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A novel approach based on the Muntz-Legendre polynomials for solving a class of the generalized Abel integral equations

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Abstract

This work is about a suitable numerical method based on the Muntz-Legendre polynomials for solving a type of the generalized Abel integral equations. The propose method approximates the unknown function with the Muntz-Legendre polynomials. To do this, we apply these functions and using Muntz-Galerkin method for the numerical solutions of this problem. Also, the stability and uniqueness of the solution by this scheme will be shown. Moreover, the convergence of this algorithm will be proved by preparing some theorems. Several numerical examples are presented to show the superiority and efficiency of current method.

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Keywords: *Generalized abel integral equations, Muntz-Legendre polynomials, Muntz-Galerkin method, Unique solution.*

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1. Introduction

In last few decades, many researchers have taken much attentions to the integral equations. A lot of integral equations cannot be solved analytically, so, it is necessary to develop some numerical solution with significant accuracy to solve these types of problems [8, 19, 7, 22]. Generalized Abel integral equations which will be study in this work arise in many chemical applications and mathematical physics such as strereology, heat conduction, crystal growth and the radiation of heat from a semi-infinite solid [2, 6, 10, 13]. The most general form of this equation is as follows,

$$u(x) = f(x) + \lambda \int_0^x F(x,t,u(t))dt, \quad x \in [a,b], \tag{1.1}$$

where $u \in X = C([a,b],\mathbb{R})$ is unknown function and $F(x,t,u(t)) = \frac{K(x,t)U(t,u(t))}{(x^3-t^3)^\alpha}$; Also, $K(x,t)$ is the kernel of Eq. (1.1). Moreover, the following conditions are considered.

I) $f:[a,b] \to \mathbb{R}$ and $F: D \to \mathbb{R}$ are continuous, where $D = \{(x,t,u) \mid x \in [a,b],\ t \in [a,x),\ u \in X\}$,

II) $0 < \alpha < 1$, $a > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

In 1974, Atkinson proved a theorem for existence of solution of the Abel integral equations [2]. In 2011, Eshaghi et al, proved existence and uniqueness of solution for a nonlinear integral equation [6]. In 2014, Mokhtary and Goreishi used spectral Tau method for the numerical solution of Abel integral equation. Also, they showed stability and convergence properties of this scheme [10]. In 2015, Mokhtary et al used Tau method based on the Muntz-Legendre polynomials for solving linear multi-order fractional differential equations. [11]. In 2015, Sizikov and Sidorov used the generalized quadrature rule for solving singular integral equations of Abel type with application to infrared tomography [18]. In 2016, Mokhtary studied the Abel-Hammerstein integral equations by Muntz-Galerkin method based on the Muntz-Legendre polynomials. Also, in this work he showed the behavior of the equation, existence, uniqueness and stability of the method through some theorems [12]. In 2016, Shen and Wang developed the Muntz-Galerkin method for solving the Poisson equation with mixed Dirichlet Neumann boundary conditions [17]. In 2018, Zarei and Noeiaghdam solved generalized Abel's integral equations of the first and second kinds via Taylor-collocation method [23].

In 2019, Rao and Chakraverty presented a new scheme by using interval Legendre polynomials based on collocation method for solving uncertain differential equations in [14]. In 2020, Dixit presented an efficient algorithm based on the orthonormal Bernstein polynomial for numerical solution of a system of generalized Abel integral equations [5]. Moreover, there are several analytical methods such as Adomian decomposition, Laplace transform, successive approximations or Picard iteration which were utilized to obtain the exact solution of linear or nonlinear singular integral equations [21].

This work is organized as follows. In section 1 some necessary definitions and properties of the Muntz-Legerdre polynomials are explained. Section 2 is allocated to the existence and uniqueness of the solution for problem under study. Method of solution is presented in section 3. Section 4 is devoted to the study of convergence analysis of our algorithm. In section 5, several numerical examples are shown by utilizing this scheme. Finally, a brief conclusion is given in section 6.

Muntz-Legendre polynomials: For $m = 0,1,2,\ldots$ the Muntz-Legendre polynomials of order m are defined as follows

$$L_m(x) = \frac{1}{2\pi i}\int_I \prod_{\kappa=0}^{m-1} \frac{t+\zeta_\kappa+1}{t-\zeta_\kappa}\frac{x^t}{t-\zeta_m}dt, \tag{1.2}$$

where I contain all the zeros of denominator of the integrand.

The numbers $0=\zeta_0<\zeta_1<\zeta_2<\ldots<\zeta_m<\ldots$ are called Muntz numbers with conditions $\lim_{m\to+\infty}\zeta_m=+\infty$ and $\Sigma_{m=1}^{+\infty}\zeta_m^{-1}=+\infty.$.

These L_m s are orthogonal with the weight function $w(x)=1$, in other words, we have the following relation

$$\int_I L_i(x)L_j(x)dx = \frac{\delta_{i,j}}{2\zeta_i+1}, \quad i \ge j, \tag{1.3}$$

where $\delta_{i,j}$ is the delta Kronecker function.

Muntz space: The space generated by $\{x^{\zeta_0}, x^{\zeta_1}, x^{\zeta_2}, \ldots\}$ is called the Muntz space and the Muntz-Legendre polynomials can be constructed by the following recursive relations,

$$L_0(x) = 2\zeta_0+1, \quad x\in(0,1], \tag{1.4}$$

$$L_m(x) = L_{m-1}(x) - (\zeta_\kappa+\zeta_j+1)x^{\zeta_m}\int_x^1 t^{-\zeta_m-1}L_{m-1}(t)dt, \; m=1,2,\ldots, \; x\in(0,1]. \tag{1.5}$$

For example, if we assume $\zeta_i = \frac{i}{2}$ for all $n\in\mathbb{N}\cup\{0\}$, then the Muntz-Legendre polynomials can be written as

$$
\begin{aligned}
&L_0(x)=1,\\
&L_1(x)=-2+3x^{\frac{1}{2}},\\
&L_2(x)=3-12x^{\frac{1}{2}}+10x,\\
&L_3(x)=-4+30x^{\frac{1}{2}}-60x+35x^{\frac{3}{2}},\\
&L_4(x)=5-60x^{\frac{1}{2}}+210x-280x^{\frac{3}{2}}+126x^2,\\
&\vdots\\
&.
\end{aligned}
\tag{1.6}
$$

Another form of the Muntz-Legendre polynomials is [3]

$$L_m(x)=\Sigma_{\kappa=0}^{m}\rho_{\kappa,m}x^{\zeta_m}, \tag{1.7}$$

where,

$$\rho_{\kappa,m}(x)=\prod_{j=0}^{m-1}(\zeta_\kappa+\zeta_j+1)/\prod_{j=0,\ j\neq\kappa}^{m}(\zeta_\kappa-\zeta_j),\ \ m=0,1,2,.... \tag{1.8}$$

One can see more properties of the Muntz-Legendre polynomials in [20].

Also, a piece of natural numbers N_k for all $k\in\mathbb{N}$ is defined as:

$$N_k=\{1,2,3,...,k\}.$$

2. Existence and uniqueness of the solution

In this section, we discuss about the existence and uniqueness of solution for Eq. (1.1). Let $\{L_0,L_1,L_2,...,L_n\}$ be the Muntz-Legendre basis and $\Im_n=span\{L_0,L_1,L_2,...,L_n\}$.

