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Edge reflexive irregularity strength of m^k -graph of path graph

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Abstract

In graph theory we define the total labeling, such that the edge labels and vertex labels are positive integers and even integers respectively and for different edges we have distinct weights, whereas the weight of an edge is the sum of labels of that edge and adjacent vertices, then labeling is referred as edge irregular reflexive total labeling. In this paper we calculated the exact values of reflexive edge irregularity strength for m^k-graph of path graph mP_n for $k = 1$ with $n \geq 3$, $m \geq 4$.

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1. Introduction

A graph give us a simple way to analyze the particular problem. Graph analysis can be used to find systemic hazards, catch fraud, and

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improve financial procedures. A mathematical issue served as the inspiration for graph theory. The mathematical structure of networks, which has edges connecting their nodes, vertices, or connections, is studied using graph theory. The hub of graph theory launched with the problem of Koinsber Bridge in 1735. The problem cause to the perception of Eulerian graph. Euler used the dots and lines to give the idea of vertices and edges respectively. Graphs are used to manage airline networks, to allocate channels to a television station.

A collection of elements in $V(G)$ and $E(G)$ make up a graph G. Each edge in edge set is connected to either one or more vertices of vertex set. The size and order are terms used to describe the number of edges and vertices respectively. Ayache and Alameri in [4] established an interesting graph, suppose there is a connected and simple graph $G(V, E)$. For $k = 1$ we can drive mG graph with by taking similar m-copies of graph G, for $k = 2$ we can drive $m(mG) = m^2(G)$ graph by taking similar m-copies of graph mG and for $k = 3$ we can drive $m(m(mG)) = m^3(G)$ graph by taking similar m-copies of graph $m(mG)$ and so on with $m \geq 2$. Then join each vertex of each copy with an edge to the corresponding vertices of all other copies. After that we get a new graph known as m^k-graph and is symbolized by $m^k(G)$.

Let q is the size and p is the order of simple graph G, then the order and size of m^k-graph of G for $k \geq 1$ and $m \geq 2$ are:

1. $|V(m^k(G)| = m^k p$
2. $|E(m^k(G)| = m^k q + \left(\frac{m(m-1)}{2}\right) pkm^{k-1}$

Labeling in graph theory is used to make the graph simpler to monitor the data flow and identify any illegal access or efforts to manipulate the network. The labeling can be used to assign distinct labels to each communication channel. The labeling can also be found in a variety of industries, such as computer science, chemistry and telecommunication. Labels on the edges represent the distances between vertices or nodes or the cost of transferring data and are used to describe network routing issues in computer technology. The first concept of graph labeling established by Rosa [15] in 1966. Golomb [8] stated in 1972, the same concept of graph labeling as graceful labeling, the numbers are allocated only to vertices, then it is known as vertex labeling. If the numbers are allocated only to edges, then it is known as edge labeling and if numbers are allocated to both, then it is known as total labeling.

Chartrand et al. [7] stated as the labeling is irregular, with distinct weights of vertices and/or edges and the minimum largest label of graph G is the graph strength $s(G)$. Irregular labeling is further divided into irregular edge labeling and irregular vertex labeling. The labeling $h : V(G) \to \{1,2,3,\cdots,k\}$ of a simple graph G is referred as an irregular edge labeling, if the weight $wt_h(v_i v_j) = h(v_i) + h(v_j)$ of edge $v_i v_j \in E(G)$ with $(i \neq j)$ are distinct and the labeling $h : E(G) \to \{1,2,3,\cdots,k\}$ is referred as an irregular vertex labeling, if the distinct weight of a vertex v_i i.e. $wt_h(v_i) = \sum h(v_i v_j)$.

Miškuf et al. in [12] stated a labeling defined as irregular vertex total labeling and irregular edge total labeling. Marzuki et al. put forward in [13] a new labeling named as totally irregular labeling. Baa et al. in [6] introduced the concept of edge irregular total k-labeling, with vertex and edge labeling of graph J, $h : V \cup E \to \{1,2,3,\cdots,k\}$, such that the total edge weights $wt_h(v_i v_j) = h(v_i) + h(v_i v_j) + h(v_j)$ are distinct for all edges $v_i v_j$, $v_i' v_j' \in E(G)$ with $v_i v_j \neq v_i' v_j'$. The shortest value of k is the strength of total edge irregular labeling and is symbolized by $tes(G)$. Rosyida et al. study the irregularity strength of general uniform cactus chain graphs [16], Imran et al., found the total vertex irregularity strength of generalized prism graphs [9]. Some known results can be found in [1, 5, 10, 11].

