Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 46 (2025), No. 7, pp. 2047-2056
DOI : 10.47974/JIOS-1314

On Leonardo Octonions

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Abstract

In this article, we give a new family of Octonions. We describe the Leonardo Octonions. Then, we derive certain algebraic characteristics for Leonardo Octonions including the generating function, Binet's formula, exponential generating function, some summation formulas, Catalan's, Cassini's and d'Ocagne's identities.

Subject Classification: *11B37, 11B39, 11R52.*

Keywords: *Octonions, Fibonacci Octonions, Lucas Octonions, Leonardo numbers, Leonardo Octonions.*

1. Introduction

Let O be the Octonion algebra over $\mathbb{R}$. According to Cayley-Dickson procedure, any $z \in O$ can be shown such that

$$z = x + y\boldsymbol{e},$$

where x and y are quaternions belong to the real quaternion division algebra

$$H = \{s_0 + s_1\boldsymbol{i} + s_2\boldsymbol{j} + s_3\boldsymbol{k} : \ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \boldsymbol{ijk} = -1, s_0, s_1, s_2, s_3 \in \mathbb{R}\}.$$

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Let $z_1 = x_1 + y_1\boldsymbol{e}, z_2 = x_2 + y_2\boldsymbol{e}$ be two Octonions and $\overline{x_2}, \overline{y_2}$ be conjugates of the quaternions x_2, y_2, respectively. In relation to Octonions, we list the following operations:

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\boldsymbol{e} \quad (addition),$$
$$z_1 z_2 = (x_1 x_2 - \overline{y_2} y_1) + (y_2 x_1 + y_1 \overline{x_2})\boldsymbol{e} \quad (multiplication).$$

O is an 8-dimensional $\mathbb{R}$-linear space with basis $B = \{\boldsymbol{e_i} : i = 0,1,\dots,7\}$. The basis B complies with multiplication rules in Table 1.

Table 1
Multiplication rules of B.

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Now, we present some basic characteristics of Octonions.

1. For all $z \in O$, we can denote as
$$z = \sum_{k=0}^{7} r_k \mathbf{e_k}, \ r_k \in \mathbb{R}.$$
2. Let $Re(z) = r_0$ and $Im(z) = \sum_{k=1}^{7} r_k \mathbf{e_k}$ for $z \in O$. Then we can denote as
$$z = Re(z) + Im(z).$$
3. The conjugate of $z \in O$ is
$$\overline{z} = Re(z) - Im(z).$$
4. For all $z, w \in O$, the conjugate operation provides
$$\overline{\overline{z}} = z, \qquad \overline{(z+w)} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{wz}.$$
5. The norm of any $z \in O$ is defined by
$$N_z = z\overline{z} = \overline{z}z = \sum_{k=0}^{7} r_k^2.$$
6. The inverse of any $0 \neq z \in O$ is
$$z^{-1} = \frac{\overline{z}}{N_z}.$$
7. For all $z, w \in O$,
$$N_{zw} = N_z N_w,$$
$$(zw)^{-1} = w^{-1} z^{-1}.$$

8. O is alternative

$$z(zw)=z^2w, \qquad (wz)z=wz^2, \qquad (zw)z=z(wz)=zwz,$$

for all $z,w\in O$. On the other hand O is non-commutative and non-associative (see, for example, [8] and [10]).

Integer sequences are commonly utilized in research. Fibonacci and Lucas sequences are some of the most prominent of them. For $n\geq 2$, these sequences are described by

$$\begin{aligned} F_n &= F_{n-1}+F_{n-2}, \qquad F_0=0, F_1=1, \\ L_n &= L_{n-1}+L_{n-2}, \qquad L_0=2, L_1=1. \end{aligned}$$