Definition 2.1: [9] Let u is the exact solution and $v\in\Im_n$ be the best approximation for the problem (1.1), then we have

$$\|u-v\|\leq\|u-w\|,\ \ \forall w\in\Im_n. \tag{2.1}$$

Lemma 2.2: [4] *Let* $I=[0,T]$ *and* $z,f,K\in X$ *are functions such that* K *be nonnegative and satisfies the following relation*

$$z(x)\leq f(x)+\int_0^x K(s)z(s)ds,\ \ \forall x\in[a,b]. \tag{2.2}$$

Therefore, we have

$$z(x) \le f(x) + \int_0^x K(\tau) f(\tau).exp(\int_\tau^x K(\mu)d\mu)d\tau, \ \forall x \in [a,b]. \tag{2.3}$$

Also, if *f* is not strictly decreasing on $[a,b]$, one obtains

$$z(x) \le f(x).\exp(\int_\tau^x K(\mu)d\mu)d\tau, \ \forall x \in [a,b]. \tag{2.4}$$

Lemma 2.3: [15] *(Lebesgue dominated convergence theorem) Let's* $\{h_n\}_{n\in\mathbb{N}}$ *be a sequence of functions on* $\mathcal{I}$ *which are Lebesgue-integrable. Suppose that* $\lim_{n\to+\infty} h_n = h$ *on an interval* $\mathcal{I}$ *and h is Lebesgue-integrable. Also, suppose Lebesgue-integrable function k exists such that*

$$|h_n(x)| \le k(x), \quad \forall x \in \mathcal{I}, \ \forall n \in \mathbb{N}, \tag{2.5}$$

then

$$\int_{\mathcal{I}} h_n d\mu \to \int_{\mathcal{I}} h d\mu, \tag{2.6}$$

as $n \to +\infty$, where μ is Lebesgue measure.

Theorem 2.4: *(Existence and uniqueness): Suppose that F satisfies the following Lipschitz condition for all* $x,t \in [a,b]$, *and all* $u_1, u_2 \in X$

$$|F(x,t,u_1(t)) - F(x,t,u_2(t))| \le C_L |u_1(t) - u_2(t)|, \tag{2.7}$$

where $=\{(x,t,w):(x,t)\in[a,b]^2, w\in X$ and $|(w-f)(t)| \le A\}$, $A \in \mathbb{R}^{\ge 0}$ and C_L is Lipschitz coefficient. Then, there is a unique continuous solution for the Eq. (1.1) on $[a,b]$.

Proof: The Picard recursion sequence for the Eq. (1.1) has the form

$$u_{n+1}(x) = f(x) + \int_0^\tau F(x,\tau,u_n(\tau))d\tau, \ \forall n \in \mathbb{N} \cup \{0\}, \ \forall x \in [a,b], \tag{2.8}$$

with initial term $u_0 = f$. We prove that each u_n, satisfies the following inequality for all $n \in \mathbb{N} \cup \{0\}$

$$|(u_n - f)(t)| \le \mathcal{M}_A t \le A, \ \forall t \in [a,b], \tag{2.9}$$

in which $\mathcal{M}_A = max\{F(x,\tau,u):(x,\tau,u)\in\Omega\}$. Obviously, Eq. (2.8) holds for $n=0$. Suppose that (2.9) is true for $k \in \mathbb{N} \cup \{0\}$. Thus, for $n = k+1$, we get

$$|(u_{k+1} - f)(t)| \le \int_0^t |F(t,\tau,u_k(\tau))| d\tau \le \mathcal{M}_A t \le A, \ \forall t \in [a,b]. \tag{2.10}$$

Now, we prove that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $[a,b]$. To do this, we assume that $y_n = u_{n+1} - u_n$. Hence, inequality

$$|y_n(x)| \le \frac{\mathcal{M}_A C_L^n x^{n+1}}{(n+1)!},\ \forall x \in [0,b],\ \forall n = 0,1,2,\ldots, \tag{2.11}$$

is hold [4]. Define partial sum $S_n = \sum_{\ell=0}^{n} \frac{(C_L x)^\ell}{\ell!}$ on $\{0\} \cup \mathbb{N}$ for arbitrary x in $[a,b]$. So, $\lim_{n\to+\infty} S_n = e^{C_L x}$ based on the relation $\sum_{\ell=0}^{\infty} \frac{(C_L x)^\ell}{\ell!} = e^{C_L x}$. Since $\{S_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, for $\frac{C_L}{\mathcal{M}_A}\varepsilon > 0$,

$$\exists N \in \mathbb{N} \ni i,j > N \to |S_i - S_j| < \frac{C_L}{\mathcal{M}_A}\varepsilon. \tag{2.12}$$

Using the following relation

$$u_n(x) - u_m(x) = \sum_{\ell=m}^{n-1} (u_{\ell+1}(x) - u_\ell(x)) = \sum_{\ell=m}^{n-1} y_\ell(x), \tag{2.13}$$

and (2.12), results in

$$\begin{aligned} |u_n(x) - u_m(x)| \le \left|\sum_{\ell=m}^{n-1} y_\ell(x)\right| &\le \sum_{\ell=m}^{n-1} |y_\ell(x)| \\ &\le \mathcal{M}_A \sum_{\ell=m}^{n-1} \frac{C_L^\ell x^{\ell+1}}{(\ell+1)!} \\ &= \mathcal{M}_A C_L^{-1} \sum_{\ell=m+1}^{n} \frac{C_L^\ell x^\ell}{\ell!} \\ &= \mathcal{M}_A C_L^{-1} |S_n - S_m| \\ &< \mathcal{M}_A C_L^{-1} . \frac{C_L}{\mathcal{M}_A}\varepsilon = \varepsilon. \end{aligned} \tag{2.14}$$

Then, $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. So, u_n is convergent to a function u in $[a,b]$. This means that,

$$\lim_{n\to+\infty} u_n(x) = u(x). \tag{2.15}$$

According to the Lipschitz condition(2), we have

$$\begin{aligned} \left|\int_0^x (F(x,t,u_n(t)) - F(x,t,u(t)))dt\right| \le C_L \int_0^x |u_n(t) \\ -u(t)|\, dt \quad \forall x \in [a,b]\ \forall n = 0,1,2,\ldots. \end{aligned} \tag{2.16}$$

Using relation (2.15), one obtains $\lim_{n\to+\infty} \int_0^x |u_n(t) - u(t)|\, dt = 0,\ x \in [a,b]$. Therefore,

$$\lim_{n\to+\infty} \int_0^x F(x,t,u_n(t)) = F(x,t,u(t))dt. \tag{2.17}$$

Taking the limit from both sides of (2.8), results in

$$\begin{aligned} u(t) &= \lim_{n\to+\infty} u_{n+1}(x) \\ &= \lim_{n\to+\infty}(f(x)+\int_0^x F(x,t,u_n(t))dt) \\ &= f(x)+\lim_{n\to+\infty}\int_0^x F(x,t,u_n(t))dt. \end{aligned} \tag{2.18}$$

Using lemma (2.3), we get

$$\begin{aligned} u(x) &= f(x)+\int_0^x F(x,t,\lim_{n\to+\infty}u_n(t)) \\ &= f(x)+\int_0^x F(x,t,u(t))dt. \end{aligned} \tag{2.19}$$

Therefore, $u(x)$ is a solution for Eq. (1.1).