In the light of work in [3] and [2], Tanna et al. in [14] stated an edge irregular reflexive total k-labeling, with vertex and edge labeling

$$h_v : V(G) \to \{0,2,4,\cdots,2q_v\}$$

$$h_e : E(G) \to \{1,2,3,\cdots,q_e\}$$

Where $k = max\{q_e, 2q_v\}$, and for every two distinct edges $v_i v_j$ and $v_i' v_j'$ of graph G, such that the weights of edges are

$$wt_h(v_i v_j) = h_v(v_i) + h_e(v_i v_j) + h_v(v_j) \neq wt_h(v_i' v_j') = h_v(v_i') + h_e(v_i' v_j') + h_v(v_j')$$

The smallest k is the edge reflexive irregularity strength already proved as lemma in [14] and is symbolized by $res(G)$.

Lemma 1: *For each and every graph G,*

$$res(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & \text{if } |E(G)| \not\equiv 2,3 (\bmod 6) \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & \text{if } |E(G)| \equiv 2,3 (\bmod 6) \end{cases}$$

The bound of strength of edge reflexive total labeling of graph G, where the edge set $|E(G)|$ is the size of every graph G.

2. Problem formulation

Suppose we have a simple and smallest path graph P_3 and for the formulation of required graph. we will take m-copies of P_3 graph when $k=1$ and join vertices of each copy of path graph with an edge to the corresponding vertices of all other copies of path graph to get the result of $m^1 P_3 = mP_3$ and similarly when $k=2$ we will further take m-copies of mP_3 graph and join vertices of each copy of mP_3 graph with an edge to the corresponding vertices of all other copies of mP_3 graph and get the result of $m^2 P_3$ and so on with $m \geq 4$. Let $m^k(P_n)$ is the m^k-graph of path graph P_n with $(n \geq 3)$, $(m \geq 2)$ and for a non-negative integer k, we can see the Table 1 in which we use the path graph with three vertices P_3 and $1 \leq k \leq 3$.

Table 1

$m^k(P_3)$ **for** $1 \leq k \leq 3$

k	$m^k(P_3)$	Formulation of m^k-graph
1	mP_3	$m(P_3)$
2	$m^2(P_3)$	$m(m(P_3))$
3	$m^3(P_3)$	$m(m(m(P_3)))$

In this paper, we are not allow to use $m=2$ and $m=3$ when $k=1$ because If we take $m=2$ in the m^k-graph of path graph P_n with $k=1$, then we get the Cartesian product of path graph with 2 vertices P_2 and path graph with n vertices P_n i.e. $2P_n = P_2 \times P_n$. If we take $m=3$ in the m^k-graph of path graph P_n with $k=1$, then we get the Cartesian product of cycle graph with 3 vertices C_3 and path graph with n vertices P_n i.e. $3P_n = C_3 \times P_n$ and couple of researchers have already worked on it.

The graph mP_n with $(m \geq 4)$ is formed by vertex set $V(mP_n) = \{a_i^j : 1 \leq j \leq m;\ 1 \leq i \leq n\}$ and edge set $E(mP_n) = \{a_i^j a_{i+1}^j : 1 \leq i \leq n-1;\ 1 \leq j \leq m\} \cup \{a_i^t a_i^{j+t} : 1 \leq j \leq m-t;\ 1 \leq i \leq n;\ 1 \leq t \leq m-1\}$, see Figure 1. The order and size of this graph are:

1. $|V(mP_n)| = mn$
2. $|E(mP_n)| = \frac{m(nm+m-2)}{2}$

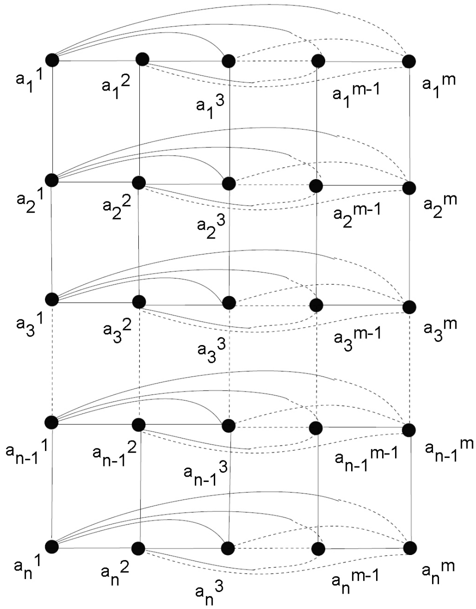

Figure 1

m^k -Graph of Path Graph mP_n with $k=1$

The bounds of the strength of edge irregular reflexive total labeling of m^k-graph of P_n graph symbolized as $res(mP_n)$ can be got through Theorem 1.

