0009-0006-1291-0484

^{*} ORCID-ID: 0000-0003-2460-2682

[†] E-mail: paulinshiny@gmail.com

^{*} E-mail: drtbharathi2024@gmail.com (Corresponding Author)

model, a PFG is designed to use the strength of connections between nodes [3]. Various matrices and energies have been established for graphs [4], [5]. This concept was extended to the energy of fuzzy graphs by Narayanan, and Mathew, in 2013 [6]. The Laplacian matrix and Laplacian energy of fuzzy graphs were defined and explored by Rahimi, and Fayazi, in 2014 [7]. Similarly, for intuitionistic fuzzy graph [8], interval-valued fuzzy graph [9], picture fuzzy graph [10], and hesitancy fuzzy graph [11], the Laplacian matrix and its energy are defined and discussed. Laplacian energy was also studied on trees with diameter 4 by Rahman, et al. in 2021 [12] and on graphs with commutative rings by Sarathy, and Sankar, Ravi. in 2024 [13]. Some operations on matrices are done by Al-Hawary et al. in [14], and provide various results on spectra of fuzzy graphs. This motivates us to define the Laplacian matrix and Laplacian energy of power fuzzy graphs and discuss some of the results based on power fuzzy graphs. A useful algorithm is designed using the Laplacian concepts to secure communication systems in cryptography. An algorithm has been introduced using edge regular power fuzzy graph by Bharathi, et al. in 2025 [15]. The required basic definitions of the method are given in section 2. The properties based on the Laplacian matrix and energy of power fuzzy graphs are given as results in section 3. In section 4, the application of this concept to cryptosystems is discussed with an example.

2. Preliminaries

Definition 1: [6] Let $\mathcal{G}_f = (V, E, \theta, \omega)$ be a fuzzy graph with vertices $v_i \in V$ where $i = 1$ to n . The adjacency matrix $A(\mathcal{G}_f)$ is an $n \times n$ matrix defined as $A(\mathcal{G}_f) = [a_{ij}]$ where $a_{ij} = \begin{cases} \omega_{\mathcal{G}_f}(v_i, v_j), & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$. [7] Let $D(\mathcal{G}_f) = [d_{ii}]$ where $d_{ii} = \sum_{j \neq i} \omega_{\mathcal{G}_f}(v_i, v_j)$ be a degree matrix, then Laplacian matrix $L(\mathcal{G}_f) = [l_{ij}]$ is defined as $L(\mathcal{G}_f) = D(\mathcal{G}_f) - A(\mathcal{G}_f)$.

Example 2: Consider a fuzzy graph $\mathcal{G}_f$ in Figure 1 with vertex and edge membership values in Table 1 and Table 2 respectively. The degree matrix, adjacency matrix, and Laplacian matrix are

$$D(\mathcal{G}_f) = \begin{bmatrix} 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(\mathcal{G}_f) = \begin{bmatrix} 0 & 0.4 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0.2 & 0.3 & 0 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 0.1 & 0.4 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 \end{bmatrix}$$

$$L(\mathcal{G}_f) = \begin{bmatrix} 0.6 & -0.4 & -0.2 & 0 & 0 & 0 & 0 & 0 \\ -0.4 & 0.5 & 0 & -0.1 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0.7 & -0.2 & -0.3 & 0 & 0 & 0 \\ 0 & -0.1 & -0.2 & 0.4 & 0 & -0.1 & 0 & 0 \\ 0 & 0 & -0.3 & 0 & 1 & -0.4 & -0.3 & 0 \\ 0 & 0 & 0 & -0.1 & -0.4 & 0.9 & 0 & -0.4 \\ 0 & 0 & 0 & 0 & -0.3 & 0 & 0.9 & -0.6 \\ 0 & 0 & 0 & 0 & 0 & -0.4 & -0.6 & 1 \end{bmatrix}$$