To show the uniqueness of solution for Eq. (1.1), suppose that $u_1(x)$ and $u_2(x)$ are two solutions of this equation, therefore,

$$\begin{aligned} |u_1(x)-u_2(x)| &\le \int_0^x |F(x,\tau,u_1(\tau))-F(x,\tau,u_2(\tau))|d\tau \\ &\le C_L\int_0^x |u_1(\tau)-u_2(\tau)|d\tau, \ \forall x\in[a,b]. \end{aligned} \tag{2.20}$$

Setting $y(x)=|u_1(x)-u_2(x)|$, $f(x)=0$ and $K(x)=C_L$ on $[a,b]$; also using Lemma (2.2), one obtains

$$|u_1(x)-u_2(x)|\le 0\exp(C_L x)=0, \ \forall x\in[a,b]. \tag{2.21}$$

Therefore, $\|u_1-u_2\|_\infty=0$. This implies that $u_1=u_2$ and the proof is completed.

Furthermore, Eq. (1.1) has solution $u(x)$ whose first derivative is not defined at $x=0$ and has behaviour with the following form [4]

$$u'(x)\simeq \frac{1}{(x^3)^\alpha}, \ 0<\alpha<1.$$

So, smoothness of Eq. (1.1) in compare with $\Im_n$ is concluded and is investigated for weakly singular integral equations [4]. In this case, convergence property of the method will be investigated in section 4.

3. Method of solution

Implementation the Galerkin method for solving the generalized Abel intagral equation can lead to some problems such as, high computational cost and low accuracy. In this work, we provide a well-posed upper triangular algebraic system with forward substitution. By using this technique, one is faced with easy computation and so a very low CPU times. The other problem to obtain numerical solution of the generalized Abel integral equations with desirable accuracy by Galerkin methods based on polynomials is in its singularities at some points in the considered domain.

To overcome on this difficulty we use the Galerkin method based on non-polynomial functions named Muntz-Legendre polynomials. These non-polynomials functions are orthogonal with respect to the weight function $w(x)=1$. In this part, we explain Muntz-Galerkin method based on Muntz-Legendre Polynomials for solving the generalized Abel integral equation. One can see the capability of this scheme in section 5.

Lemma 3.1: **S**uppose that $u_n(x)$ is the solution of Eq. (1.1), then we have

$$u_n^{\ell}(x)=\underline{u}LQ^{\ell-1}\underline{X}, \quad \forall x\in[a,b], \; \forall n\in\mathbb{N}, \tag{3.1}$$

where

$$\underline{u}=\begin{bmatrix} u_0 & u_1 & u_2 & \ldots & u_{2n} \end{bmatrix}, \tag{3.2}$$

and

$$\underline{X}=\begin{bmatrix} 1 & x^{\zeta_1} & x^{\zeta_2} & \ldots & x^{\zeta_{2n}} \end{bmatrix}^T, \tag{3.3}$$

in which $\{\zeta_i\}_{i\in\mathbb{N}}$ is a sequence of Muntz numbers,

$$Q=\begin{bmatrix} \underline{\underline{u_0}} & \underline{\underline{u_1}} & \underline{\underline{u_2}} & \cdots & \underline{\underline{u_n}} & & & & \\ & \underline{\underline{u_0}} & \underline{\underline{u_1}} & \cdots & \underline{\underline{u_{n-1}}} & \underline{\underline{u_n}} & & O & \\ & & \underline{\underline{u_0}} & \cdots & \underline{\underline{u_{n-2}}} & \underline{\underline{u_{n-1}}} & \underline{\underline{u_n}} & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & \underline{\underline{u_0}} & \underline{\underline{u_1}} & \underline{\underline{u_2}} & \cdots & \underline{\underline{u_n}} \\ & & & & & \underline{\underline{u_0}} & \underline{\underline{u_1}} & \cdots & \underline{\underline{u_{n-1}}} \\ & & O & & & & \underline{\underline{u_0}} & \cdots & \underline{\underline{u_{n-2}}} \\ & & & & & & & \ddots & \vdots \\ & & & & & & & & \underline{\underline{u_0}} \end{bmatrix}, \; \underline{\underline{u_i}}=\underline{u}\vec{L}_i, \; \forall i\in\{0,1,2,\ldots,n\}, \tag{3.4}$$

also, $\vec{L}_i=[L_{ij}]_{i=0}^{2n}$ for all $i\in\{0,1,2,\ldots,n\}$ and $L=[L_{ij}]_{2n+1,2n+1}$ is a matrix of Muntz-Legendre polynomial coefficients.

Proof: This formula will be proved by induction. For $\ell=1$, one can easily see that (3.1) holds. Now, assume that (3.1) is true for $k\in\mathbb{N}$. So, for $k+1$, we have

$$\begin{aligned} u_n^{k+1}(x) &= u_n^k(x)u_n(x) \\ &= (\underline{u}LQ^{k-1}\underline{X})(\underline{u}L\underline{X}) \\ &= \underline{u}LQ^{k-1}(\underline{X}\times(\underline{u}L\underline{X})). \end{aligned} \tag{3.5}$$

If one show that $\underline{X}\times(\underline{u}L\underline{X})=Q\underline{X}$, the proof is completed. To show this equality, we have

$$\begin{aligned} \underline{X}\times(\underline{u}L\underline{X}) &= \underline{X}\times((\underline{u}L)\underline{X}) \\ &= \underline{X}([\Sigma_{s=0}^{2n}u_rL_{r,0} \quad \Sigma_{s=0}^{2n}u_rL_{r,1} \quad \Sigma_{s=0}^{2n}u_rL_{r,2} \quad \ldots \quad \Sigma_{s=0}^{2n}u_rL_{r,n}]\underline{X}) \\ &= \underline{X}\sum_{s=0}^{n}(\sum_{r=0}^{n}u_rL_{r,s})x^{\zeta_s} \\ &= ([\sum_{s=0}^{n}(\sum_{r=0}^{n}u_rL_{r,s}x^{\zeta_s+\zeta_i})]_{i=0}^{2n-1})^T. \end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned} Q\underline{X} &= ([\sum_{s=0}^{n}(\underline{u}L_sx^{\zeta_{s+i}})]_{i=0}^{n})^T \\ &= ([\sum_{s=0}^{n}(\sum_{r=0}^{n}u_rL_{r,s}x^{\zeta_{s+i}})]_{i=0}^{2n-1})^T \\ &= ([\sum_{s=0}^{n}(\sum_{r=0}^{n}u_rL_{r,s}x^{\zeta_s+\zeta_i})]_{i=0}^{2n-1})^T. \end{aligned} \tag{3.7}$$

Therefore, the validity of (3.1) is resulted.

Remark 1: Attention that, this relation is implemented for nonlinear problems.