3. Calculations of the Results

Theorem 1: *Let mP_n be the m^k-graph with $k=1$ of the path graph, Then for $n \geq 3,\ m \geq 4,$*

$$res(mP_n) = \begin{cases} \frac{mn(m+1)}{6} - 2\left\lceil \frac{m-5}{6} \right\rceil, & \text{if } m \equiv 0,3,8,11 (\text{mod}\,12) \\ (2m-6)n + 2\left\lceil \frac{(m^2-11m+36)n-2m}{12} \right\rceil, & \text{if } m \equiv 1,10 (\text{mod}12) \\ (2m-6)n + 2\left\lceil \frac{(m^2-11m+36)n}{12} - \frac{1}{2}\left\lceil \frac{m-2}{3} \right\rceil \right\rceil, & \text{if } m \equiv 2,5,6,9 (\text{mod}12) \\ (2m-6)n + 2\left\lceil \frac{(m^2-11m+36)n}{12} - \frac{1}{3}\left\lceil \frac{m-1}{2} \right\rceil \right\rceil, & \text{if } m \equiv 4,7 (\text{mod}12) \end{cases}$$

Proof: Let mP_n be the m^k-graph with $k=1$ of the path graph, then for $n \geq 3,\ m \geq 4,$ then the lower bounds of reflexive edge strength is given by Lemma 1 are

$$res(mP_n) \geq \begin{cases} \left\lceil \frac{m(nm+n-2)}{6} \right\rceil, & \text{if } |E(mP_n)| \not\equiv 2,3 \pmod 6 \\ \left\lceil \frac{m(nm+n-2)}{6} \right\rceil + 1, & \text{if } |E(mP_n)| \not\equiv 2,3 \pmod 6 \end{cases}$$

Vertex Labeling

The process of assigning labels or values to a graph's vertices (or nodes) is known as "vertex labeling." In graph theory, a graph is made up of vertices and the edges connect them. we define labeling for vertices of mP_n graph

$$V(mP_n) = \{a_i^j : 1 \leq j \leq m; 1 \leq i \leq n\}$$

and we can see that vertex labeling depends only on $\{i : 1 \leq i \leq n\}$ and $m \geq 4$ because all the vertices $\{a_i^1, a_i^2, \cdots, a_i^{m-1}, a_i^m\}$ have same vertex labels and we define vertex labeling h_v such that

$$h_v(a_i^m) = \begin{cases} 0, & \text{if } i = 1,\ m \geq 4 \\ \frac{m^2-1}{4}, & \text{if } i = 2,\ m \equiv 1,3 \pmod 4 \\ \frac{m^2}{4}, & \text{if } i = 2,\ m \equiv 0 \pmod 4 \\ \frac{m^2}{4} - 1, & \text{if } i = 2,\ m \equiv 2 \pmod 4 \end{cases}$$

For $(i \geq 3)$

$$h_v(a_i^m) = \begin{cases} \frac{mi(m+1)}{6} - 2\left\lceil \frac{m-5}{6} \right\rceil, & \text{if } m \equiv 0,3,8,11 \pmod{12} \\ (2m-6)i + 2\left\lceil \frac{(m^2-11m+36)i-2m}{12} \right\rceil, & \text{if } m \equiv 1,10 \pmod{12} \\ (2m-6)i + 2\left\lceil \frac{(m^2-11m+36)i}{12} - \frac{1}{2}\left\lceil \frac{m-2}{3} \right\rceil \right\rceil, & \text{if } m \equiv 2,5,6,9 \pmod{12} \\ (2m-6)i + 2\left\lceil \frac{(m^2-11m+36)i}{12} - \frac{1}{3}\left\lceil \frac{m-1}{2} \right\rceil \right\rceil, & \text{if } m \equiv 4,7 \pmod{12} \end{cases}$$

Edge Labeling

The process of assigning labels or values to a graph's edges (or internodes) is known as "edge labeling." In graph theory, a graph is made up of edges and the vertices that are connected by edges. we define labeling for edges of mP_n graph

$$E(mP_n) = \{a_i^j a_{i+1}^j : 1 \le i \le n-1; 1 \le j \le m\} \cup \{a_i^t a_i^{j+t} : 1 \le i \le n; 1 \le t \le m-1; 1 \le j \le m-t\}$$

and we generalized the edge labeling for the edges $\{a_i^t a_i^{j+t} : 1 \le i \le n;$ $1 \le t \le m-1;\ 1 \le j \le m-t\}$ and using

$$q(t) = \frac{(2m+1)t - t^2 - 2m}{2}$$

For $1 \le i \le n; 1 \le j \le m-t; 1 \le t \le m-1$ we define edge labeling h_e such that

