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Efficient descent direction of a primal-dual interior point algorithm for convex quadratic optimization

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Abstract

We introduce a new interior-point method for solving convex quadratic programming under full-Newton step. The method involves an equivalent algebraic transformation applied to the system defining the central path. This approach provides an efficient search direction for the considered algorithm. Furthermore, we show that the introduced method produces an optimal solution within polynomial time. The established numerical tests conclude that the newly proposed algorithm is not only polynomial but requires a number of iterations clearly lower than that obtained theoretically.

Subject Classification: *90C20, 90C51.*

Keywords: *Convex quadratic programming, Primal-dual interior point method, Descent direction.*

1. Introduction

Primal-dual path-following methods are among the most efficient algorithms of interior-point methods (IPMs) for solving wide classes of optimization problems thanks to their polynomial complexity and their numerical efficiency [16, 22]. The success of these methods for solving linear optimization (LO) leads researchers to extend it naturally to other

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important problems such as semidefinite programs (SDP) [12], quadratic programs (QP) [2, 18], complementarity problems (CP) [3] and conic optimization problems (COP).

Search directions computation is an important aspect of IPM. In recent time, primal-dual methods that utilize kernel functions garnered significant interest among researchers. This has led to a series of publications that explore novel research directions, aiming to provide polynomial complexity for both small and large-update algorithms. In this sense, Bai et al. [4] presented a large class of eligible kernel functions, which is fairly general and includes the classical logarithmic functions and the self-regular functions, as well as many non-self-regular functions as special cases. For some other related kernel function, we refer to [5, 6, 7, 11, 13, 14, 15].

Another technique for defining other search directions for LO was given by Darvay [8]. This approach relies on transforming the centering equations by an algebraic equivalent transformation (AET). Achache [1] generalized this method for convex quadratic optimization (CQO), while Wang and Bai [19, 20, 21] for semidefinite optimization (SDO), second-order cone optimization (SOCO) and symmetric optimization (SO).

In 2018, Darvay and Takács [10] proposed another technique for obtaining a new class of research directions for solving linear problem (LP). The technique is based on applying the function φ on both sides of a specific algebraic equivalent transformation of the centering equations defined by Zhang and Xu [23].

Motivated by the above works, in this paper, we extend the technique given in [10] to the convex quadratic programming (CQP) case. In our analysis, we demonstrate that the obtained algorithm possess a polynomial complexity, specifically achieving by $O\left(\log\frac{\left(n+\frac{9}{2}\right)}{\varepsilon}\right)$ iteration bound.

The paper is organized as follows: In Section 2, we discuss the notion of the central path for CQP. Section 3, explores the new descent direction followed by an overview of the associated algorithm. Section 4 is dedicated to the complexity analysis study, the convergence result and the total number of iteration for optimality. Section 5, includes some numerical experiments with accompanying observations. Section 6 presents the conclusion of the paper.

2. The problem statement

Consider the CQP problem in its standard form:

$$\begin{cases} \min \frac{1}{2} x^T G x + r^T x, \\ Mx = b, \\ x \in \mathbb{R}^n_+. \end{cases} \tag{CQP}$$

Where G represents a symmetric positive semidefinite matrix with dimensions $n \times n$, $M \in \mathbb{R}^{m \times n}$, such that $Rank(M) = m < n$, $r \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The dual problem of (CQP) will be expressed as:

$$\begin{cases} \max b^T \lambda - \frac{1}{2} x^T G x, \\ M^T \lambda - Gx + w = r, \\ \lambda \in \mathbb{R}^m, w \in \mathbb{R}^n_+. \end{cases} \tag{CQD}$$

Suppose that there exist an initial point (x^0, λ^0, w^0) such that

$$Mx^0 = b, M^T \lambda^0 - Gx^0 + w^0 = r, x^0 > 0 \text{ and } w^0 > 0.$$

This last condition is named the interior-point condition (IPC) for both primal and dual problems.

Clearly, under the previous assumptions, finding an optimal solution for both (CQP) and (CQD) is the same as solving a system of nonlinear equations

$$\begin{cases} Mx = b, \\ M^T \lambda - Gx + w = r, \\ xw = 0, \; x, w \geq 0. \end{cases} \tag{1}$$

Here, xw means the coordinatewise product of x and w, hence

$$xw = (x_1 w_1, x_2 w_2, \dots, x_n w_n)^T.$$

The concept of primal-dual path-following IPMs is to replace the last equation in system (1) with the parameterized equation $xw = \eta e$, where e represents a vector of all ones with a length of n and $\eta > 0$. This leads us to consider the system

$$\begin{cases} Mx = b, \\ M^T \lambda - Gx + w = r, \\ xw = \eta e, \; x, w > 0. \end{cases} \tag{2}$$

If the IPC holds, then for a fixed $\eta > 0$, the η-center given by Sonnevend [17] is the unique solution of system (2). For different η, the central path generate a sequence which converges to a primal-dual optimal solution for both problems (CQP) and (CQD), when $\eta \to 0$.