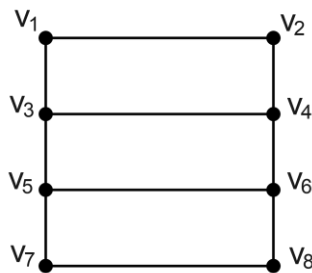


Figure 1
Fuzzy Ladder graph L_4

Table 1
Vertex membership values of $\mathcal{G}_f$ in Figure 1

Vertices	Membership value
v_1	$\theta(v_1) = 0.4$
v_2	$\theta(v_2) = 0.6$
v_3	$\theta(v_3) = 0.3$
v_4	$\theta(v_4) = 0.2$
v_5	$\theta(v_5) = 0.4$
v_6	$\theta(v_6) = 0.5$
v_7	$\theta(v_7) = 0.6$
v_8	$\theta(v_8) = 0.7$

Table 2
Edge membership values of $\mathcal{G}_f$ of Figure 1

Edge	Edge membership value
(v_1, v_2)	$\omega(v_1, v_2) = 0.4$
(v_1, v_3)	$\omega(v_1, v_3) = 0.2$
(v_2, v_4)	$\omega(v_2, v_4) = 0.1$
(v_3, v_4)	$\omega(v_3, v_4) = 0.2$
(v_3, v_5)	$\omega(v_3, v_5) = 0.3$
(v_4, v_6)	$\omega(v_4, v_6) = 0.1$
(v_5, v_6)	$\omega(v_5, v_6) = 0.4$
(v_5, v_7)	$\omega(v_5, v_7) = 0.3$
(v_6, v_8)	$\omega(v_6, v_8) = 0.4$
(v_7, v_8)	$\omega(v_7, v_8) = 0.6$

Definition 3: [7] Let $\mathcal{G}_f = (V, E, \theta, \omega)$ with n vertices be a fuzzy graph and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigen values of $\mathcal{G}_f$. Then the Laplacian energy of $\mathcal{G}_f$ is give by

$$LE(\mathcal{G}_f) = \sum_{i=1}^n \left| \lambda_i - \frac{2 \sum_{(u,v) \in E} \omega(u,v)}{n} \right|$$

Example 4: The Laplacian eigenvalues of Figure 1 are 0, 1, 0.1116, 0.3646, 0.6097, 0.8486, 1.229, 1.836. The Laplacian energy $LE(\mathcal{G}_f) = 0.75 + 0.25 + 0.6384 + 0.3854 + 0.1403 + 0.0986 + 0.479 + 1.086 = 3.8277$

Definition 5: [16] Let $\mathcal{G}_f = (V, E, \theta, \omega)$ be a fuzzy graph. Then density of a fuzzy graph is

$$De(\mathcal{G}_f) = \frac{2 \sum_{(u,v) \in E} \omega(u,v)}{\sum_{u,v \in V} \theta(u) \wedge \theta(v)}.$$

Example 6: The density of $\mathcal{G}_f$ in the Figure 1 is $\frac{2 \times 3}{3.5} = 1.7142$.

Definition 7: [17] Let $\mathcal{G}_f = (V, E, \theta, \omega)$ be a fuzzy graph. The size of $\mathcal{G}_f$ is defined as $size(\mathcal{G}_f) = \sum_{(u,v) \in E} \omega(u, v)$.

Definition 8: [3] Consider a fuzzy graph $\mathcal{G}_f = (V, E, \theta, \omega)$ of a graph $\mathcal{G} = (V, E)$. Then the definition of power fuzzy graph (PFG) of $\mathcal{G}_f$ is $\mathcal{G}_f^k = (V, E_k, \theta, \omega_k)$, where $E_k = E \cup E^*$, for every independent vertices $u, v \in V$ in $\mathcal{G}_f$ $\exists$ edges $uv \in E^*$ such that $\mathcal{L}(u, v) \leq k$ where $\mathcal{L}$ is the length between u and v and $k > 1$ provided

$$\min(\theta(u)^k, \theta(v)^k) \leq \omega_k(u, v) \leq \min(\theta(u)^{k-1}, \theta(v)^{k-1}).$$

Definition 9: [18] Let $\mathcal{G}_f^k = (V, E_k, \theta, \omega_k)$ be PFG. If $\deg_{\mathcal{G}_f^k}(v) = r$ for all $v \in V$, (i.e.), every vertex has the same degree r in $\mathcal{G}_f^k$, then $\mathcal{G}_f^k$ is said to be regular.