$$h_e(a_i^t a_i^{j+t}) = \begin{cases}
j + q(t), & \text{if } i = 1,\ m \ge 4 \\
j + \left\lceil \frac{m}{2} \right\rceil + q(t), & \text{if } m \equiv 2, 6, 10 (\text{mod } 12) \text{ then } i = 2 \\
j + \frac{m+4}{2} + q(t), & \text{if } m \equiv 2, 6, 10 (\text{mod } 12) \text{ then } i = 2 \\
j + \frac{(m^2+m)i - 3m^2 + m - 16}{6} + q(t), & \text{if } m \equiv 4 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 3) \\
 & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 6) \\
 & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 3 (\text{mod } 6) \\
j + \frac{(m^2+m)i - 3m^2 + m}{6} + q(t), & \text{if } m \equiv 0 (\text{mod } 12) \text{ then } i \ge 3 \\
 & \text{if } m \equiv 4 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 3) \\
 & \text{if } m \equiv 9 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 2) \\
 & \text{if } m \equiv 6 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 2) \\
 & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 4 (\text{mod } 6) \\
 & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 6) \\
j + \frac{(m^2+m)i - 3m^2 + m - 8}{6} + q(t), & \text{if } m \equiv 8 (\text{mod } 12) \text{ then } i \ge 3 \\
 & \text{if } m \equiv 4 (\text{mod } 12) \text{ then } i \equiv 2 (\text{mod } 3) \\
 & \text{if } m \equiv 5 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 2) \\
 & \text{if } m \equiv 2 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 2) \\
 & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 2 (\text{mod } 6) \\
 & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 5 (\text{mod } 6) \\
j + \frac{(m^2+m)i - 3m^2 + m - 4}{6} + q(t), & \text{if } m \equiv 7 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 3) \\
 & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 3 (\text{mod } 6) \\
 & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 6) \\
j + \frac{(m^2+m)i - 3m^2 + m - 12}{6} + q(t), & \text{if } m \equiv 3 (\text{mod } 12) \text{ then } i \ge 3 \\
 & \text{if } m \equiv 7 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 3) \\
 & \text{if } m \equiv 6 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 2) \\
 & \text{if } m \equiv 9 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 2) \\
 & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 6) \\
 & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 4 (\text{mod } 6) \\
j + \frac{(m^2+m)i - 3m^2 + m - 20}{6} + q(t), & \text{if } m \equiv 11 (\text{mod } 12) \text{ then } i \ge 3 \\
 & \text{if } m \equiv 7 (\text{mod } 12) \text{ then } i \equiv 2 (\text{mod } 3) \\
 & \text{if } m \equiv 2 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 2) \\
 & \text{if } m \equiv 5 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 2) \\
 & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 5 (\text{mod } 6) \\
 & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 2 (\text{mod } 6)
\end{cases}$$

And for the edges $\{a_i^j a_{i+1}^j : 1 \le i \le n-1; 1 \le j \le m\}$ we define edge labeling h_e such that

$$h_e(a_i^j a_{i+1}^j) = \begin{cases} j + \left\lceil \frac{m^2-2m}{4} \right\rceil, & \text{if } m \not\equiv 2,6,10 (\text{mod } 12) \text{ then } i=1 \\ j + \frac{m^2-2m+4}{4}, & \text{if } m \equiv 2,6,10 (\text{mod } 12) \text{ then } i=1 \\ j + \left\lceil \frac{3m^2-2m-8}{12} \right\rceil, & \text{if } m \not\equiv 1,3,4,10,11 (\text{mod } 12) \text{ then } i=2 \\ j + \left\lceil \frac{3m^2-2m-17}{12} \right\rceil, & \text{if } m \equiv 1,3,4,11 (\text{mod } 12) \text{ then } i=2 \\ j + \frac{3m^2-2m+8}{12}, & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i=2 \\ j + \frac{(m^2+m)i-m^2-3m-8}{6}, & \text{if } m \equiv 8 (\text{mod } 12) \text{ then } i \ge 3 \\ & \text{if } m \equiv 4,7 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 3) \\ j + \frac{(m^2+m)i-m^2-3m}{6}, & \text{if } m \equiv 0 (\text{mod } 12) \text{ then } i \ge 3 \\ j + \frac{(m^2+m)i-m^2-3m-4}{6}, & \text{if } m \equiv 4,7 (\text{mod } 12) \text{ then } i \equiv 1 (\text{mod } 3) \\ j + \frac{(m^2+m)i-m^2-3m-12}{6}, & \text{if } m \equiv 3 (\text{mod } 12) \text{ then } i \ge 3 \\ & \text{if } m \equiv 4,7 (\text{mod } 12) \text{ then } i \equiv 2 (\text{mod } 3) \end{cases}$$

$$h_e(a_i^j a_{i+1}^j) = \begin{cases} j + \frac{(m^2+m)i-m^2-3m-20}{6}, & \text{if } m \equiv 11 (\text{mod } 12) \text{ then } i \ge 3 \\ j + \frac{(m^2+m)i-m^2-3m-14}{6}, & \text{if } m \equiv 2,5 (\text{mod } 12) \text{ then } i \ge 3 \\ & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 6) \\ & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 3 (\text{mod } 6) \\ j + \frac{(m^2+m)i-m^2-3m-6}{6}, & \text{if } m \equiv 6,9 (\text{mod } 12) \text{ then } i \ge 3 \\ & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 2 (\text{mod } 6) \\ & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 5 (\text{mod } 6) \\ j + \frac{(m^2+m)i-m^2-3m-2}{6}, & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 3 (\text{mod } 6) \\ & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 0 (\text{mod } 6) \\ j + \frac{(m^2+m)i-m^2-3m-10}{6}, & \text{if } m \equiv 1,10 (\text{mod } 12) \text{ then } i \equiv 1,4 (\text{mod } 6) \\ j + \frac{(m^2+m)i-m^2-3m-18}{6}, & \text{if } m \equiv 10 (\text{mod } 12) \text{ then } i \equiv 5 (\text{mod } 6) \\ & \text{if } m \equiv 1 (\text{mod } 12) \text{ then } i \equiv 2 (\text{mod } 6) \end{cases}$$