3. New search direction

In this section, we generalize the technique introduced by Darvay et al. [10] to get new search direction for CQP. Since $(x,w,\eta)>0,$ we deduce from (2)

$$\frac{xw}{\eta}=e\Leftrightarrow\sqrt{\frac{xw}{\eta}}=e\Leftrightarrow\frac{xw}{\eta}=\sqrt{\frac{xw}{\eta}},$$

with, $\frac{xw}{\eta}=\left(\frac{x_1w_1}{\eta},\frac{x_2w_2}{\eta},\ldots,\frac{x_nw_n}{\eta}\right)^T>0$ and $\sqrt{\frac{xw}{\eta}}=\left(\sqrt{\frac{x_1w_1}{\eta}},\sqrt{\frac{x_2w_2}{\eta}},\ldots,\sqrt{\frac{x_nw_n}{\eta}}\right)^T.$

Now, we can express the perturbed central path equivalently as follows

$$\begin{cases}Mx=b,\\ M^T\lambda-Gx+w=r,\\ \frac{xw}{\eta}=\sqrt{\frac{xw}{\eta}}.\end{cases}\tag{3}$$

Now, let's define a continuously differentiable function φ on (κ^2,∞), such that $0\leq\kappa<1,$ and $2t\varphi'(t^2)-\varphi'(t)>0,$ $\forall t>\kappa^2.$ Applying the AET technique on system (3), we obtain

$$\begin{cases}Mx=b,\\ M^T\lambda-Gx+w=r,\\ \varphi\left(\frac{xw}{\eta}\right)=\varphi\left(\sqrt{\frac{xw}{\eta}}\right)\end{cases}\tag{4}$$

Newton iteration applied to system (4) define:

$$(x_+,\lambda_+,w_+)=(x+\Delta x,\lambda+\Delta\lambda,w+\Delta w),$$

here $(\Delta x,\Delta\lambda,\Delta w)$ is the search direction solution of the linear system

$$\begin{cases}M\Delta x=0,\\ M^T\Delta\lambda-G\Delta x+\Delta w=0,\\ \frac{1}{\eta}(w\Delta x+x\Delta w)=\dfrac{-\varphi\left(\frac{xw}{\eta}\right)+\varphi\left(\sqrt{\frac{xw}{\eta}}\right)}{\varphi'\left(\frac{xw}{\eta}\right)-\frac{1}{2\sqrt{\frac{xw}{\eta}}}\varphi'\left(\sqrt{\frac{xw}{\eta}}\right)}.\end{cases}\tag{5}$$

Defining the scaled search directions

$$d_x=X^{-1}V\Delta x,\ \text{and}\ d_w=W^{-1}V\Delta w,$$

with

$$X = diag(x), W = diag(w) \text{ and } V = diag(v), \text{where } v = \sqrt{\frac{xw}{\eta}}.$$

Here, the matrices $diag(x)$, $diag(w)$, and $diag(v)$ are diagonal matrices that have the components of x, w and v, respectively. Hence, we obtain

$$w\Delta x + x\Delta w = \eta v(d_x + d_w) \text{ and } \eta d_x d_w = \Delta x \Delta w. \tag{6}$$

Clearly, using the last notations, the system (5) can be represented as

$$\begin{cases} \overline{M} d_x = 0 \\ \overline{M}^T \Delta\lambda - \overline{G} d_x + d_w = 0 \\ d_x + d_w = d_v. \end{cases} \tag{7}$$

Where

$$d_v = \frac{2\varphi(v) - 2\varphi(v^2)}{2v\varphi'(v^2) - \varphi'(v)},$$

$$\overline{M} = \frac{1}{\eta} M V^{-1} X \text{ and } \overline{G} = \frac{1}{\eta} V^{-1} X G V^{-1} X.$$

Generic Primal-dual IPM for CQP
Input:

a proximity parameter $\tau > 1$ (default $\tau = \frac{1}{10}$);
an accuracy parameter $\varepsilon > 0$;
an update parameter ϑ, $0 < \vartheta < 1$ (default $\vartheta = \frac{1}{12\sqrt{n}}$);
a strictly feasible point (x^0, y^0, w^0); $\eta^0 = \frac{(x^0)^T w^0}{n}$; $v^0 = \sqrt{\frac{x^0 w^0}{\eta^0}} > \frac{e}{\sqrt{2}}$;
begin
$(x, y, w) = (x^0, y^0, w^0)$;
while $x^T w \geq \varepsilon$ **do**
-Take $\eta = (1 - \vartheta)\eta$;
-Compute $(\Delta x, \Delta\lambda, \Delta w)$ from (7) via (5);
-Take $(x, \lambda, w) = (x + \Delta x, \lambda + \Delta\lambda, w + \Delta w)$;
end while
end.