Let $u_n(x)$ be the solution of Eq. (1.1) in nonlinear case as follows:

$$u_n(x)=\Sigma_{i=0}^{2n}u_iL_i(x)=\underline{u}L\underline{X}, \tag{3.8}$$

where $\underline{u}$, L and $\underline{X}$ are as in Lemma (3.1). Assume that f and F are approximated by the following finite series.

$$f(x)\simeq\Sigma_{i=0}^{2n}f_ix^{\zeta_i}=\underline{f}\underline{X}, \tag{3.9}$$

where

$$\underline{f}=\begin{bmatrix}f_0 & f_1 & f_2 & \ldots & f_{2n}\end{bmatrix}^T, \tag{3.10}$$

and

$$K(x,t)U(t,u(t))\simeq\Sigma_{\ell=0}^{n}z_\ell(x,t)u^\ell(t), \tag{3.11}$$

in which, we set

$$z_{\ell}(x,t)=\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\kappa_{ij}^{\ell}x^{i}t^{j},\ \forall\ell\in\mathbb{N}_{n}\cup\{0\}. \tag{3.12}$$

Substituting these relations in Eq. (1.1), one obtains

$$u_{n}(x)=\underline{f}\,\underline{X}+\int_0^x\frac{z_0(x,t)}{(x^3-t^3)^{\alpha}}dt+\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{\ell=1}^{n}\int_0^x\frac{\kappa_{ij}^{\ell}x^it^j}{(x^3-t^3)^{\alpha}}u_n^{\ell}(t)dt. \tag{3.13}$$

Using (3.1), we get

$$\begin{aligned}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{\ell=1}^{n}\int_0^x\frac{\kappa_{ij}^{\ell}x^it^j}{(x^3-t^3)^{\alpha}}u_n^{\ell}(t)dt &=\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{\ell=1}^{n}\int_0^x\frac{\kappa_{ij}^{\ell}x^it^j}{(x^3-t^3)^{\alpha}}\underline{u}LQ^{\ell-1}\underline{X_t}dt\\&=\underline{u}L\sum_{\ell=1}^{n}Q^{\ell-1}(\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\kappa_{ij}^{\ell}x^i\{\int_0^x\frac{t^{j+\zeta_k}}{(x^3-t^3)^{\alpha}}dt\}_{k=0}^{2n}),\end{aligned} \tag{3.14}$$

where $\underline{X_t}=X|_{x=t}$.

By calculating the integrals in this relation, results in

$$\begin{aligned}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{\ell=1}^{n}]\int_0^x\frac{\kappa_{ij}^{\ell}x^it^j}{(x^3-t^3)^{\alpha}}u_n^{\ell}(t)dt &=\underline{u}L\sum_{\ell=1}^{n}Q^{\ell-1}\left(\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\kappa_{ij}^{\ell}x^i\left\{\frac{\Gamma(1-\alpha)\Gamma\left(\frac{j+\zeta_k+1}{3}\right)}{3\Gamma\left(1-\alpha+\frac{j+\zeta_k+1}{3}\right)}x^{j+\zeta_k+1-3\alpha}\right\}_{k=0}^{2n}\right)\\&=\underline{u}L\sum_{\ell=1}^{n}Q^{\ell-1}\left(\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\kappa_{ij}^{\ell}\{A_j^kx^{i+j+\zeta_k+1-3\alpha}\}_{k=0}^{2n}\right)\\&\simeq\underline{u}L\left(\sum_{\ell=1}^{n}Q^{\ell-1}A_{\ell}\right)\underline{X}\\&=\underline{u}L\underline{Q}\underline{X},\end{aligned} \tag{3.15}$$

where

$$\underline{Q}=\sum_{\ell=1}^{n}Q^{\ell-1}A_{\ell},[A_{\ell}]_{p,2q+p+1}=\sum_{j=0}^{q}k_{j,\ell-j}^{\ell}A_{\ell-j}^{p},\forall p\forall q\in\{0,1,2,\dots,2n\}, \tag{3.16}$$

such that $2q+p+1\le 2n$ and $A_j^p=\frac{\Gamma(1-\alpha)\Gamma\left(\frac{j+\zeta_p+1}{3}\right)}{3\Gamma\left(1-\alpha+\frac{j+\zeta_p+1}{3}\right)}$. In the similar way, we set $z_0(x,t)=\omega(x,t)\vec{z}\underline{X_t}$. Therefore,

$$\begin{aligned}\int_0^x\frac{z_0(x,t)}{(x^3-t^3)^{\alpha}}dt&=\int_0^x\frac{\omega(x,t)\vec{z}\underline{X_t}}{(x^3-t^3)^{\alpha}}dt\\&=\vec{z}\int_0^x\frac{\omega(x,t)\underline{X_t}}{(x^3-t^3)^{\alpha}}dt\end{aligned}$$

$$
\begin{aligned}
&= \vec{z}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\int_0^x \frac{\kappa_{ij}^0 x^i t^j}{(x^3-t^3)^\alpha}\{t^{\zeta_k}\}_{k=0}^{2n} dt \\
&= \vec{z}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\kappa_{ij}^0 x^i \left\{\int 7_0^x \frac{t^{j+\zeta_k}}{(x^3-t^3)^\alpha} dt\right\}_{k=0}^{2n} \qquad (3.17)\\
&= \vec{z}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\kappa_{ij}^0 \left\{\frac{\Gamma(1-\alpha)\Gamma\left(\frac{j+\zeta_k+1}{3}\right)}{3\Gamma\left(1-\alpha+\frac{j+\zeta_k+1}{3}\right)} x^{i+j+\zeta_k+1-3\alpha}\right\}_{k=0}^{2n} \\
&= \vec{z}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\kappa_{ij}^0 \left\{A_j^k x^{i+j+\zeta_k+1-3\alpha}\right\}_{k=0}^{2n} \\
&\simeq \vec{z}A_0\underline{X},
\end{aligned}
$$

where

$$
[A_0]_{p,2q+p+1} = \sum_{j=0}^{q} k_{j,\ell-j}^0 A_{\ell-j}^p,\ \forall p \forall q \in \{0,1,2,\dots,2n\}, \qquad (3.18)
$$

such that $2q+p+1 \le 2n$. Now, we rewrite Eq. (3.13) in the following matrix form

$$
\underline{u}L(I_{2n+1}-\underline{Q})\underline{X} = (\underline{f}+A_0)\underline{X}, \qquad (3.19)
$$

where I_{2n+1} is an identity matrix of order $2n+1$. Utilizing $L^{-1}\underline{L}$ instead of $\underline{X}$ in (3.19) and multiplying the result by $\underline{L}^T$ from right and integrating both sides on $[a,b]$, results in

$$
\underline{u}L(I_{2n+1}-\underline{Q})L^{-1} = (\underline{f}+A_0)L^{-1}. \qquad (3.20)
$$

It should be noted that $\underline{L}\,\underline{L}^T$ is a diagonal matrix; since $L_0, L_1, L_2, \dots$ are orthogonal polynomials.

In particular case considering $U(t,u(t)) = u^\ell(t)$ with $\ell \in \mathbb{N}\setminus\{1\}$, Eq. (3.20) will be as follows

$$
\underline{\underline{u}}(I_{2n+1}-\underline{Q}) = \underline{f}, \qquad (3.21)
$$

where $\underline{\underline{u}} = \underline{u}L$ and $\underline{Q} = Q^{\ell-1}A_\ell$.