Edge Weights

The edge weights are typically non-negative integers. Graphs that are weighted can be either directed or undirected. We find the edge weights of mP_n graph by adding labels of the two vertices and included edge. we define the weights wt_h of all edges for mP_n graph such that

For $\{a_i^t a_i^{j+t} : 1 \le i \le n; 1 \le j \le m-t; 1 \le t \le m-1\}$ we have

$$wt_h(a_i^t a_i^{j+t}) = j + \frac{(m^2+m)i-(m^2+m)}{2} + \frac{(2m+1)t-t^2-2m}{2}$$

And for $\{a_i^j a_{i+1}^j : 1 \le j \le m; 1 \le i \le n-1\}$ we have

$$wt_h(a_i^j a_{i+1}^j) = j + \frac{(m^2+m)i-2m}{2}$$

It is simple to check the different and consecutive edge weights $\{1, 2, 3, \cdots\cdots, \frac{m(nm+m-2)}{2}\}$.

For the analytical and graphical verification of the strength of mP_n graph i.e. $res(mP_n)$.

$$res(mP_n) = \begin{cases} \frac{mn(m+1)}{6} - 2\left\lceil \frac{m-5}{6} \right\rceil, & \text{if } m \equiv 0,3,8,11 (\text{mod} 12) \\ (2m-6)n + 2\left\lceil \frac{(m^2-11m+36)n-2m}{12} \right\rceil, & \text{if } m \equiv 1,10 (\text{mod} 12) \\ (2m-6)n + 2\left\lceil \frac{(m^2-11m+36)n}{12} - \frac{1}{2}\left\lceil \frac{m-2}{3} \right\rceil \right\rceil, & \text{if } m \equiv 2,5,6,9 (\text{mod} 12) \\ (2m-6)n + 2\left\lceil \frac{(m^2-11m+36)n}{12} - \frac{1}{3}\left\lceil \frac{m-1}{2} \right\rceil \right\rceil, & \text{if } m \equiv 4,7 (\text{mod} 12) \end{cases}$$

Now we define vertex and edge labeling in the different cases of mP_n -graph with $m \ge 4$ for every positive integer n, $n \ge 3$

Case-1 (when $m = 4$ for $4P_n$ graph):

In this case, we use 4-copies of path graph P_n with $n \ge 3$. Following are the vertex labels, edge labels and edge weights respectively as follows:

Vertex Labeling

For the labeling h_v of vertices a_i^j for $4P_n$ graph such that:

$$a_i^j = \{a_1^1, a_1^2, a_1^3, a_1^4, a_2^1, a_2^2, a_2^3, a_2^4, , ..., a_n^1, a_n^2, a_n^3, a_n^4\}$$

$$h_v(a_i^j)=\begin{cases}0, & \text{if } i=1,\ 1\le j\le 4\\ 4, & \text{if } i=2,\ 1\le j\le 4\\ 2i+2\left\lceil\frac{2i-2}{3}\right\rceil, & \text{if } i\ge 3,\ 1\le j\le 4\end{cases}$$

Edge Labeling

For the labeling h_e of edges $a_i^t a_i^{j+t}$ and $a_i^j a_{i+1}^j$ for $4P_n$ graph such that:

$$a_i^t a_i^{j+t}=\{a_1^1a_1^2, a_1^1a_1^3, a_1^1a_1^4, a_1^2a_1^3, a_1^2a_1^4, a_1^3a_1^4, a_2^1a_2^2, a_2^1a_2^3, a_2^1a_2^4, a_2^2a_2^3, a_2^2a_2^4, a_2^3a_2^4, \ldots\ldots, a_n^1a_n^2, a_n^1a_n^3, a_n^1a_n^4, a_n^2a_n^3, a_n^2a_n^4, a_n^3a_n^4\}$$

$$a_i^j a_{i+1}^j=\{a_1^1a_2^1, a_1^2a_2^2, a_1^3a_2^3, a_1^4a_2^4, a_2^1a_3^1, a_2^2a_3^2, a_2^3a_3^3, a_2^4a_3^4, \ldots\ldots, a_{n-1}^1a_n^1, a_{n-1}^2a_n^2, a_{n-1}^3a_n^3, a_{n-1}^4a_n^4\}$$

with $1\le i\le n; 1\le j\le 4-t; 1\le t\le 3$ we have

$$h_e(a_i^t a_i^{j+t})=\begin{cases} j+\frac{9t-t^2-8}{2}, & \text{if } i=1\\ j+2+\frac{9t-t^2-8}{2}, & \text{if } i=2\\ j+\frac{10i-30}{3}+\frac{9t-t^2-8}{2}, & \text{if } i\equiv 0(\operatorname{mod} 3)\\ j+\frac{10i-22}{3}+\frac{9t-t^2-8}{2}, & \text{if } i\equiv 1(\operatorname{mod} 3)\\ j+\frac{10i-26}{3}+\frac{9t-t^2-8}{2}, & i\text{if } i\equiv 2(\operatorname{mod} 3)\end{cases}$$