Figure 1
Generic algorithm for CQP

Let $\varphi:\left(\frac{1}{\sqrt{2}},\infty\right)\to\mathbb{R},\ \ \varphi(t)=t^2$. So

$$d_v=\frac{v-v^3}{2v^2-e}. \tag{8}$$

The condition $2t\varphi'(t^2)-\varphi'(t)>0, \forall t>\kappa^2$ is holds for $\kappa^2=\frac{1}{\sqrt{2}}$. Introducing a proximity measure

$$\sigma(v)=\sigma(xw,\eta)=\frac{\|d_v\|}{2}=\frac{1}{2}\left\|\frac{v-v^3}{2v^2-e}\right\|, \tag{9}$$

here, $\|.\|$ signifies the Euclidean norm.

We describe the generic algorithm in Figure 1.

The next section contains some results of the complexity analysis of our algorithm.

4. Convergence and complexity analysis

Lemma 4.1: *Let* $c_v=d_x-d_w$. *Then,*

$$\|c_v\|\le\|d_v\|. \tag{10}$$

Proof: From the last equation of system (7), we obtain

$$d_x=\frac{d_v+c_v}{2} \text{ and } d_w=\frac{d_v-c_v}{2}.$$

Those give

$$d_xd_w=\frac{d_v^2-c_v^2}{4}, \tag{11}$$

on the other hand, we have

$$\|d_v\|^2=\|d_x+d_w\|^2=\|d_x-d_w\|^2+4d_x^Td_w=\|c_v\|^2+4d_x^Td_w.$$

Since $\bar{M}d_x=0$ and $\bar{G}$ is a positive semidefinite matrix, then

$$d_x^Td_w=d_x^T(\bar{G}d_x-\bar{M}^T\Delta\lambda)=d_x^T\bar{G}d_x-(\bar{M}d_x)^T\Delta\lambda\ge 0,$$

which gives $\|c_v\|\le\|d_v\|$.

The next lemma gives a condition that guarantees the feasibility of the full Newton step.

Lemma 4.2: *If* $v>\frac{1}{\sqrt{2}}e$ *and* $\sigma=\sigma(xw,\eta)<1$. *Then*

$$(x_+,w_+)>0.$$

Proof: Let $\rho \in [0,1]$. Denote by $(x_+(\rho), w_+(\rho)) = (x+\rho\Delta x, w+\rho\Delta w)$. So,

$$x_+(\rho)w_+(\rho) = xw + \rho(w\Delta x + x\Delta w) + \rho^2 \Delta x \Delta w.$$

In view of (6), we observe that

$$\frac{x_+(\rho)w_+(\rho)}{\eta} = \frac{xw}{\eta} + \rho v(d_x + d_w) + \rho^2 d_x d_w. \tag{12}$$

Based on (7) and (11), we express

$$\frac{x_+(\rho)w_+(\rho)}{\eta} = (1-\rho)v^2 + \rho(v^2 + vd_v) + \rho^2\left(\frac{d_v^2 - c_v^2}{4}\right). \tag{13}$$

In addition, from (8) we obtain

$$v^2 + vd_v = \frac{v^4}{2v^2 - e} \geq e. \tag{14}$$

This last inequality follows directly from

$$(v^2 - e)^2 \geq 0 \Leftrightarrow v^4 \geq 2v^2 - e.$$

Then

$$\begin{aligned}\frac{x_+(\rho)w_+(\rho)}{\eta} &\geq (1-\rho)v^2 + \rho e + \rho^2\left(\frac{d_v^2}{4} - \frac{c_v^2}{4}\right) - \rho\frac{d_v^2}{4} \\ &\geq (1-\rho)v^2 + \rho\left(e - \left((1-\rho)\frac{d_v^2}{4} + \rho\frac{c_v^2}{4}\right)\right).\end{aligned} \tag{15}$$

$x_+(\rho)w_+(\rho) > 0$ when $\left\|(1-\rho)\frac{d_v^2}{4} + \rho\frac{c_v^2}{4}\right\|_\infty < 1$, Consequently,

$$\begin{aligned}\left\|(1-\rho)\frac{d_v^2}{4} + \rho\frac{c_v^2}{4}\right\|_\infty &\leq (1-\rho)\frac{\|d_v^2\|_\infty}{4} + \rho\frac{\|c_v^2\|_\infty}{4} \\ &\leq (1-\rho)\frac{\|d_v\|_2^2}{4} + \rho\frac{\|c_v\|_2^2}{4},\end{aligned}$$

from (10), we get

$$\begin{aligned}\left\|(1-\rho)\frac{d_v^2}{4} + \rho\frac{c_v^2}{4}\right\|_\infty &\leq (1-\rho)\frac{\|d_v\|_2^2}{4} + \rho\frac{\|d_v\|_2^2}{4} \\ &\leq \frac{\|d_v\|_2^2}{4} = \sigma^2 < 1.\end{aligned}$$

So, $x_+(\rho)w_+(\rho) > 0 \;\; \forall \rho \in [0,1]$. This indicates that the linear functions $x_+(\rho)$ and $w_+(\rho)$ do not switch signs within the interval $[0,1]$ and for

$\rho=0$, we have $(x_+(0), w_+(0)) = (x,w) > 0$. As a result, it follows that $(x_+(1), w_+(1)) = (x_+, w_+) > 0$.