Furthermore, we present a relationship

$$F_n+L_n=2F_{n+1}. \tag{1.1}$$

In this work, we investigate the Leonardo sequences. These sequences have properties analogou to the Fibonacci sequences. In [3], the authers have studied Leonardo numbers and provided several properties of them. For $n\geq 2$, Leonardo sequences are described as follows:

$$Le_n=Le_{n-1}+Le_{n-2}+1, \tag{1.2}$$

where $Le_0=Le_1=1$. The Binet's formula [3] for Le_n is

$$Le_n=2\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)-1, \tag{1.3}$$

where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$. For $n\geq 2$, there is a relation as follows:

$$Le_{n+1}=2Le_n-Le_{n-2}. \tag{1.4}$$

In addition, the relationship between Leonardo and Fibonacci numbers is as follows:

$$Le_n=2F_{n+1}-1, n\geq 0 \tag{1.5}$$

In [9], Leonardo numbers have been generalized. The authors [4] have described incomplete Leonardo numbers and derived several properties for them. Alp and Kocer presented some characteristics of Leonardo numbers in [1] and introduced hybrid Leonardo numbers in [2]. In [7], the authors introduced Leonardo polynomials and hybrinomials, and presented several characteristics of them. Furthermore, the authors investigated the 2 -dimensional and 3-dimensional associations of Leonardo numbers (see, for example, [12] and [13]). There are several characteristics for Fibonacci, Lucas and Leonardo numbers. We refer to [1], [3], [6], [11] for more information. In [5], the authors defined Fibonacci and Lucas Octonions and provided several characteristics of them. In this article, we provide a new family of Octonions. We introduce Leonardo Octonions. The Leonardo Octonions are associated with Fibonacci and Lucas Octonions. We provide the generating function, Binet's formula, some summation formulas and some important identities of Leonardo Octonions.

2. Preliminaries

With our paper, we present several well-known definitions and features in this part. Throughout this paper, let

$$\begin{aligned}\widehat{\boldsymbol{\alpha}} &= \textstyle\sum_{k=0}^{7} \alpha^k \boldsymbol{e_k},\\ \widehat{\boldsymbol{\beta}} &= \textstyle\sum_{k=0}^{7} \beta^k \boldsymbol{e_k},\\ OF_{-1} &= \boldsymbol{e_0}+\boldsymbol{e_2}+\boldsymbol{e_3}+2\boldsymbol{e_4}+3\boldsymbol{e_5}+5\boldsymbol{e_6}+8\boldsymbol{e_7}.\end{aligned}$$

In [5], the authors described Fibonacci Octonions OF_n and Lucas Octonions OL_n such that

$$OF_n = \textstyle\sum_{k=0}^{7} F_{n+k}\boldsymbol{e_k}, \tag{2.1}$$

$$OL_n = \textstyle\sum_{k=0}^{7} L_{n+k}\boldsymbol{e_k}, \tag{2.2}$$

with the initial conditions

$$\begin{aligned}OF_0 &= \boldsymbol{e_1}+\boldsymbol{e_2}+2\boldsymbol{e_3}+3\boldsymbol{e_4}+5\boldsymbol{e_5}+8\boldsymbol{e_6}+13\boldsymbol{e_7},\\ OF_1 &= \boldsymbol{e_0}+\boldsymbol{e_1}+2\boldsymbol{e_2}+3\boldsymbol{e_3}+5\boldsymbol{e_4}+8\boldsymbol{e_5}+13\boldsymbol{e_6}+21\boldsymbol{e_7},\end{aligned}$$

and

$$\begin{aligned}OL_0 &= 2\boldsymbol{e_0}+\boldsymbol{e_1}+3\boldsymbol{e_2}+4\boldsymbol{e_3}+7\boldsymbol{e_4}+11\boldsymbol{e_5}+18\boldsymbol{e_6}+29\boldsymbol{e_7},\\ OL_1 &= \boldsymbol{e_0}+3\boldsymbol{e_1}+4\boldsymbol{e_2}+7\boldsymbol{e_3}+11\boldsymbol{e_4}+18\boldsymbol{e_5}+29\boldsymbol{e_6}+47\boldsymbol{e_7}.\end{aligned}$$