The matrices A_ℓ and $Q^{\ell-1}$ are upper triangular; So, $\underline{Q}$ is an upper triangular matrix as

$$
\underline{Q} = \begin{bmatrix}
0 & \underline{Q}_{00}(\underline{\underline{u_0}}) & \underline{Q}_{01}(\underline{\underline{u_0}},\underline{\underline{u_1}}) & \cdots & \underline{Q}_{0,2n-1}(\underline{\underline{u_0}},\underline{\underline{u_1}},\dots,\underline{\underline{u_{2n-1}}}) \\
0 & 0 & \underline{Q}_{11}(\underline{\underline{u_0}}) & \cdots & \underline{Q}_{1,2n-1}(\underline{\underline{u_0}},\underline{\underline{u_1}},\dots,\underline{\underline{u_{2n-2}}}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \underline{Q}_{2n-1,2n-1}(\underline{\underline{u_0}}) \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \qquad (3.22)
$$

where

$$\underline{Q}_{ij}(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_j}}),\ \forall i,j\in\{0\}\cup\mathbb{N}, \tag{3.23}$$

are $(j+1)$-variable functions and not linear necessarily. Therefore with details

$$\underline{\underline{u}}\,\underline{Q}=\begin{bmatrix}\mathfrak{F}_1(\underline{\underline{u_0}}) & \mathfrak{F}_2(\underline{\underline{u_0}},\underline{\underline{u_1}}) & \mathfrak{F}_3(\underline{\underline{u_0}},\underline{\underline{u_1}},\underline{\underline{u_2}}) & \cdots & \mathfrak{F}_{2n+1}(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{2n}}})\end{bmatrix}. \tag{3.24}$$

Also, $\mathfrak{F}_i(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{i-1}}}),\forall i\in\mathbb{N}$ are $(i-1)$-variable and not linear necessarily. Utilizing (3.24) in (3.21), one obtains the following vector form equation

$$\begin{bmatrix}\underline{\underline{u_0}}\\ \underline{\underline{u_1}}\\ \vdots\\ \underline{\underline{u_{2n}}}\end{bmatrix}=\begin{bmatrix}f_0\\ f_1\\ \vdots\\ f_{2n}\end{bmatrix}+\begin{bmatrix}0\\ \mathfrak{F}_1(\underline{\underline{u_0}})\\ \mathfrak{F}_2(\underline{\underline{u_0}},\underline{\underline{u_1}})\\ \vdots\\ \mathfrak{F}_{2n}(\underline{\underline{u_0}},\underline{\underline{u_1}},\ldots,\underline{\underline{u_{2n-1}}})\end{bmatrix}. \tag{3.25}$$

Then, $\underline{\underline{u}}$ is determined by solving this system and the vector function $\underline{u}$ is obtained by the following relation,

$$\underline{\underline{u}}^T=L\underline{u}^T. \tag{3.26}$$

Also, we obtain an approximation for $\sum_{i=0}^{2n}u_iL_i.$. In the case of $\ell=1$, Eq. (3.21) is reduced to

$$\underline{\underline{u}}(I_{n+1}-A_1)=\underline{f}. \tag{3.27}$$

By solving this linear system, $\underline{\underline{u}}$ and $\underline{u}$ are determined. Finally, one obtains the approximate solution as follows

$$u_n(x)=\sum_{i=0}^{n}u_iL_i. \tag{3.28}$$

4. Convergence analysis

The convergence property is very important for a numerical algorithm. So, in this section this property will be shown for our presented scheme.

Lemma 4.1: [1] *Suppose* $\underline{u}^T=\vec{X}$ *and L is coefficient matrix of (3.26). Let P is coefficient error matrix and* $\vec{\tilde{X}}$ *is computed solution of* $(L+P)\vec{\tilde{X}}=\underline{\underline{u}}^T$, *then*

$$\| \vec{X} - \tilde{\vec{X}} \|_\infty \leq C(L) \| \tilde{\vec{X}} \|_\infty \frac{\|P\|_\infty}{\|L\|_\infty}, \tag{4.1}$$

where $C(L)$ *is condition number of L.*

Theorem 4.2: *Under assumption of the Eq. (1.1), we have*

$$\| u - \tilde{u}_n \|_\infty \leq \frac{M(b-a)^{n+1}}{(n+1)!} + B_n C(L) \| \tilde{\vec{X}} \|_\infty O(\varepsilon), \tag{4.2}$$

where $M = \max_{n\in\mathbb{N}} \max_{x\in[a,b]} | u^{(n)}(x) |$, $B_n = \max_{x\in[a,b]} | L_i(x) |$, $C(L)$ *is condition number of L and* $\tilde{u}_n = \sum_{i=0}^{2n} \tilde{u}_i L_i$ *is an approximate solution by the presented method. So, the convergence property of our scheme is concluded.*

Proof: Using Taylor expansion, we get

$$T_n(x) = u(x_0) + u'(x_0)(x - x_0) + u''(x_0)\frac{(x-x_0)^2}{2} + \ldots + u^{(n)}(x_0)\frac{(x-x_0)^n}{n!}, \tag{4.3}$$

for $x_0 \in (a,b)$. Therefore,

$$u(x) - T_n(x) = \frac{1}{(n+1)!} u^{(n+1)}(\eta)(x - x_0)^{n+1}, \tag{4.4}$$

for $\eta \in (a,b)$. According to triangular inequality, we have

$$\| u - \tilde{u}_n \|_\infty \leq \| u - \bar{\bar{u}}_n \|_\infty + \| \bar{\bar{u}}_n - \tilde{u}_n \|_\infty, \tag{4.5}$$

in which $\bar{\bar{u}}_n = \sum_{i=0}^{2n} \tilde{u}_i L_i$ is the best approximation of $u(x)$. Using the Taylor expansion of $u(x)$ for the first term in the right side of (4.5), we get

$$\begin{aligned} \| u - \bar{\bar{u}}_n \|_\infty &\leq \| u - T_n \|_\infty \\ &\leq \frac{M_n (x-x_0)^{n+1}}{(n+1)!} \\ &\leq \frac{\max_{n\in\mathbb{N}} M_n (x-x_0)^{n+1}}{(n+1)!} \\ &= \frac{M(b-a)^{n+1}}{(n+1)!}, \end{aligned} \tag{4.6}$$

where $M_n = \max_{x\in[a,b]} | u^{(n)}(x) |$ and $M = \max_{n\in\mathbb{N}} M_n$. Now, for the second term of (4.5), we have,

$$| \tilde{u}_{2n}(x) - \bar{\bar{u}}_n(x) | = | \sum_{i=0}^{n} (\bar{\bar{u}}_i - \tilde{u}_i) L_i(x) | \leq \max_{i\in\mathbb{N}\cup\{0\}} | \bar{\bar{u}}_i - \tilde{u}_i | \max_{x\in[a,b]} \sum_{i=0}^{2n} | L_i(x) |. \tag{4.7}$$