The following lemma will be beneficial in the upcoming part of our analysis.

Lemma 4.3: *[9, Lemma 5.2] Let the decreasing function* $g:[u,\infty) \to (0,\infty)$, *such that* $u>0$. *Moreover, Let* $\zeta \in \mathbb{R}^n_+$ *where* $min(\zeta)>u$. *So*

$$\| g(\zeta)(e-\zeta^2) \| \leq g(\min(\zeta)) \| e-\zeta^2 \| \leq g(u) \| e-\zeta^2 \|.$$

In the next result, we give the conditions of quadratically convergent of the proximity σ.

Lemma 4.4: *Suppose* $v>\frac{1}{\sqrt{2}}e$ *and* $\sigma=\sigma(xw,\eta)<\frac{1}{\sqrt{2}}$. *So* $v_+=\sqrt{\frac{x_+w_+}{\eta}}>\frac{1}{\sqrt{2}}e$ *also*

$$\sigma(x_+w_+,\eta) \leq \frac{5\sigma^2}{1-2\sigma^2}\sqrt{1-\sigma^2}.$$

Proof: According to Lemma 4.2, we have $(x_+,s_+)>0$, this means that $v_+=\sqrt{\frac{x_+w_+}{\eta}}$ is well defined.

Let's set $\rho=1$. Consequently, from equation (13), it can be deduced that

$$v_+^2=\frac{x_+w_+}{\eta}=v^2+vd_v+\frac{d_v^2}{4}-\frac{c_v^2}{4}. \tag{16}$$

Now, in view of (14) we have

$$v^2+vd_v=e+\frac{(v^2-e)^2}{2v^2-e}. \tag{17}$$

From (8), (16) and (17) we may write

$$\begin{aligned} v_+^2 &= e+\frac{(v^2-e)^2}{2v^2-e}+\frac{1}{4}\left(\frac{v-v^3}{2v^2-e}\right)^2-\frac{c_v^2}{4} \\ &= e+\frac{(v^2-e)^2}{2v^2-e}+\frac{(v(e-v^2))^2}{4(2v^2-e)^2}-\frac{c_v^2}{4} \\ &= e+\frac{(v^2-e)^2}{2v^2-e}\left(e+\frac{v^2}{4(2v^2-e)}\right)-\frac{c_v^2}{4} \\ &= e+\frac{(v^2-e)^2(9v^2-4e)}{4(2v^2-e)^2}-\frac{c_v^2}{4}. \end{aligned} \tag{18}$$

Using (8) and (18) we get

$$\begin{aligned} e - v_+^2 &= \frac{c_v^2}{4} - \frac{(v^2-e)^2(9v^2-4e)}{4(2v^2-e)^2} \\ &= \frac{c_v^2}{4} - \frac{(9v^2-4e)}{v^2} \cdot \frac{v^2(e-v^2)^2}{4(2v^2-e)^2} \\ &= \frac{c_v^2}{4} - \frac{(9v^2-4e)}{v^2} \cdot \frac{v^2+v^6-2v^4}{4(2v^2-e)^2} \\ &= \frac{c_v^2}{4} - \frac{(9v^2-4e)}{v^2} \cdot \frac{(v-v^3)^2}{4(2v^2-e)^2} \\ &= \frac{c_v^2}{4} - \frac{(9v^2-4e)}{v^2} \cdot \frac{d_v^2}{4}. \end{aligned} \tag{19}$$

Since $\frac{9v^2-4e}{v^2} < 9e$, for all $v > \frac{1}{\sqrt{2}}e$, and from (10), we get

$$\| e - v_+^2 \| \le \left\| \frac{c_v^2}{4} \right\| + \left\| \frac{(9v^2-4e)}{v^2} \cdot \frac{d_v^2}{4} \right\| < \frac{\|d_v\|^2}{4} + 9\frac{\|d_v\|^2}{4} = 10\sigma^2 \tag{20}$$

Beside these, we know that $\frac{d_v^2}{4} > 0$ and we also have from (14) that $v^2 + vd_v = \frac{v^4}{2v^2-e} \ge e$. As a result, these inferences indicate that, based on (16)

$$v_+^2 = v^2 + vd_v + \frac{d_v^2}{4} - \frac{c_v^2}{4} \ge e - \frac{c_v^2}{4}.$$