In [5], the Catalan's identity [5] for OF_n is provided by

$$OF_n^2 - OF_{n+r}OF_{n-r} = (-1)^{n+r}[F_r^2 OL_0 - F_{2r}(OF_0 - 14\boldsymbol{e_5} - 14\boldsymbol{e_6} - 7\boldsymbol{e_7})]. \tag{2.3}$$

In [5], the d'Ocagne's identity for OF_n is provided by

$$OF_{m+1}OF_n - OF_mOF_{n+1} = (-1)^m[F_{n-m}OL_0 + L_{n-m}(OF_0 - 14\boldsymbol{e_5} - 14\boldsymbol{e_6} - 7\boldsymbol{e_7})]. \tag{2.4}$$

Moreover, some summations (see, for example, [3] and [5]) are given as follows:

$$\textstyle\sum_{t=0}^{n} (Le_t)^2 = (Le_n - 1)(Le_{n+1} - 1) + (n+1), \tag{2.5}$$

$$\textstyle\sum_{t=0}^{n} OF_t = OF_{n+1} - OF_1, \tag{2.6}$$

$$\textstyle\sum_{t=1}^{n} OF_{2t-1} = OF_{2n} - OF_0, \tag{2.7}$$

$$\textstyle\sum_{t=0}^{n} OF_{2t} = OF_{2n+1} - OF_{-1}, \tag{2.8}$$

$$\textstyle\sum_{t=0}^{n} \binom{n}{t} OF_t = OF_{2n}, \tag{2.9}$$

$$\textstyle\sum_{t=0}^{n} (-1)^t \binom{n}{t} OF_t = (-1)^n OF_{-n}, \tag{2.10}$$

where $F_{-n} = (-1)^{n+1}F_n$, $OF_{-n} = \sum_{t=0}^{7} (-1)^{n-t+1}F_{n-t}\boldsymbol{e_t}$.

3. Leonardo Octonions

In this part, we describe a new class for Octonions. Leonardo Octonions are the name given to such Octonions. For the Leonardo Octonions, we present Binet's formula as well as several additional identities. Throughout this paper, let

$$\begin{aligned}\widehat{\boldsymbol{e}} &= \textstyle\sum_{k=0}^{7} \boldsymbol{e_k},\\ OLe_0 &= \boldsymbol{e_0}+\boldsymbol{e_1}+3\boldsymbol{e_2}+5\boldsymbol{e_3}+9\boldsymbol{e_4}+15\boldsymbol{e_5}+25\boldsymbol{e_6}+41\boldsymbol{e_7},\\ OLe_1 &= \boldsymbol{e_0}+3\boldsymbol{e_1}+5\boldsymbol{e_2}+9\boldsymbol{e_3}+15\boldsymbol{e_4}+25\boldsymbol{e_5}+41\boldsymbol{e_6}+67\boldsymbol{e_7}.\end{aligned}$$

Definition 3.1: The n-th Leonardo Octonions are defined by

$$OLe_n = \sum_{k=0}^{7} Le_{n+k}\boldsymbol{e_k},\ for\ n \geq 0.$$

The conjugate Octonion $\overline{OLe_n}$ of OLe_n is

$$\overline{OLe_n} = Le_n - \sum_{k=1}^{7} Le_{n+k}\boldsymbol{e_k}. \tag{3.1}$$

Proposition 3.1: For $n \geq 1$, the norm of Leonardo Octonions OLe_n is given by

$$N_{OLe_n} = (Le_{n+7} - 1)(Le_{n+8} - 1) - (Le_{n-1} - 1)(Le_n - 1) + 8.$$

Proof: By using the identity (2.5), we can easily establish the norm of OLe_n such that

$$\begin{aligned} N_{OLe_n} &= OLe_n\overline{OLe_n} \\ &= \sum_{k=0}^{7} (Le_{n+k})^2 \\ &= \sum_{k=0}^{n+7} (Le_k)^2 - \sum_{k=0}^{n-1} (Le_k)^2 \\ &= (Le_{n+7} - 1)(Le_{n+8} - 1) - (Le_{n-1} - 1)(Le_n - 1) + 8. \end{aligned}$$