Assuming $\mathcal{N} = \sum_{i=0}^{2n} | L_i(x) |$, results that

$$\| \tilde{u}_{2n}(x) - \bar{\bar{u}}_n(x) \|_\infty \leq \mathcal{N} \max_{i \in \mathbb{N} \cup \{0\}} | \bar{\bar{u}}_i - \tilde{u}_i | = \mathcal{N} \| \vec{X} - \tilde{\vec{X}} \|_\infty, \tag{4.8}$$

where $\vec{X} = [\bar{\bar{u}}_0, \bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots, \bar{\bar{u}}_{2n}]^T$, $\tilde{\vec{X}} = [\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{2n}]^T$ and $\| \vec{X} - \tilde{\vec{X}} \|_\infty = \max_{i \in \mathbb{N} \cup \{0\}} | \bar{\bar{u}}_i - \tilde{u}_i |$. $\vec{X}$ is the exact solution of $L\vec{X} = E$ and for $\tilde{\vec{X}}$ there exist a matrix P such that $(L+P)\tilde{\vec{X}} = E$. According to (4.1),

$$\| \vec{X} - \tilde{\vec{X}} \|_\infty \leq C(L) \| \tilde{\vec{X}} \|_\infty O(\varepsilon), \tag{4.9}$$

in which $O(\varepsilon) = \frac{\|P\|_\infty}{\|L\|_\infty}$. Consequently

$$\| \bar{\bar{u}}_n - \tilde{u}_n \|_\infty \leq \mathcal{N} C(L) \| \tilde{\vec{X}} \|_\infty O(\varepsilon). \tag{4.10}$$

By summing the both sides of inequalities (4.6) and (4.10), the proof is completed.

5. Numerical experiments

In this part, to show the performance of our presented method some examples are presented and in all of them we use Wolfram mathematica 12.2.

Example 1: Suppose the nonlinear generalized Abel integral equation

$$u(x) = x^5 - \frac{1}{3} x^{\frac{41}{4}} + \lambda \int_0^x \frac{x^{-\frac{1}{4}}}{(x^3 - t^3)^{\frac{1}{4}}} u^2(t) dt, \quad x \in [\sqrt{2}, \sqrt{3}], \tag{5.1}$$

with the exact solution $u(x) = x^5$ and $\lambda = \frac{\Gamma\left(\frac{53}{12}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{11}{3}\right)}$. For calculating of error, the formula

$$\| e_n \|_2 = \sqrt{\int_a^b (e_n(x))^2 dx}, \tag{5.2}$$

is used in all examples such that $e_n = u - u_n$. The numerical results are shown in Table 1 and Figure 1. The Muntz number sequence are chosen for $\{\frac{i}{4}\}_{i \in \mathbb{N}}$. By using the presented method on (5.1) with substituting $n = 1$ in (3.8), we get

$$u_1(x) = \underline{u} L \underline{X}, \tag{5.3}$$

in which

$$\underline{u} = [u_0 \quad u_1 \quad u_2], \ L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 5 & 0 \\ 10 & -30 & 21 \end{bmatrix} \text{ and } \underline{X} = [x^0 \quad x^{\frac{1}{4}} \quad x^{\frac{1}{2}}]^T. \tag{5.4}$$

Other calculations are as follows

$$A=\begin{bmatrix} 0 & \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{13}{12}\right)} & 0 & 0 \\ 0 & 0 & \frac{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{7}{6}\right)} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{5}{4}\right)} \end{bmatrix}, \quad Q=\begin{bmatrix} u_0 & u_1 & u_2 & 0 \\ 0 & u_0 & u_1 & u_2 \\ 0 & 0 & u_0 & u_1 \\ 0 & 0 & 0 & u_0 \end{bmatrix}, \tag{5.5}$$

$$\underline{f}=[0 \quad 0 \quad 0], \mathfrak{F}=\frac{\Gamma\left(\frac{53}{12}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{11}{3}\right)}\underline{\underline{u}}\,\underline{Q}A=\begin{bmatrix} 0 & \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{53}{12}\right)u_0^2}{3\Gamma\left(\frac{13}{3}\right)\Gamma\left(\frac{11}{3}\right)} & \frac{2\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{53}{12}\right)u_0u_1}{3\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{11}{3}\right)} \end{bmatrix}, \tag{5.6}$$

$$\underline{\underline{u}}=\underline{u}L. \tag{5.7}$$

In the case of $\ell=1,$ by using (3.21), one obtains

$$\underline{\underline{u}}(I_{2(1)+1}-QA_2)=\underline{f}, \tag{5.8}$$

where $A_2=\begin{bmatrix} 0 & \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{13}{12}\right)} & 0 \\ 0 & 0 & \frac{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{7}{6}\right)} \\ 0 & 0 & 0 \end{bmatrix}$. According to (3.25), we have

$$\underline{\underline{u}}=\underline{f}+\mathfrak{F}, \ i.e.; \begin{bmatrix} \underline{\underline{u_0}} \\ \underline{\underline{u_1}} \\ \underline{\underline{u_2}} \end{bmatrix}=\begin{bmatrix} 0 \\ \mathfrak{F}_1(\underline{\underline{u_0}}) \\ \mathfrak{F}_2(\underline{\underline{u_0}},\underline{\underline{u_1}}) \end{bmatrix}. \tag{5.9}$$

Solving explicit system equations (5.9), unknown elements $\underline{\underline{u_0}},\underline{\underline{u_1}}$ and $\underline{\underline{u_2}}$ will be determined. Also, it is seen that $u_0=u_1=u_2=0.$ Therefore, one obtains $\underline{u}$ as follows

$$L^{-1}\underline{\underline{u}}=\underline{u}, \tag{5.10}$$

So, $\underline{u}=\vec{0}_{1\times 3}$ and $u_1(x)=0.$ Hence, $e_1=\| u-u_1 \|_2 = \sqrt{[3ex]\int_{\sqrt{2}}^{\sqrt{3}}(u(x)-u_n(x))^2\,dx} = 5.84367$ where $\|.\|_2$ is the L_2 norm on Hilbert space $X=C([a,b],\mathbb{R})$ [9]. We have zero vector $\underline{u}$ and therefore $u_n=0.$ This is because of zero coefficients of f 's terms before x^5 and $\underline{\underline{u_i}}$'s criterion in (3.25). For the other n, we have

$$
\begin{aligned}
&n=10\to u_{10}(x)=\frac{1}{161820}(19019-750750x^{\frac{1}{4}}+11411400\sqrt{x}-91291200x^{\frac{3}{4}}\\
&+436486050x-1326917592x^{\frac{5}{4}}+2632773000x^{\frac{3}{2}}-3402352800x^{\frac{7}{4}}\\
&+2764411650x^2-1284474100x^{\frac{9}{4}}+260847048x^{\frac{5}{2}}),\\
&\vdots\\
&n=19\to u_{19}(x)=\frac{23}{960566918220}(-77+9240x^{\frac{1}{4}}-438900\sqrt{x}+11411400x^{\frac{3}{4}}\\
&-187065450x+2095133040x^{\frac{5}{4}}-16877460600x^{\frac{3}{2}}+101264763600x^{\frac{7}{4}}\\
&-463746587850x^2+1648876756800x^{\frac{9}{4}}-4604171251680x^{\frac{5}{2}}\\
&+10165053412800x^{\frac{11}{4}}-17788843472400x^3\\
&+24630706346400x^{\frac{13}{4}}-26804003965200x^{\frac{7}{2}}+22634492237280x^{\frac{15}{4}}\\
&-14518835481150x^4+6832393167600x^{\frac{17}{4}}-2223239046600x^{\frac{9}{2}}\\
&+446775310800x^{\frac{19}{4}}),\\
&n=20\to u_{20}(x)=x^5,\\
&n\ge 21\to u_n(x)=x^5. \qquad (5.11)
\end{aligned}
$$

Finally, our approximate solution which is agreement with the exact solution will be found. These results show that our presented algorithm is suitable and capable with very accuarcy for solving this kind of nonlinear generalized Abel integral equation.