Hence

$$\min(v_+^2) \ge 1 - \frac{\|c_v\|_\infty^2}{4} \ge 1 - \frac{\|c_v\|^2}{4} \ge 1 - \frac{\|d_v\|^2}{4} \ge 1 - \sigma^2,$$

and this relation yields

$$\min(v_+) \ge \sqrt{1-\sigma^2}. \tag{21}$$

On the other hand, from $\sigma < \frac{1}{\sqrt{2}}$, it can be deduced that $\sqrt{1-\sigma^2} > \frac{1}{\sqrt{2}}$. In view of (21) we get $v_+ > \frac{e}{\sqrt{2}}$. Moreover, using the definition of σ, we got

$$\sigma(v_+) = \sigma(x_+ w_+, \eta) = \frac{1}{2}\left\| \frac{v_+}{2v_+^2 - e}(e - v_+^2) \right\|$$

Let $f(t) = \frac{t}{2t^2-1}$ for all $t > \frac{1}{\sqrt{2}}$. We have $f'(t) < 0$, this implies that f is decreasing for all $t > \frac{1}{\sqrt{2}}$. Therefore, by utilizing (20), (21) and Lemma 4.3, we obtain

$$\begin{aligned} \sigma(v_+) &\le \frac{1}{2} f(\sqrt{1-\sigma^2}) \| (e - v_+^2) \| \\ &\le \frac{1}{2} \frac{\sqrt{1-\sigma^2}}{2(1-\sigma^2)-1} \| (e - v_+^2) \| \end{aligned}$$

$$\leq \frac{5\sigma^2\sqrt{1-\sigma^2}}{1-2\sigma^2}.$$

With that, we have completed the proof.

The lemma bellow examines the forte duality gap.

Lemma 4.5: *Let* $\sigma = \sigma(xw,\eta)$. *So*

$$(x_+)^T w_+ \leq \eta(n+9\sigma^2).$$

Moreover, if $\sigma < \frac{1}{\sqrt{2}}$, then $(x_+)^T w_+ < \eta\left(n+\frac{9}{2}\right)$.

Proof: (17) gives

$$\begin{aligned} v^2 + vd_v &= e + \left(\frac{4(2v^2-e)}{v^2}\right)\left(\frac{v^2(e-v^2)^2}{4(2v^2-e)^2}\right) \\ &= e + \left(8e - \frac{4e}{v^2}\right)\frac{d_v^2}{4} \\ &\leq e + 8\frac{d_v^2}{4}, \end{aligned} \tag{22}$$

and we know that

$$\begin{aligned} (x_+)^T w_+ &= \eta\sum_{i=1}^{n}(v_{+_i})^2 \\ &= \eta\sum_{i=1}^{n}\left(v_i^2 + v_i(d_v)_i + \frac{(d_v)_i^2}{4} - \frac{(c_v)_i^2}{4}\right) \\ &= \eta\sum_{i=1}^{n}(v_i^2 + v_i(d_v)_i) + \eta\left(\frac{\|d_v\|^2 - \|c_v\|^2}{4}\right) \\ &\leq \eta\sum_{i=1}^{n}\left(1 + 8\frac{(d_v)_i^2}{4}\right) + \eta\left(\frac{\|d_v\|^2 - \|c_v\|^2}{4}\right) \\ &\leq \eta n + 8\eta\frac{\|d_v\|^2}{4} + \eta\frac{\|d_v\|^2}{4} \\ &\leq \eta(n+9\sigma^2). \end{aligned}$$

Now, if $\sigma < \frac{1}{\sqrt{2}}$ then $\sigma^2 < \frac{1}{2}$ and

$$(x_+)^T w_+ < \eta\left(n+\frac{9}{2}\right).$$

With that, we have completed the proof.

Now, we show that the algorithm is well defined.

Lemma 4.6: *Suppose that,* $v > \frac{1}{\sqrt{2}}e$, $\eta_+ = (1-\vartheta)\eta$, *with* $0 < \vartheta < 1$ *and* $\sigma = \sigma(xw,\eta) < \frac{1}{\sqrt{2}}$. *In addition, let* $\tilde{v} = \sqrt{\frac{x_+ w_+}{\eta_+}}$. *So,* $\tilde{v} > \frac{1}{\sqrt{2}}e$ *and*

$$\sigma(\tilde{v})=\sigma(x_+w_+,\eta_+)<\frac{\sqrt{1-\sigma^2}}{2\sqrt{1-\vartheta}(1-2\sigma^2+\vartheta)}(\vartheta\sqrt{n}+10\sigma^2).$$

Furthermore, when $\sigma<\frac{1}{10}$ and $\vartheta=\frac{1}{12\sqrt{n}}$, then $\sigma(\tilde{v})<\frac{1}{10}$.