Proposition 3.2: *For $n \geq 0$, then we have*

$$OLe_n + \overline{OLe_n} = 2Le_n.$$

Proof: By Definition 3.1 and the identity (3.1), we can easily obtain

$$\begin{aligned} OLe_n + \overline{OLe_n} &= \sum_{k=0}^{7} Le_{n+k}\boldsymbol{e_k} + Le_n - \sum_{k=1}^{7} Le_{n+k}\boldsymbol{e_k} \\ &= 2Le_n. \end{aligned}$$

Corollary 3.1: *For $n \geq 2$, then we have*

$$OLe_n = OLe_{n-1} + OLe_{n-2} + \hat{\boldsymbol{e}}.$$

Proof: By Definition 3.1 and the identity (1.2), we can easily derive

$$\begin{aligned} OLe_n - OLe_{n-1} - OLe_{n-2} &= \sum_{k=0}^{7} Le_{n+k}\boldsymbol{e_k} - \sum_{k=0}^{7} Le_{n+k-1}\boldsymbol{e_k} - \sum_{k=0}^{7} Le_{n+k-2}\boldsymbol{e_k} \\ &= \sum_{k=0}^{7} \boldsymbol{e_k} \\ &= \hat{\boldsymbol{e}}. \end{aligned}$$

Corollary 3.2: *For $n \geq 2$, then we get*

$$OLe_{n+1} = 2OLe_n - OLe_{n-2}. \tag{3.2}$$

Proof: By Definition 3.1 and the identity (1.4), we can get

$$\begin{aligned} OLe_{n+1} - 2OLe_n + OLe_{n-2} &= \sum_{k=0}^{7} Le_{n+k+1}\boldsymbol{e_k} - 2\sum_{k=0}^{7} Le_{n+k}\boldsymbol{e_k} + \sum_{k=0}^{7} Le_{n+k-2}\boldsymbol{e_k} \\ &= \sum_{k=0}^{7} (0 \cdot \boldsymbol{e_k}) \\ &= 0. \end{aligned}$$

Corollary 3.3: *For $n \geq 0$, then we have*

$$OLe_n = 2OF_{n+1} - \hat{\boldsymbol{e}}. \tag{3.3}$$

Proof: By (1.5), (2.1) and Definition 3.1, we find that

$$
\begin{aligned}
OLe_n &= \sum_{k=0}^{7} Le_{n+k}\boldsymbol{e_k} \\
&= 2\sum_{k=0}^{7} F_{n+k+1}\boldsymbol{e_k} - \sum_{k=0}^{7} \boldsymbol{e_k} \\
&= 2OF_{n+1} - \hat{\boldsymbol{e}}.
\end{aligned}
$$

We present an association that connects Fibonacci, Lucas, and Leonardo Octonions.

Corollary 3.4: ***For*** $n \geq 0$, *we get*

$$OLe_n = OF_n + OL_n - \hat{\boldsymbol{e}}.$$

Proof: By Definition 3.1, (1.1), (1.5), (2.1) and (2.2), we can obtain

$$
\begin{aligned}
OF_n + OL_n - OLe_n &= \sum_{k=0}^{7} F_{n+k}\boldsymbol{e_k} + \sum_{k=0}^{7} L_{n+k}\boldsymbol{e_k} - \sum_{k=0}^{7} Le_{n+k}\boldsymbol{e_k} \\
&= \sum_{k=0}^{7} \boldsymbol{e_k} \\
&= \hat{\boldsymbol{e}}.
\end{aligned}
$$