And with $1\le i\le n-1; 1\le j\le 4$ we have

$$h_e(a_i^j a_{i+1}^j)=\begin{cases} j+2, & \text{if } i=1\\ j+2, & \text{if } i=2\\ j+\frac{10i-18}{3}, & \text{if } i\equiv 0(\operatorname{mod} 3)\\ j+\frac{10i-16}{3}, & \text{if } i\equiv 1(\operatorname{mod} 3)\\ j+\frac{10i-20}{3}, & \text{if } i\equiv 2(\operatorname{mod} 3)\end{cases}$$

Edge Weights

For the weights wt_h of edges $a_i^t a_i^{j+t}$ and $a_i^j a_{i+1}^j$ for $4P_n$ graph such that:

$$wt_h(a_i^t a_i^{j+t}) = \left\{ j+10i-10+\frac{9t-t^2-8}{2}, \text{if } 1 \le j \le 4-t;\ 1 \le t \le 3 \right.$$

$$wt_h(a_i^j a_{i+1}^j) = \{ j+10i-4, \text{ if } 1 \le i \le n-1;\ 1 \le j \le 4$$

It is simple to check the different and consecutive edge weights $\{1,2,3,\cdots\cdots,8n+4\}$.

So for the strength $res(mP_n)$ with $m=4$ we have

$$res(4P_n) = 2n + 2\left\lceil \frac{2n-2}{3} \right\rceil$$

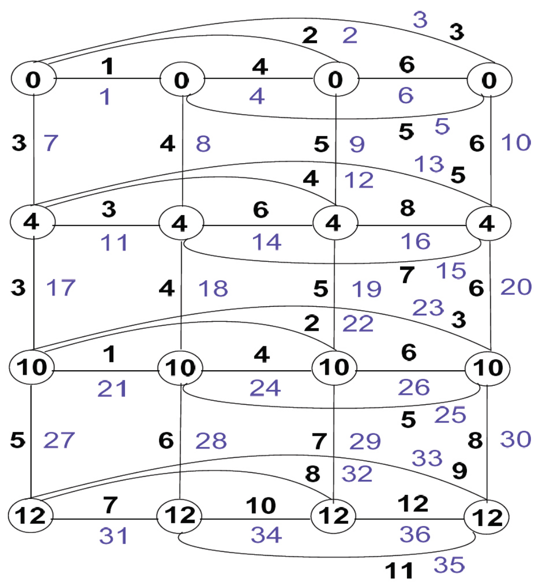

Figure 2

Edge reflexive irregular labeling of $4P_4$

Edge reflexive irregular labeling of 4P_4 is shown in Figure 2.

Case-2 (when $m=5$ for $5P_n$ graph):

In this case, we use 5-copies of path graph P_n with $n \ge 3$. Following are the vertex labels, edge labels and edge weights respectively as follows:

Vertex Labeling

For the labeling h_v of vertices a_i^j for $5P_n$ graph such that:

$$a_i^j = \{a_1^1, a_1^2, a_1^3, a_1^4, a_1^5, a_2^1, a_2^2, a_2^3, a_2^4, a_2^5, a_3^1, a_3^2, a_3^3, a_3^4, a_3^5 \ldots\ldots, a_n^1, a_n^2, a_n^3, a_n^4, a_n^5\}$$

we have

$$h_v(a_i^j) = \begin{cases} 0, & \text{if } i=1,\ 1 \le j \le 5 \\ 6, & \text{if } i=2,\ 1 \le j \le 5 \\ 4i + 2\left\lceil \frac{i-1}{2} \right\rceil, & \text{if } i \ge 3,\ 1 \le j \le 5 \end{cases}$$

Edge Labeling

For the labeling h_e of edges $a_i^t a_i^{j+t}$ and $a_i^j a_{i+1}^j$ for $5P_n$ graph such that:

$$a_i^t a_i^{j+t} = \{a_1^1 a_1^2, a_1^1 a_1^3, a_1^1 a_1^4, a_1^1 a_1^5, a_1^2 a_1^3, a_1^2 a_1^4, a_1^2 a_1^5, a_1^3 a_1^4, a_1^3 a_1^5, a_1^4 a_1^5, \ldots\ldots, a_n^1 a_n^2, a_n^1 a_n^3, \\ a_n^1 a_n^4, a_n^1 a_n^5, a_n^2 a_n^3, a_n^2 a_n^4, a_n^2 a_n^5, a_n^3 a_n^4, a_n^3 a_n^5, a_n^4 a_n^5\}$$