Proof: By utilizing Lemma, 4.4 we have $v_+>\frac{1}{\sqrt{2}}e$. From $\tilde{v}=\sqrt{\frac{x_+w_+}{\eta_+}}$, we deduce

$$\tilde{v}=\sqrt{\frac{x_+w_+}{\eta_+}}=\frac{1}{\sqrt{1-\vartheta}}v_+>\frac{1}{\sqrt{2}}e,\forall 0<\vartheta<1.$$

Additionally

$$\sigma(\tilde{v})=\frac{1}{2}\left\|\frac{\tilde{v}-\tilde{v}^3}{2\tilde{v}^2-e}\right\|. \tag{23}$$

We have

$$2\tilde{v}^2-e=\frac{2v_+^2-(1-\vartheta)e}{1-\vartheta}, \tag{24}$$

also

$$\begin{aligned}\tilde{v}-\tilde{v}^3&=\tilde{v}(e-\tilde{v}^2)\\&=\frac{1}{\sqrt{1-\vartheta}}v_+\left(e-\frac{1}{1-\vartheta}v_+^2\right)\\&=\frac{1}{(1-\vartheta)\sqrt{1-\vartheta}}v_+((1-\vartheta)e-v_+^2).\end{aligned} \tag{25}$$

Then

$$\begin{aligned}\frac{\tilde{v}-\tilde{v}^3}{2\tilde{v}^2-e}&=\frac{1}{(1-\vartheta)\sqrt{1-\vartheta}}v_+((1-\vartheta)e-v_+^2)\cdot\frac{1-\vartheta}{2v_+^2-(1-\vartheta)e}\\&=\frac{1}{\sqrt{1-\vartheta}}\cdot\frac{v_+}{2v_+^2-(1-\vartheta)e}((1-\vartheta)e-v_+^2).\end{aligned} \tag{26}$$

Let $f(t)=\frac{t}{2t^2-(1-\vartheta)}$ for all $t>\frac{1}{\sqrt{2}}$. Since $f'(t)<0$, we conclude that f is decreasing for all $t>\frac{1}{\sqrt{2}}$. Moreover, considering Lemma 4.3 and (26), we get

$$\sigma(\tilde{v})=\frac{1}{2}\left\|\frac{\tilde{v}-\tilde{v}^3}{2\tilde{v}^2-e}\right\|\le\frac{\sqrt{1-\sigma^2}}{2\sqrt{1-\vartheta}(1-2\sigma^2+\vartheta)}\|(1-\vartheta)e-v_+^2\|. \tag{27}$$

From (20) we have

$$\|(1-\vartheta)e-v_+^2\|\le\|-\vartheta e\|+\|e-v_+^2\|<\vartheta\sqrt{n}+10\sigma^2. \tag{28}$$

Hence, by using (27) and (28) we obtain

$$\sigma(\tilde{v}) < \frac{\sqrt{1-\sigma^2}}{2\sqrt{1-\vartheta}(1-2\sigma^2+\vartheta)}(\vartheta\sqrt{n}+10\sigma^2),$$

with that, we conclude the first part of the proof.

Now assume that $\sigma < \frac{1}{10}$ and $\vartheta = \frac{1}{12\sqrt{n}}$. In addition, we have $\sqrt{1-\sigma^2} < 1$ and $\vartheta > 0$. Then, we get

$$\begin{aligned}\sigma(\tilde{v}) &< \frac{\vartheta\sqrt{n}+10\sigma^2}{2\sqrt{1-\vartheta}(1-2\sigma^2)} = \frac{\frac{1}{12}+10\sigma^2}{2\sqrt{1-\vartheta}(1-2\sigma^2)} \\ &< \frac{1}{2\sqrt{1-\vartheta}}\frac{50}{49}\left(\frac{1}{12}+\frac{1}{10}\right) = \frac{1}{2\sqrt{1-\vartheta}}\left(\frac{55}{294}\right).\end{aligned}$$

For $n \in \mathbb{N}^*$, we get $2\sqrt{1-\vartheta} = 2\sqrt{1-\frac{1}{12\sqrt{n}}} = \sqrt{4-\frac{1}{3\sqrt{n}}} \geq \sqrt{\frac{11}{3}}$. Taking all these inequalities into consideration, we conclude

$$\sigma(\tilde{v}) < \sqrt{\frac{3}{11}}\left(\frac{55}{294}\right) = \frac{5\sqrt{33}}{294} < \frac{1}{10}$$

With that, we have completed the proof.