Table 1

The results of the Muntz-Galerkin method for Example 1.

n	Numerical errors
1	5.84367

Contd...

⋮	5.84367
9	5.84367
10	4.16292×10^{1}
11	1.98671×10^{-1}
12	8.29645×10^{-2}
13	2.98097×10^{-2}
14	9.02055×10^{-3}
15	2.23380×10^{-3}
16	4.34411×10^{-4}
17	6.21835×10^{-5}
18	5.82323×10^{-6}
19	2.67555×10^{-7}
≥20	0

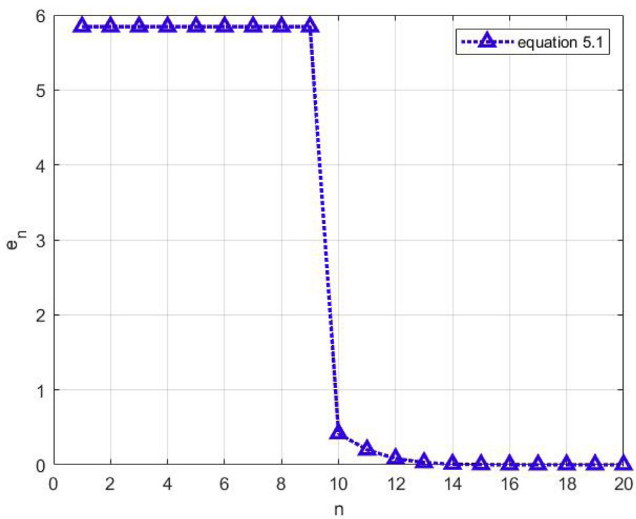

Figure 1

The error values for Example 1 by the Muntz-Galerkin method.

Example 2: [21] Consider the following generalized Abel integral equations:

$$u(x) = \frac{7}{10}x^5 + \int_0^x \frac{1}{(x^3 - t^3)^{\frac{1}{3}}} u(t)dt, \ x \in [0,1]. \tag{5.12}$$

with the exact solution $u(x)=x^5$. The Muntz number sequence chosen for this equation is $\{i\}_{i\in\mathbb{N}}$. Using the presented method for $n=1$, in (5.12), results in

$$u_2(x)=\underline{u}L\underline{X}, \tag{5.13}$$

in which

$$\underline{u}=[u_0 \quad u_1],\ L=\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix} \text{ and } \underline{X}=[x^0 \quad x^1]^T. \tag{5.14}$$

The calculations of this technique is as follows

$$A=\begin{bmatrix}\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) & 0\\ 0 & \frac{\Gamma^2\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{4}{3}\right)}\end{bmatrix},\ \underline{f}=[0 \quad 0] \text{ and } \underline{\underline{u}}=\underline{u}L. \tag{5.15}$$

On the other hand (3.27) becames

$$\underline{\underline{u}}(I_{1+1}-A_1)=\underline{f}. \tag{5.16}$$

Solving explicit system of equations (5.16), unknown elements $\underline{\underline{u_0}}$ and $\underline{\underline{u_1}}$ will be determined. Therefore, it is seen that $u_0=u_1=0$ which gets $\underline{u}=\vec{0}_{1\times 2}$, $u_1(x)=0$ and $e_1=0.301511$. Determining the vector $\underline{u}=0$, we get $u_n=0$, and $e_n=0.301511$, for $n=2,3,4$. Setting $n=5$,

$$u_5(x)=\underline{u}L\underline{X}, \tag{5.17}$$

in which

$$\underline{u}=[u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4],\ L=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ -1 & 2 & 0 & 0 & 0\\ 1 & -6 & 6 & 0 & 0\\ -1 & 12 & -30 & 20 & 0\\ 1 & -20 & 90 & -140 & 70\end{bmatrix}, \tag{5.18}$$

and $\underline{X}=[x^0 \ \ x^1 \ \ x^2 \ \ x^3 \ \ x^4]$. Also, we have

$$A=\begin{bmatrix} \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma^2\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{4}{3}\right)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{5}{3}\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{3}\right)}{3\Gamma\left(\frac{7}{3}\right)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{8}{3}\right)} \end{bmatrix}, \ \underline{f}=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{7}{10} \end{bmatrix}, \tag{5.19}$$

and $\underline{\underline{u}}=\underline{u}L.$ Considering relation (3.27), we conclude that

$$\underline{\underline{u}}(I_{5+1}-A)=\underline{f}. \tag{5.20}$$

and

$$L^{-1}\underline{\underline{u}}=\underline{u}, \tag{5.21}$$

Obviously, it is seen that

$$\underline{u}=\left[\frac{7}{60\left(1-\frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{8}{3}\right)}\right)} \quad \frac{1}{4\left(1-\frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{8}{3}\right)}\right)} \quad \frac{5}{24\left(1-\frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{8}{3}\right)}\right)} \quad \frac{7}{72\left(1-\frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{8}{3}\right)}\right)} \quad \frac{1}{40\left(1-\frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{8}{3}\right)}\right)} \quad \frac{1}{360\left(1-\frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{8}{3}\right)}\right)}\right].$$

Therefore, the approximate solution $u_5(x)=\sum_{i=0}^{5}u_iL_i(x)=x^5$ will be obtained. Also, for $n\geq 6,$ one gets the exact solution. The results of this example reveal that Muntz-Galerkin method is very efficient and capable to obtain numerical solution of the generalized Abel integral equation.

Example 3: Assume the following generalized Abel integral equation

$$u(x)=f(x)+\int_0^x \frac{x^{\frac{1}{4}}+x^{\frac{3}{4}}+x^{\frac{9}{4}}-x^{\frac{5}{4}}-x^{\frac{7}{4}}}{(x^3-t^3)^{\frac{1}{4}}}u^3(t)dt, \quad x\in[1,1.12], \tag{5.22}$$

with $f(x)=x-\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)(-1+\sqrt{x})^2(1+\sqrt{x})x^4}{3\Gamma\left(\frac{25}{12}\right)}$. The exact solution is $u(x)=x$ and the Muntz number sequence is chosen as $\left\{\frac{i}{4}\right\}_{i\in\mathbb{N}}$. To solve this equation by the presented method for $n=1$, we get the error bound $e_1=3.673\times10^{-1}$. Applying the suggested method with $n=2$, one obtains

$$u_2(x)=\underline{u}L\underline{X}, \tag{5.23}$$

in which

$$\underline{u}=[u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4],\ L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 5 & 0 & 0 & 0 \\ 10 & -30 & 21 & 0 & 0 \\ -20 & 105 & -168 & 84 & 0 \\ 35 & -280 & 756 & -840 & 330 \end{bmatrix}, \tag{5.24}$$

and $\underline{X}=[x^0 \ \ x^1 \ \ x^2 \ \ x^3 \ \ x^4]$. Using the proposed method, result in