$$a_i^j a_{i+1}^j = \{a_1^1 a_2^1, a_1^2 a_2^2, a_1^3 a_2^3, a_1^4 a_2^4, a_1^5 a_2^5, a_2^1 a_3^1, a_2^2 a_3^2, a_2^3 a_3^3, a_2^4 a_3^4, a_2^5 a_3^5, \ldots\ldots, a_{n-1}^1 a_n^1, \\ a_{n-1}^2 a_n^2, a_{n-1}^3 a_n^3, a_{n-1}^4 a_n^4, a_{n-1}^5 a_n^5\}$$

with $1 \le i \le n; 1 \le j \le 5-t; 1 \le t \le 4$ we have

$$h_e(a_i^t a_i^{j+t}) = \begin{cases} j + \frac{11t - t^2 - 10}{2}, & \text{if } i=1 \\ j + 3 + \frac{11t - t^2 - 10}{2}, & \text{if } i=2 \\ j + 5i - 15 + \frac{11t - t^2 - 10}{2}, & \text{if } i \ge 4, i \equiv 0 (\text{mod} 2) \\ j + 5i - 13 + \frac{11t - t^2 - 10}{2}, & \text{if } i \equiv 1 (\text{mod} 2) \end{cases}$$

And with $1 \le i \le n-1; 1 \le j \le 5$ we have

$$h_e(a_i^j a_{i+1}^j) = \begin{cases} j+4, & \text{if } i=1 \\ j+5, & \text{if } i=2 \\ j+5i-9, & \text{if } i \ge 3 \end{cases}$$

Edge Weights

For the weights wt_h of edges $a_i^t a_i^{j+t}$ and $a_i^j a_{i+1}^j$ for $5P_n$ graph such that:

$$wt_h(a_i^t a_i^{j+t}) = \Big\{ j + 15i - 15 + \frac{11t - t^2 - 10}{2}, \text{ if } 1 \le j \le 5-t;\ 1 \le t \le 4$$

$$wt_h(a_i^j a_{i+1}^j) = \{ j + 15i - 5, \text{if } 1 \le i \le n-1;\ 1 \le j \le 5$$

It is simple to check the different and consecutive edge weights $\{1,2,3,\cdots\cdots,\frac{5(5n+3)}{2}\}$.

So for the strength $res(mP_n)$ with $m=5$ we have

$$res(5P_n)=4n+2\lceil\tfrac{n-1}{2}\rceil$$

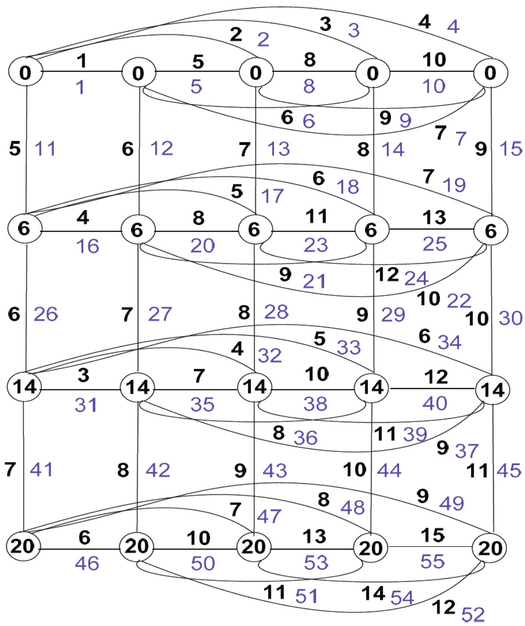

Figure 3

Edge reflexive irregular labeling of $5P_4$

Edge reflexive irregular labeling of 5P_5 is shown in Figure 3.

Case-3 (when m = 8 for $8P_n$ graph):

In this case, we use 8-copies of path graph P_n with $n\geq 3$. Following are the vertex labels, edge labels and edge weights respectively as follows:

Vertex Labeling

For the labeling h_v of vertices a_i^j for $8P_n$ graph such that:

For $1\leq i\leq n;1\leq j\leq 8$ we have

$$h_v(a_i^j)=\begin{cases}0, & \text{if } i=1,\ 1\leq j\leq 8\\ 16, & \text{if } i=2,\ 1\leq j\leq 8\\ 12i-2, & \text{if } i\geq 3,\ 1\leq j\leq 8\end{cases}$$