Lemma 4.7: *Assume* (x^0, w^0) *a strictly feasible point with* $\sigma(x^0w^0, \eta^0) < \frac{1}{\sqrt{2}}$*, and* $\eta^0 = \frac{(x^0)^T w^0}{n}$*. Additionally,* (x^k, w^k) *is the obtained pair vectors after k iterations. So* $(x^k)^T w^k < \varepsilon$ *holds when*

$$k \geq \left\lceil \frac{1}{\vartheta}\log\left(\frac{\eta^0\left(n+\frac{9}{2}\right)}{\varepsilon}\right)\right\rceil.$$

Proof: After k iterations, $\eta^k = (1-\vartheta)^k\eta^0$. From Lemma 4.5 and $\sigma(xw,\eta) < \frac{1}{\sqrt{2}}$, it becomes

$$(x^k)^T w^k < \eta^k\left(n+\frac{9}{2}\right) = (1-\vartheta)^k\eta^0\left(n+\frac{9}{2}\right).$$

Hence, $(x^k)^T w^k < \varepsilon$ is satisfied if

$$\eta^0\left(n+\frac{9}{2}\right)(1-\vartheta)^k \leq \varepsilon.$$

By introducing logarithms of both sides, and take into consideration $\vartheta \leq -\log(1-\vartheta)$, the above inequality holds when

$$k\vartheta \geq \log\left(\eta^0\left(n+\frac{9}{2}\right)\right) - \log(\varepsilon) = \log\left(\frac{\eta^0\left(n+\frac{9}{2}\right)}{\varepsilon}\right).$$

Hence,

$$k \geq \left[\frac{1}{\vartheta} \log\left(\frac{\eta^0 \left(n + \frac{9}{2} \right)}{\varepsilon} \right) \right].$$

With that, we have completed the proof.

Theorem 4.1: *Suppose that* $x^0 = w^0 = e$ *and consider the default values of* ϑ *and* τ, *the algorithm presented in Figure requires at most*

$$O\left(\sqrt{n} \log \frac{\left(n + \frac{9}{2} \right)}{\varepsilon} \right)$$

interior-point iterations. Furthermore $(x^k)^T w^k < \varepsilon$.

Proof: The case $x^0 = w^0 = e$ gives $\eta^0 = \frac{(x^0)^T w^0}{n} = 1$. Substitute this value into Lemma , the result remains valid.

5. Numerical experiments

This section deals with numerical experiments of our algorithm for solving some CQP problems. We designate by x^* the optimal solution of (CQP), (λ^*, w^*) the optimal solution of (CQD), $T(s)$ the execution time required to achieve optimality in second and $Iter$ the number of iterations This implementation is manipulated using Matlab. The chosen accuracy parameter is $\varepsilon = 10^{-4}$.

5.1 *Examples with fixed size*

The examples are taken from the literature [7]. We take $\vartheta \in \{0.2,\ 0.5,\ 0.7,\ 0.9\}$ and $\eta^0 \in \{\frac{(x^0)^T w^0}{n},\ 0.5, 0.1\}$.

Example 5.1: $M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, r = \begin{pmatrix} -4 \\ -6 \\ 0 \\ 0 \end{pmatrix}, G = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

The starting points are:

$$x^0 = \begin{pmatrix} 0.9683 \\ 0.5775 \\ 0.4543 \\ 1.1444 \end{pmatrix}, \lambda^0 = \begin{pmatrix} -0.9184 \\ -1.1244 \end{pmatrix}, w^0 = \begin{pmatrix} 0.7612 \\ 0.9141 \\ 0.9185 \\ 1.1244 \end{pmatrix}.$$

The obtained primal-dual optimal solutions are:

$$x^* = \begin{pmatrix} 1.1290 \\ 0.7742 \\ 0.0969 \\ 0.0000 \end{pmatrix}, \lambda^* = \begin{pmatrix} -0.0001 \\ -1.0321 \end{pmatrix}, w^* = \begin{pmatrix} 0.0000 \\ 0.0000 \\ 0.0002 \\ 1.0321 \end{pmatrix}.$$

For different values of ϑ and η^0, we summarize the obtained results of Example 5.1 in Tables 1, 2 and 3.

Table 1

Results of Example 5.1 for $\eta^0 = \frac{(x^0)^T w^0}{n} = 0.7423$

ϑ	0.2	0.5	0.7	0.9
Iter	47	15	9	5
$T(s)$	0.014892	0.017392	0.008403	0.033754

Table 2

Results of Example 5.1 for $\eta^0 = 0.5$

ϑ	0.2	0.5	0.7	0.9
Iter	45	15	9	5
$T(s)$	0.014564	0.008402	0.008468	0.006912

Table 3

Results of Example 5.1 for $\eta^0 = 0.1$

ϑ	0.2	0.5	0.7	0.9
Iter	38	12	7	4
$T(s)$	0.013937	0.010756	0.023374	0.006627

Example 5.2: $M = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix}, b = \begin{pmatrix} 9.31 \\ 5.45 \\ 6.60 \end{pmatrix}, r = \begin{pmatrix} 1 \\ -1.5 \\ 2 \\ 1.5 \\ 3 \end{pmatrix},$

$$G = \begin{pmatrix} 20 & 1.2 & 0.5 & 0.5 & -1 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix}.$$