Theorem 10: [18] The size of a r -regular PFG, $\mathcal{G}_f^k = (V, E_k, \theta, \omega_k)$ is $\frac{nr}{2}$, where n is the number of vertices.

Note 11: The trace of a square matrix A is the sum of the diagonal elements of A .

3. Results and discussion

Theorem 12: The trace of the Laplacian matrix of a PFG $\mathcal{G}_f^t$ for $1 < t \leq k_m$ is equal to twice the sum of edge membership values of $\mathcal{G}_f^t$.

Proof: Let $\mathcal{G}_f^t = (V, E_t, \theta, \omega_t)$ be a PFG whose Laplacian matrix is denoted by $L(\mathcal{G}_f^t)$ and trace denoted by $\text{trace}(L(\mathcal{G}_f^t))$. By the Definition 1, $L(\mathcal{G}_f^t) = D(\mathcal{G}_f^t) - A(\mathcal{G}_f^t)$.

$$\Rightarrow \text{trace}(L(\mathcal{G}_f^t)) = \text{trace}(D(\mathcal{G}_f^t)) - \text{trace}(A(\mathcal{G}_f^t))$$

Since $a_{ii} = 0$ for all i in $A(\mathcal{G}_f^t)$, the $\text{trace}(A(\mathcal{G}_f^t)) = 0$. Also, by the Note 11 we have $\text{trace}(D(\mathcal{G}_f^t)) = 2 \sum_{(u,v) \in E_t} \omega_t(u, v)$. Hence $\text{trace}(L(\mathcal{G}_f^t)) = 2 \sum_{(u,v) \in E_t} \omega_t(u, v)$.

Example 13: Let $\mathcal{G}_f^1 = (V, E_1, \theta, \omega_1)$, $\mathcal{G}_f^2 = (V, E_2, \theta, \omega_2)$ and $\mathcal{G}_f^3 = (V, E_3, \theta, \omega_3)$ be the 4-path PFGs in Figures 2, 3 and 4 respectively. We can find $\sum_{(u,v) \in E_1} \omega_1(u, v) = 0.6$, $\sum_{(u,v) \in E_2} \omega_2(u, v) = 0.79$ and $\sum_{(u,v) \in E_1} \omega_1(u, v) = 0.539$. Then the Laplacian matrices are

$$L(\mathcal{G}_f^1) = \begin{bmatrix} 0.3 & -0.3 & 0 & 0 \\ -0.3 & 0.4 & -0.1 & 0 \\ 0 & -0.1 & 0.3 & -0.2 \\ 0 & 0 & -0.2 & 0.2 \end{bmatrix}$$

$$L(\mathcal{G}_f^2) = \begin{bmatrix} 0.39 & -0.3 & -0.09 & 0 \\ -0.3 & 0.65 & -0.15 & -0.2 \\ -0.09 & -0.15 & 0.29 & -0.05 \\ 0 & -0.2 & -0.05 & 0.25 \end{bmatrix}$$

$$L(\mathcal{G}_f^3) = \begin{bmatrix} 0.41 & -0.08 & -0.03 & -0.3 \\ -0.08 & 0.189 & -0.009 & -0.1 \\ -0.03 & -0.009 & 0.059 & -0.02 \\ -0.3 & -0.1 & -0.02 & 0.42 \end{bmatrix}$$

We can find trace of the matrices by adding the diagonal elements. Hence $\text{trace}(L(\mathcal{G}_f^1)) = 1.2$, $\text{trace}(L(\mathcal{G}_f^2)) = 1.58$ and $\text{trace}(L(\mathcal{G}_f^3)) = 1.078$. This implies $\text{trace}(L(\mathcal{G}_f^t)) = 2 \sum_{(u,v) \in E_t} \omega_t(u, v)$.