Edge Labeling

For the labeling h_e of edges $a_i^t a_i^{j+t}$ and $a_i^j a_{i+1}^j$ for $8P_n$ graph such that:

$$\begin{aligned} a_i^t a_i^{j+t} = \{ & a_1^1a_1^2, a_1^1a_1^3, a_1^1a_1^4, a_1^1a_1^5, a_1^1a_1^6, a_1^1a_1^7, a_1^1a_1^8, a_1^2a_1^3, a_1^2a_1^4, a_1^2a_1^5, a_1^2a_1^6, \\ & a_1^2a_1^7, a_1^2a_1^8, a_1^3a_1^4, a_1^3a_1^5, a_1^3a_1^6, a_1^3a_1^7, a_1^3a_1^8, a_1^4a_1^5, a_1^4a_1^6, a_1^4a_1^7, a_1^4a_1^8, \\ & a_1^5a_1^6, a_1^5a_1^7, a_1^5a_1^8, a_1^6a_1^7, a_1^6a_1^8, a_1^7a_1^8, \ldots\ldots, a_n^1a_n^2, a_n^1a_n^3, a_n^1a_n^4, a_n^1a_n^5, a_n^2a_n^3, \\ & a_n^2a_n^4, a_n^2a_n^5, a_n^3a_n^4, a_n^3a_n^5, a_n^4a_n^5\} \end{aligned}$$

$$\begin{aligned} a_i^j a_{i+1}^j = \{ & a_1^1a_2^1, a_1^2a_2^2, a_1^3a_2^3, a_1^4a_2^4, a_1^5a_2^5, a_1^6a_2^6, a_1^7a_2^7, a_1^8a_1^8, \ldots\ldots, a_{n-1}^1a_n^1, a_{n-1}^2a_n^2, \\ & a_{n-1}^3a_n^3, a_{n-1}^4a_n^4, a_{n-1}^5a_n^5, a_{n-1}^6a_n^6, a_{n-1}^7a_n^7, a_{n-1}^8a_n^8\} \end{aligned}$$

with $1 \le i \le n; 1 \le j \le 8-t; 1 \le t \le 7$ we have

$$h_e(a_i^t a_i^{j+t}) = \begin{cases} j + \frac{17t - t^2 - 16}{2}, & \text{if } i = 1 \\ j + 4 + \frac{17t - t^2 - 16}{2}, & \text{if } i = 2 \\ j + 12i - 32 + \frac{17t - t^2 - 16}{2}, & \text{if } i \ge 3 \end{cases}$$

And with $1 \le i \le n-1; 1 \le j \le 8$ we have

$$h_e(a_i^j a_{i+1}^j) = \begin{cases} j + 12, & \text{if } i = 1 \\ j + 14, & \text{if } i = 2 \\ j + 12i - 16, & \text{if } i \ge 3 \end{cases}$$

Edge Weights

For the weights wt_h of edges $a_i^t a_i^{j+t}$ and $a_i^j a_{i+1}^j$ for $8P_n$ graph such that:

$$wt_h(a_i^t a_i^{j+t}) = \left\{ j + 36i - 36 + \frac{17t - t^2 - 16}{2}, \text{if} 1 \le j \le 8 - t;\ 1 \le t \le 7 \right.$$

$$wt_h(a_i^j a_{i+1}^j) = \{ j + 36i - 8, \text{if} 1 \le i \le n - 1;\ 1 \le j \le 8$$

It is simple to check the different and consecutive edge weights $\{1, 2, 3, \cdots\cdots, 32n + 20\}$.

So for the strength $res(mP_n)$ with $m = 8$ we have

$$res(8P_n) = 12n - 2$$

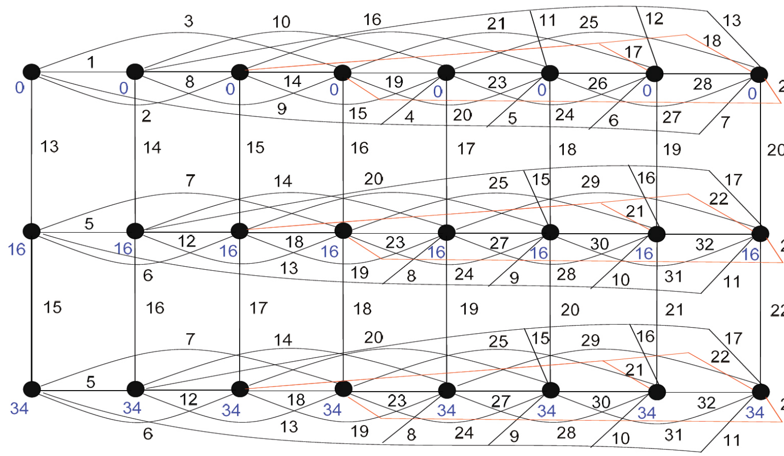

Figure 4

Edge reflexive irregular labeling of $8P_3$

Edge reflexive irregular labeling of 8P_3 is shown in Figure 4. So, h is the irregular reflexive labeling for different edge weights of m^k graph of Path graph P_n for $m \geq 4$, $k = 1$ and $n \geq 3$. Which completes the proof.

4. Conclusion

In this paper we have calculated graph labels, edge-weights and strength of graph. We have calculated the bounds of the edge reflexive irregularity strength $res(mP_n)$ of m^k-graph mP_n with $k = 1$ for path graph P_n with $n \geq 3$ and $m \geq 4$.

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