Theorem 3.1: The generating function of Leonardo Octonions OLe_n is

$$g(u) = \frac{1}{1-2u+u^3}\left[OLe_0 + (OLe_1 - 2OLe_0)u + (OLe_2 - 2OLe_1)u^2\right].$$

Proof: Let

$$g(u) = OLe_0 + OLe_1u + OLe_2u^2 + OLe_3u^3 + \cdots + OLe_iu^i + \cdots \tag{3.4}$$

be the generating function for OLe_n. Therefore we have

$$-2g(u)u = -2OLe_0u - 2OLe_1u^2 - 2OLe_2u^3 - 2OLe_3u^4 - \cdots - 2OLe_iu^{i+1} - \cdots \tag{3.5}$$

$$g(u)u^3 = OLe_0u^3 + OLe_1u^4 + OLe_2u^5 + OLe_3u^6 + \cdots + OLe_iu^{i+3} + \cdots. \tag{3.6}$$

From (3.4), (3.5), (3.6) and (3.2), we get

$$
\begin{aligned}
g(u)(1-2u+u^3) &= OLe_0 + (OLe_1 - 2OLe_0)u + (OLe_2 - 2OLe_1)u^2 \\
&\quad + (OLe_3 - 2OLe_2 + OLe_0)u^3 \\
&\quad + \cdots + (OLe_i - 2OLe_{i-1} + OLe_{i-3})u^i + \cdots + \\
&= OLe_0 + (OLe_1 - 2OLe_0)u + (OLe_2 - 2OLe_1)u^2.
\end{aligned}
$$

Hence, the proof is completed.

Theorem 3.2: The Binet's formula of Leonardo Octonions OLe_n is

$$OLe_n = 2\left(\frac{\hat{\boldsymbol{\alpha}}\alpha^{n+1} - \hat{\boldsymbol{\beta}}\beta^{n+1}}{\alpha-\beta}\right) - \hat{\boldsymbol{e}}, \text{ for } n \geq 0. \tag{3.7}$$

Proof: By (1.3) and Definition 3.1, we obtain

$$
\begin{aligned}
OLe_n &= \sum_{k=0}^{7} Le_{n+k}\boldsymbol{e_k} \\
&= \sum_{k=0}^{7} \left[2\left(\frac{\alpha^{n+k+1} - \beta^{n+k+1}}{\alpha-\beta}\right) - 1\right]\boldsymbol{e_k} \\
&= \frac{2}{\alpha-\beta}\left(\alpha^{n+1}\sum_{k=0}^{7} \alpha^k\boldsymbol{e_k} - \beta^{n+1}\sum_{k=0}^{7} \beta^k\boldsymbol{e_k}\right) - \sum_{k=0}^{7} \boldsymbol{e_k} \\
&= 2\left(\frac{\alpha^{n+1}\hat{\boldsymbol{\alpha}} - \beta^{n+1}\hat{\boldsymbol{\beta}}}{\alpha-\beta}\right) - \hat{\boldsymbol{e}}.
\end{aligned}
$$

Theorem 3.3: *The exponential generating function for Leonardo Octonions OLe_n is*

$$\sum_{t=0}^{\infty} OLe_t \frac{u^t}{t!} = 2\left(\frac{\alpha\widehat{\boldsymbol{\alpha}}e^{\alpha u}-\beta\widehat{\boldsymbol{\beta}}e^{\beta u}}{\alpha-\beta}\right) - \widehat{\boldsymbol{e}}e^{u}.$$