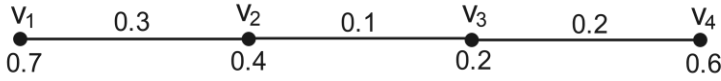


Figure 2
PFG $\mathcal{G}_f^1$

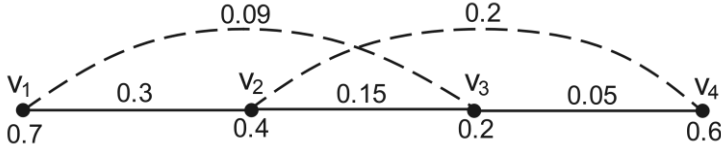


Figure 3
 $(\mathcal{G}_f^2)$ - PFG of $\mathcal{G}_f^1$

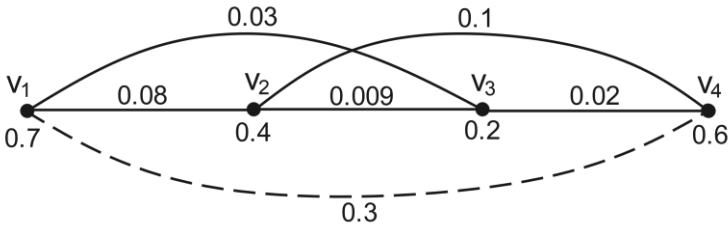


Figure 4
 $(\mathcal{G}_f^3)$ - PFG of $\mathcal{G}_f^2$

Corollary 14: The size of a PFG $\mathcal{G}_f^t$ for $1 < t \leq k_m$ is equal to half of the trace of its Laplacian matrix.

Proof: By the Definition 7, $size(\mathcal{G}_f^t) = \sum_{(u,v) \in E_t} \omega_t(u,v)$. Hence from the Theorem 12, $size(\mathcal{G}_f^t) = \frac{trace(L(\mathcal{G}_f^t))}{2}$.

Note 15: For a PFG $\mathcal{G}_f^k$ and Laplacian matrix $L(\mathcal{G}_f^k)$ associated with $\mathcal{G}_f^k$. Then, for trace of $L(\mathcal{G}_f^k)$, the Laplacian energy $LE(\mathcal{G}_f^k)$ of $\mathcal{G}_f^k$ and the density $De(\mathcal{G}_f^k)$ of $\mathcal{G}_f^k$, the following inequality holds:

$$De(\mathcal{G}_f^k) < LE(\mathcal{G}_f^k) < trace(L(\mathcal{G}_f^k))$$

Remark 16: The Laplacian energy sequence of the PFGs is $LE(\mathcal{G}_f^2) > LE(\mathcal{G}_f^3) > \dots > LE(\mathcal{G}_f^{k_m})$.

Theorem 17: Let $\mathcal{G}_f^k$ be a regular PFG then $LE(\mathcal{G}_f^k) = \sum_{i=1}^n |\lambda_i - r|$, where $r = deg_{\mathcal{G}_f^k}(v), \forall v \in V$.

Proof: By the Definition 3, we have $LE(\mathcal{G}_f^k) = \sum_{i=1}^n \left| \lambda_i - \frac{2 \sum_{(u,v) \in E_k} \omega_k(u,v)}{n} \right|$.

$$\Rightarrow LE(\mathcal{G}_f^k) = \sum_{i=1}^n \left| \lambda_i - \frac{2size(\mathcal{G}_f^k)}{n} \right| \quad (\text{By the Definition 7})$$

$$\Rightarrow LE(\mathcal{G}_f^k) = \sum_{i=1}^n |\lambda_i - r| \quad (\text{By the Theorem 10}).$$

Note 18: For a regular PFG, the diagonal elements are constant and equal to r .

4. Application

We construct an asymmetric algorithm using the concepts of graph theory, fuzzy logic, and mathematical structures to enhance the security and efficiency of cryptographic systems. The complexity introduced by this model increases the difficulty of predicting or breaking the cryptographic scheme. We can use this algorithm based on security levels, context, user roles, or data sensitivity.