The starting points are:

$$x^0 = \begin{pmatrix} 2.4539 \\ 0.7875 \\ 1.5838 \\ 2.4038 \\ 1.3074 \end{pmatrix}, \lambda^0 = \begin{pmatrix} 20.5435 \\ 9.4781 \\ 4.3927 \end{pmatrix}, w^0 = \begin{pmatrix} 7.1215 \\ 7.9763 \\ 8.3150 \\ 6.8686 \\ 7.9750 \end{pmatrix}.$$

The obtained primal-dual optimal solutions are:

$$x^* = \begin{pmatrix} 2.6321 \\ 0.7019 \\ 1.3994 \\ 2.4643 \\ 1.0846 \end{pmatrix}, \lambda^* = \begin{pmatrix} 25.2686 \\ 11.7725 \\ 5.2567 \end{pmatrix}, w^* = 10^{-4} \times \begin{pmatrix} 0.0759 \\ 0.2846 \\ 0.1427 \\ 0.0811 \\ 0.1842 \end{pmatrix}.$$

For different values of ϑ and η^0, we summarize the obtained results of Example 5.2 in Tables 4, 5 and 6

Table 4

Results of Example 5.2 for $\eta^0 = \frac{(x^0)^T w^0}{n} = 12.7727$

ϑ	0.2	0.5	0.7	0.9
Iter	61	23	20	20
$T(s)$	0.021739	0.011469	0.009939	0.009890

Table 5

Results of Example 5.2 for $\eta^0 = 0.5$

ϑ	0.2	0.5	0.7	0.9
Iter	46	20	20	20
$T(s)$	0.014880	0.010996	0.010186	0.010035

Table 6

Results of Example 5.2 for $\eta^0 = 0.1$

ϑ	0.2	0.5	0.7	0.9
Iter	39	20	20	20
$T(s)$	0.013760	0.023863	0.010266	0.009855

Comment 1: According to the numerical experiments, we observe that the iterations number and the time required for achieving optimality with the algorithm are influenced by the values of the parameters ϑ and η^0. Surprisingly, it is observed that $\vartheta = 0.9$ and $\eta^0 = 0.1$ offer the smallest iterations number with minimal time.

Now, let's provide an example with a variable size to demonstrate the efficiency of our obtained algorithm when dealing with problems of large sizes.

5.2 *Example with variable size*

In this example, we take the best choice of ϑ and η^0, namely $\vartheta = 0.9$ and $\eta^0 = 0.1$. We note by Z^* the optimal value of (CQP).

Example 5.3: Assume that $n = 2m$, $M[i,j] = \begin{cases} 1 & \text{if} i=j \text{ or } i+m=j \\ 0 & \text{else} \end{cases}$,

$$b[i] = 2 \quad \forall \ 1 \le i \le m, \ \ r[i] = \begin{cases} -1 & \forall 1 \le i \le m \\ 0 & \forall m+1 \le i \le n \end{cases}, \ \ G = I_{n \times n}.$$

The starting points are:

$$x^0[i] = x^0[i+m] = 1, \ \ w^0[i] = 1, w^0[i+m] = 2, \lambda^0[i] = -1 \ \ \forall \ \ 1 \le i \le m.$$

The obtained primal-dual optimal solutions are:

$$x^*[i] = \tfrac{3}{2}, x^*[i+m] = \tfrac{1}{2}, \ \ w^*[i] = w^*[i+m] = 0, \lambda^*[i] = \tfrac{1}{2} \ \ \forall \ \ 1 \le i \le m.$$

In Table 7, we provide a summary of the Example 5.3 results for various sizes of (m, n).

Table 7

Numerical results of Example 5.3

(m, n)	*Iter*	$T(s)$	Z^*
(10, 20)	19	0.075756	-2.5000
(25, 50)	20	0.083204	-6.2500
(50, 100)	21	0.225048	-12.5000
(100, 200)	22	1.635784	-25.0000
(250, 500)	24	22.911453	-62.5000
(500, 1000)	25	163.186343	-125.0000
(600, 1200)	25	287.015770	-150.0000
(750, 1500)	25	526.791898	-187.5000

Comment 2: The numerical experiments demonstrate the efficiency of the algorithm that we have proposed. This efficiency is measured by the small iterations number and the reduced time to obtain the optimal primal-dual solution (x^*, λ^*, w^*). Indeed, the number of iterations recorded in all the examples tested is clearly lower than the theoretical number.

6. Conclusion

We introduced a novel primal-dual interior point method for solving convex quadratic programming problem. Our approach is a generalization of [10] for linear optimization (LO). We demonstrated that the resulting algorithm necessitates polynomial time to solve the considered problem. Additionally, we established some numerical experiments, which were acceptable and encouraging. Finally, we point out that implementing the algorithm with the update parameter ϑ significantly reduces the number of iterations required by our proposed algorithm and leads it to reach their real numerical performances. The obtained results consolidate and confirm our theoretical purpose.

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