Proof: By Binet's formula of OLe_n in (3.7), we obtain

$$\begin{aligned}
\sum_{t=0}^{\infty} OLe_t \frac{u^t}{t!} &= \sum_{t=0}^{\infty} \left[2\left(\frac{\widehat{\boldsymbol{\alpha}}\alpha^{t+1}-\widehat{\boldsymbol{\beta}}\beta^{t+1}}{\alpha-\beta}\right) - \widehat{\boldsymbol{e}}\right]\frac{u^t}{t!} \\
&= 2\left(\frac{\alpha\widehat{\boldsymbol{\alpha}}}{\alpha-\beta}\sum_{t=0}^{\infty}\frac{(\alpha u)^t}{t!} - \frac{\beta\widehat{\boldsymbol{\beta}}}{\alpha-\beta}\sum_{t=0}^{\infty}\frac{(\beta u)^t}{t!}\right) - \widehat{\boldsymbol{e}}\sum_{t=0}^{\infty}\frac{u^t}{t!} \\
&= 2\left(\frac{\alpha\widehat{\boldsymbol{\alpha}}e^{\alpha u}-\beta\widehat{\boldsymbol{\beta}}e^{\beta u}}{\alpha-\beta}\right) - \widehat{\boldsymbol{e}}e^{u}.
\end{aligned}$$

Now, we provide a list of several associations for Leonardo Octonions OLe_n.

Proposition 3.3: The identities listed below are satisfied:

(i) $\sum_{t=0}^{n} OLe_t = OLe_{n+1} - OLe_1 - (n+1)\widehat{\boldsymbol{e}}$

(ii) $\sum_{t=0}^{n} OLe_{2t} = OLe_{2n+1} + OLe_0 - OLe_1 - (n+2)\widehat{\boldsymbol{e}}$

(iii) $\sum_{t=1}^{n} OLe_{2t-1} = OLe_{2n} + OLe_0 - OLe_1 - 2OF_{-1} - (n+2)\widehat{\boldsymbol{e}}$

(iv) $\sum_{t=1}^{n} \binom{n}{t} OLe_{t-1} = OLe_{2n-1} + OLe_0 - OLe_1 - 2^n\widehat{\boldsymbol{e}}$

(v) $\sum_{t=1}^{n} (-1)^t \binom{n}{t} OLe_{t-1} = OLe_0 - OLe_1 + 2(-1)^n OF_{-1} + \widehat{\boldsymbol{e}}$

Proof: From (3.3) and summation formulas (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), we find

(i)
$$\begin{aligned}
\sum_{t=0}^{n} OLe_t &= \sum_{t=0}^{n} (2OF_{t+1} - \widehat{\boldsymbol{e}}) \\
&= 2\sum_{t=1}^{n+1} OF_t - (n+1)\widehat{\boldsymbol{e}} \\
&= 2(OF_{n+1} + OF_{n+2} - OF_1 - OF_0) - (n+1)\widehat{\boldsymbol{e}} \\
&= 2OF_{n+3} - 2OF_2 - (n+1)\widehat{\boldsymbol{e}} \\
&= OLe_{n+1} - OLe_1 - (n+1)\widehat{\boldsymbol{e}}
\end{aligned}$$

(ii)
$$\begin{aligned}
\sum_{t=0}^{n} OLe_{2t} &= \sum_{t=0}^{n} (2OF_{2t+1} - \widehat{\boldsymbol{e}}) \\
&= 2\sum_{t=1}^{n+1} OF_{2t-1} - (n+1)\widehat{\boldsymbol{e}} \\
&= 2(OF_{2n+1} + OF_{2n} - OF_0) - (n+1)\widehat{\boldsymbol{e}} \\
&= 2OF_{2n+2} - 2OF_2 + 2OF_1 - (n+1)\widehat{\boldsymbol{e}} \\
&= OLe_{2n+1} + OLe_0 - OLe_1 - (n+2)\widehat{\boldsymbol{e}}
\end{aligned}$$

(iii)
$$\begin{aligned}
\sum_{t=1}^{n} OLe_{2t-1} &= \sum_{t=1}^{n} (2OF_{2t} - \widehat{\boldsymbol{e}}) \\
&= 2\left(\sum_{t=0}^{n} OF_{2t} - OF_0\right) - (n+1)\widehat{\boldsymbol{e}} \\
&= 2(OF_{2n+1} - OF_{-1} - OF_0) - (n+1)\widehat{\boldsymbol{e}} \\
&= OLe_{2n} + OLe_0 - OLe_1 - 2OF_{-1} - (n+2)\widehat{\boldsymbol{e}}
\end{aligned}$$