4.1 Asymmetric algorithm with Laplacian energy as plain text:

Step 1: Let PFG $\mathcal{G}_f^k = (V, E_k, \theta, \omega_k)$ be a PFG with n vertices and corresponding membership values $\theta(v_i)$ and $\omega_k(v_i, v_j)$ such that $v_i \neq v_j$ and $1 < k_r \leq k_m$. The Laplacian energy of $\mathcal{G}_f^{k_r} = (V, E_{k_r}, \theta, \omega_{k_r})$ be the plain text. where $\mathcal{G}_f^{k_r}$ is regular.

Step 2: Generate a public key k_r and n be the number of vertices such that $1 < k_r \leq k_m$, where $\mathcal{G}_f^{k_r}$ is regular.

Step 3: Let the server private/secret key be the size of $\mathcal{G}_f^{k_r}$.

Step 4: Encrypting the Laplacian matrix of $\mathcal{G}_f^{k_r}$, the cipher text sent to the receiver is Laplacian Eigenvalues, Laplacian Eigenvectors.

Step 5: Using the secret key size of $\mathcal{G}_f^{k_r}$ we compute p. Using cipher text we get Laplacian energy.

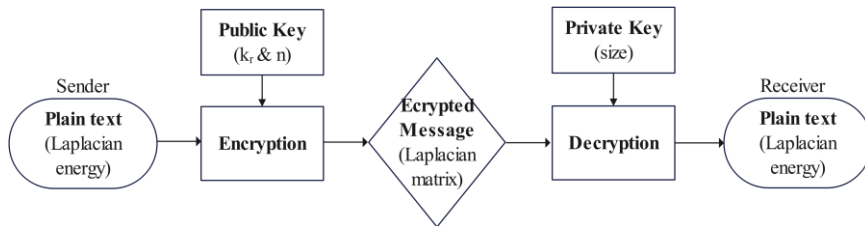


Figure 5
Block diagram of asymmetric encryption

4.2 Illustration

Consider a PFGs $\mathcal{G}_f^1$ and $\mathcal{G}_f^2$ with 6 vertices with their corresponding membership values $\theta(v_i)$ and $\omega_1(v_i, v_j)$, $\omega_2(v_i, v_j)$ in Figures 6 and 7 respectively. Let $\mathcal{G}_f^{k_r} = (V, E_{k_r}, \theta, \omega_{k_r})$ be regular.

Step 1: The plain text is Laplacian energy $LE(\mathcal{G}_f^{k_r})$ of

$$L(\mathcal{G}_f^{k_r}) = \begin{bmatrix} 1.53 & -0.3 & -0.415 & 0 & -0.415 & -0.4 \\ -0.3 & 1.53 & -0.4 & -0.415 & 0 & -0.415 \\ -0.415 & -0.4 & 1.53 & -0.3 & -0.415 & 0 \\ 0 & -0.415 & -0.3 & 1.53 & -0.4 & -0.415 \\ -0.415 & 0 & -0.415 & -0.4 & 1.53 & -0.3 \\ -0.4 & -0.415 & 0 & -0.415 & -0.3 & 1.53 \end{bmatrix}$$

Step 2: The public keys are $k_r = 2$ and $n = 6$.