$$A=\begin{bmatrix} 0 & \frac{\Gamma\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{13}{12}\right)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{7}{6}\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{5}{4}\right)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{4}{3}\right)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{17}{12}\right)} \end{bmatrix},\ \underline{f}=[0 \quad 0 \quad 0 \quad 0 \quad 1], \tag{5.25}$$

and $\underline{\underline{u}}=\underline{u}L$. On the other hand, consdering (3.21), we get

$$\underline{\underline{u}}(I_{2(2)+1}-Q^2A_3)=\underline{f}, \tag{5.25}$$

where, $A_3 = \begin{bmatrix} 0 & \frac{\Gamma\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{13}{12}\right)} & 0 & 0 & 0 \\ 0 & 0 & \frac{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{7}{6}\right)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{5}{4}\right)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{3}{4}\right)}{3\Gamma\left(\frac{4}{3}\right)} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Moreover, the following matrix-vector form according to (3.25), is obtained as

$$\underline{\underline{u}} = \underline{f} + \mathfrak{F},\ i.e.; \begin{bmatrix} \underline{\underline{u_0}} \\ \underline{\underline{u_1}} \\ \underline{\underline{u_2}} \\ \underline{\underline{u_3}} \\ \underline{\underline{u_4}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathfrak{F}_1(\underline{\underline{u_0}}) \\ \mathfrak{F}_2(\underline{\underline{u_0}}, \underline{\underline{u_1}}) \\ \mathfrak{F}_3(\underline{\underline{u_0}}, \underline{\underline{u_1}}, \underline{\underline{u_2}}) \\ \mathfrak{F}_4(\underline{\underline{u_0}}, \underline{\underline{u_1}}, \underline{\underline{u_2}}, \underline{\underline{u_3}}) \end{bmatrix}. \tag{5.26}$$

By solving this implicit system of equations, unknown elements $\underline{\underline{u_0}}$, $\underline{\underline{u_1}}$, $\underline{\underline{u_2}}$, $\underline{\underline{u_3}}$ and $\underline{\underline{u_4}}$ will be determined. Therefore,

$$\underline{u} = \left[\frac{1}{2}\ \frac{1}{3}\ \frac{2}{15}\ \frac{1}{33}\ \frac{1}{330}\right],$$

and the approximate solution $u_2(x)$ is obtained as

$$u_2(x) = \sum_{i=0}^{4} u_i L_i(x) = x,$$

which is the exact solution of problem (5.22). In this example, one can see the capacity of our presented scheme for solving the generalized nonlinear Abel integral equation.

Conclusion

In this article, the Muntz-Legendre polynomials was applied to obtain the numerical solution for the generalized Abel integral equation. We explained these polynomials to utilize in our procedure. Using this method, the generalized Abel integral equation has been reduced to a system of explicit algebraic equations. In this algorithm, the consedered problem was approximated with a linear combination of Muntz-Legendre polynomials with respect to the suitable Muntz number sequence. Also, existence, uniqueness and convergence of the proposed method were discussed through preparing some theorems. Several numerical experiments were presented to show the superiority, eï¬ƒciency and accuracy of our scheme.

References

[1] G. Allaire, S. M. Kaber, K. Trabelsi, and G. Allaire, *Numerical Linear Algebra*, New York: Springer, Dec. 17 (2008).

[2] S. Atkinson, "An existence theorem for Abel integral equations," *SIAM J. Math. Anal.*, vol. 5, no. 5, pp. 729–736 (1974).

[3] P. Borwein, T. Erdelyi, and J. Zhang, "Muntz systems and orthogonal Muntz-Legendre polynomials," *Amer. Math. Soc.*, vol. 342, no. 2, pp. 523–542 (1994).

[4] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge, U.K.: Cambridge University Press, Nov. 15 (2004).

[5] S. Dixit, "An efficient algorithm for numerical inversion of system of generalized Abel integral equations," *Appl. Appl. Math.*, vol. 15, no. 2, pp. 1275–1290 (2020).

[6] M. Eshaghi, H. Baghani, and O. Baghani, "On existence and uniqueness of solutions of a nonlinear integral equation," *J. Appl. Math.*, vol. 15, no. 2, pp. 1–7 (2011).

[7] E. Hesameddini and M. Shahbazi, "Solving system of Volterra-Fredholm integral equations with Bernstein polynomials and hybrid Bernstein Block-Pulse functions," *J. Comput. Appl. Math.*, vol. 315, pp. 182–194 (2017).

[8] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus*, CRC Press, May 19 (2015).

[9] Z. Li, Z. Qiao, and T. Tang, *Numerical Solution of Differential Equations: Introduction to Finite Difference and Finite Element Methods*, Cambridge University Press (2018).

[10] P. Mokhtary and F. Ghoreishi, "Convergence analysis of operational Tau method for Abel type Volterra integral equations," *Electron. Trans. Numer. Anal.*, vol. 41, pp. 289–305 (2014).

[11] P. Mokhtary, "High-order modified Tau method for non-smooth solutions of Abel integral equations," *Electron. Trans. Numer. Anal.*, vol. 44, pp. 462–471 (2015).

[12] P. Mokhtary, "Operational Muntz-Galerkin approximation for Abel-Hammerstein integral equations of the second kind," *Electron. Trans. Numer. Anal.*, vol. 45, pp. 183–200 (2016).

[13] J. Mouley, M. M. Panja, and B. N. Mandal, "Approximate solution of Abel integral equation in Daubechies wavelet basis," *Cubo A Math. J.*, vol. 23, no. 02, pp. 245–264 (2021).

[14] T. D. Rao and S. Chakraverty, "Solving uncertain differential equations using interval Legendre polynomials based collocation method," *J. Interdiscip. Math.*, vol. 22, no. 4, pp. 473–491 (2019).

[15] W. Rudin, *Principles of Mathematical Analysis*, New York: McGraw-Hill, Jan. (1976).

[16] T. J. Rivlin, *An Introduction to the Approximation of Functions*, Courier Corporation (1981).

[17] J. Shen and Y. Wang, "Muntz-Galerkin methods and applications to mixed Dirichlet-Neumann boundary value problems," *SIAM J. Sci. Comput.*, vol. 38, no. 4, pp. 2357–2381 (2015).

[18] V. S. Sizikov and D. N. Sidorov, "Generalized quadrature for solving singular integral equations of Abel type in application to infrared tomography," *Appl. Numer. Math.*, vol. 106, pp. 69–78 (2016).

[19] L. Song and W. Wang, "A new improved Adomian decomposition method and its application to fractional differential equations," *Appl. Math. Model.*, vol. 37, no. 3, pp. 1590–1598 (2013).

[20] U. F. Stefánsson, "Asymptotic behavior of MÜntz orthogonal polynomials," *Construct. Approx.*, vol. 32, no. 2, pp. 193–220 (2009).

[21] A. M. Wazwaz, *Linear and Nonlinear Integral Equations: Methods and Applications*, Higher Education Press (2011).

[22] M. Yi and J. Ma, "CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel," *Int. J. Comput. Math.*, vol. 92, no. 8, pp. 1715–1728 (2015).

[23] E. Zarei and S. Noeiaghdam, "Solving generalized Abel's integral equations of the first and second kinds via Taylor-collocation method" (2018), [Online]. Available: http://arxiv.org/abs/1804.08571.

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