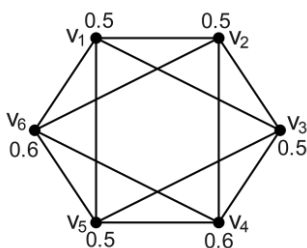


Figure 6
6-cycle PFG $\mathcal{G}_f^1$

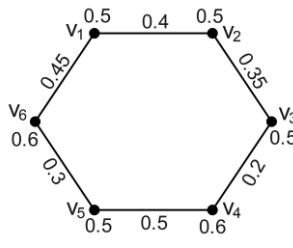


Figure 7
6-cycle PFG $\mathcal{G}_f^2$

Table 3
Edge membership values of $\mathcal{G}_f^2$ in Figure 7

Edge	Interval	Edge membership value
(v_1, v_2)	$[0.25, 0.5]$	$\omega_2(v_1, v_2) = 0.3$
(v_1, v_3)	$[0.25, 0.5]$	$\omega_2(v_1, v_3) = 0.415$
(v_1, v_5)	$[0.25, 0.5]$	$\omega_2(v_1, v_5) = 0.415$
(v_1, v_6)	$[0.25, 0.5]$	$\omega_2(v_1, v_6) = 0.4$
(v_2, v_3)	$[0.25, 0.5]$	$\omega_2(v_2, v_3) = 0.4$
(v_2, v_4)	$[0.25, 0.5]$	$\omega_2(v_2, v_4) = 0.415$
(v_2, v_6)	$[0.25, 0.5]$	$\omega_2(v_2, v_6) = 0.415$
(v_3, v_4)	$[0.25, 0.5]$	$\omega_2(v_3, v_4) = 0.3$
(v_3, v_5)	$[0.25, 0.5]$	$\omega_2(v_3, v_5) = 0.415$
(v_4, v_5)	$[0.25, 0.5]$	$\omega_2(v_4, v_5) = 0.4$
(v_4, v_6)	$[0.36, 0.6]$	$\omega_2(v_4, v_6) = 0.415$
(v_5, v_6)	$[0.25, 0.5]$	$\omega_2(v_5, v_6) = 0.3$

Step 3: The private key $size(\mathcal{G}_f^{kr}) = \frac{trace(\mathcal{G}_f^{kr})}{2} = \sum_{(u,v) \in E_k} \omega_k(u,v) = 4.59$, using the edge membership values from the Table 3.

Step 4: Using the $L(\mathcal{G}_f^{kr})$, the cipher text Laplacian eigenvalues $\{0, 1.4, 1.584, 1.584, 2.306, 2.306\}$ are encrypted.

Step 5: Using public key $n = 6$, private key $size(\mathcal{G}_f^{kr}) = 4.59$, we compute $p = \frac{size(\mathcal{G}_f^{kr})}{n} = 1.53$. Using the cipher text Laplacian eigenvalues the plain text $LE(\mathcal{G}_f^{kr})$ is found using $LE(\mathcal{G}_f^{kr}) = \sum_{i=1}^n |\lambda_i - p| = 1.53 + 0.13 + 0.054 + 0.776 + 0.054 + 0.776 = 3.32$.

5. Conclusion

The relation between the size of a PFG and the trace of the Laplacian matrix has been established. The inequality between density, Laplacian energy, and trace of the PFGs has been determined. In conclusion, cryptosystem models based on PFGs provide a new approach to secure communications by the means of flexibility, adaptability, and enhanced security through the model of PFG theory.

References

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353 (Jun. 1965). [https://doi.org/10.1016/s0019-9958\(65\)90241-x](https://doi.org/10.1016/s0019-9958(65)90241-x).
- [2] A. Rosenfeld, "Fuzzy Graphs," Fuzzy sets and their applications to cognitive and decision processes, In: Proceedings of the US-Japan Seminar on Fuzzy Sets and their Applications, pp. 77-95 (Jan. 1975). <https://doi.org/10.1016/b978-0-12-775260-0.50008-6>
- [3] T. Bharathi, S. Shiny Paulin, and Bijan Davvaz, "A novel discussion on PFGs and their application in decision making," *Journal of applied mathematics & informatics*, vol. 42, no. 1, pp. 123-137, 2024. <https://doi.org/10.14317/jami.2024.123>
- [4] R. Balakrishnan, "The energy of a graph," *Linear Algebra and its Applications*, vol. 387, pp. 287-295 "Aug. 2004". <https://doi.org/10.1016/j.laa.2004.02.038>
- [5] I. Gutman, "The Energy of a Graph: Old and New Results," In: Betten, A., Kohnert, A., Laue, R., Wassermann, A. (eds) *Algebraic Combinatorics and Applications*, Springer, Berlin, Heidelberg, pp. 196-211

- "Jan. 2001". https://doi.org/10.1007/978-3-642-59448-9_13https://doi.org/10.1007/978-3-642-59448-9_13.
- [6] A. Narayanan and S. Mathew, "Energy of a fuzzy graph," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 3, pp. 455-465 (2013).
- [7] Rahimi Sharbaf Sadegh and Fayazi Fatmeh, "Laplacian Energy of a Fuzzy Graph," *Iranian journal of mathematical chemistry. Iranian journal of mathematical chemistry*, vol. 5, no. 1, pp. 1-10 (Mar. 2014) <https://doi.org/10.22052/ijmc.2014.5214><https://doi.org/10.22052/ijmc.2014.5214>
- [8] S. Sharief Basha and E. Kartheek, "Laplacian Energy of an Intuitionistic Fuzzy Graph," *Indian Journal of Science and Technology*, vol. 8, no. 33 (Dec. 2015). <https://doi.org/10.17485/ijst/2015/v8i33/79899><https://doi.org/10.17485/ijst/2015/v8i33/79899>.
- [9] A. M. Philip, S. J. Kalayathankal, and J. V. Kureethara, "On the Laplacian energy of interval valued fuzzy graphs," *AIP conference proceedings*, vol. 2261, no. 1, p. 030142 (Jan. 2020). <https://doi.org/10.1063/5.0016779><https://doi.org/10.1063/5.0016779>.
- [10] Mahima Poonia and R. K. Bajaj, "On Laplacian energy of picture fuzzy graphs in site selection problem," *Journal of Intelligent & Fuzzy Systems*, vol. 41, no. 1, pp. 481-498 (Jun. 2021). <https://doi.org/10.3233/jifs-202131><https://doi.org/10.3233/jifs-202131>.
- [11] N. Rajagopal Reddy, M. Zubair Khan, S. Sharief Basha, A. Alahmadi, A. H. Alahmadi, and C. Lal Chowdhary, "The Laplacian Energy of Hesitancy Fuzzy Graphs in Decision-Making Problems," *Computer Systems Science and Engineering*, vol. 44, no. 3, pp. 2637-2653 (2023). <https://doi.org/10.32604/csse.2023.029255><https://doi.org/10.32604/csse.2023.029255>.
- [12] J. U. Rahman, U. Ali, and M. U. Rehman, "The Laplacian energy of diameter 4 trees," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 24, no. 1, pp. 119-128 (2021). <https://doi.org/10.1080/09720529.2019.1670943> <https://doi.org/10.1080/09720529.2019.167094>.
- [13] R. Sarathy, and J. Ravi Sankar, "Laplacian energy and color based energy for graphs associated with commutative rings," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 27, no. 8, pp. 2431-2445 (2024). <https://doi.org/10.47974/JDMSC-2016> <https://doi.org/10.47974/JDMSC-2016>.

- [14] T. Al-Hawary, S. Al-Shalaldeh, and M. Akram, "Certain Matrices and Energies of Fuzzy graphs," *TWMS Journal of Applied and Engineering Mathematics*, vol. 2, no. 1 (Jan. 2023), <https://doi.org/10.30546/2219-1259.14.1.2023.50><https://doi.org/10.30546/2219-1259.14.1.2023.50>.
- [15] T. Bharathi, S. Shiny Paulin, and S. Leo, "Cryptographic approach on edge regular PFG," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 3, pp. 945-958 (2025). <https://doi.org/10.47974/JDMSC-2216> <https://doi.org/10.47974/JDMSC-2216>.
- [16] T. Al-Hawary, "Complete fuzzy graphs," *International Journal of Mathematical Combinatorics*, vol. 4, no. 4, pp. 26-34 (Jan. 2011). <https://doi.org/10.5281/zenodo.825538><https://doi.org/10.5281/zenodo.825538>
- [17] N. Gani, and B. M. Ahamed, "Order and size in fuzzy Graph," *Bulletin of Pure and Applied Sciences*, vol. 22, no. 1, pp. 145-148 (2003). <https://doi.org/10.13140/2.1.3884.0969><https://doi.org/10.13140/2.1.3884.0969>.
- [18] T. Bharathi, S. Shiny Paulin, and M. Jeba Sherlin, "On regular PFG," *Journal of Applied Mathematics, Statistics and Informatics*, vol. 20, no. 2, pp. 5-18 (Dec 2024). <https://doi.org/10.2478/jamsi-2024-0005> <https://doi.org/10.2478/jamsi-2024-0005>.

Received January, 2025