(iv)
$$\sum_{t=1}^{n} \binom{n}{t} OLe_{t-1} = \sum_{t=1}^{n} \binom{n}{t} (2OF_t - \widehat{\boldsymbol{e}})$$

$$= 2\left(\sum_{t=0}^{n}\binom{n}{t}OF_t - OF_0\right) - (2^n - 1)\hat{\boldsymbol{e}}$$
$$= 2(OF_{2n} - OF_0) - (2^n - 1)\hat{\boldsymbol{e}}$$
$$= OLe_{2n-1} + OLe_0 - OLe_1 - 2^n\boldsymbol{e}$$

(v) $\sum_{t=1}^{n}(-1)^t\binom{n}{t}OLe_{t-1} = \sum_{t=1}^{n}(-1)^t\binom{n}{t}(2OF_t - \hat{\boldsymbol{e}})$

$$= 2\left(\sum_{t=0}^{n}(-1)^t\binom{n}{t}OF_t - OF_0\right) - \left(\sum_{t=0}^{n}(-1)^t\binom{n}{t} - 1\right)\hat{\boldsymbol{e}}$$
$$= 2((-1)^n OF_{-1} - OF_0) + \hat{\boldsymbol{e}}$$
$$= OLe_0 - OLe_1 + 2(-1)^n OF_{-1} + \hat{\boldsymbol{e}}.$$

Theorem 3.4: *(Catalan's Identity) For* $n \geq 1, r \geq 1$ *and* $n \geq r$*, we get*

$$OLe_n^2 - OLe_{n+r}OLe_{n-r} = 4(-1)^{n+r+1}$$
$$[F_r^2 OL_0 - F_{2r}(OF_0 - 14\boldsymbol{e_5} - 14\boldsymbol{e_6} - 7\boldsymbol{e_7})]$$
$$+(OLe_{n+r} - OLe_n)\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}(OLe_{n-r} - OLe_n).$$

Proof: From (2.3) and (3.3), we get

$$OLe_n^2 - OLe_{n+r}OLe_{n-r}$$
$$= (2OF_{n+1} - \hat{\boldsymbol{e}})^2 - (2OF_{n+r+1} - \hat{\boldsymbol{e}})(2OF_{n-r+1} - \hat{\boldsymbol{e}})$$
$$= 4(OF_{n+1}^2 - OF_{n+r+1}OF_{n-r+1})$$
$$+(2OF_{n+r+1} - 2OF_{n+1})\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}(2OF_{n-r+1} - 2OF_{n+1})$$
$$= 4(-1)^{n+r+1}[F_r^2 OL_0 - F_{2r}(OF_0 - 14\boldsymbol{e_5} - 14\boldsymbol{e_6} - 7\boldsymbol{e_7})]$$

In Theorem 3.4, taking $r = 1$, we obtain Cassiniâ€™s identity of Leonardo Octonions OLe as follows:

Corollary 3.5: *(Cassini's Identity) For* $n \geq 1$*, we get*

$$OLe_n^2 - OLe_{n+1}OLe_{n-1} = 4(-1)^n[OL_0 - OF_0 + 14\boldsymbol{e_5} + 14\boldsymbol{e_6} + 7\boldsymbol{e_7}]$$
$$+(OLe_{n+1} - OLe_n)\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}(OLe_{n-1} - OLe_n).$$

Theorem 3.5: *(d'Ocagne's Identity) For* $n \geq 0, m \geq 1$ *and* $m > n + 1$*, we get*

$$OLe_{m+1}OLe_n - OLe_m OLe_{n+1}$$
$$= 4(-1)^{m+1}[F_{n-m}OL_0 + L_{n-m}(OF_0 - 14\boldsymbol{e_5} - 14\boldsymbol{e_6} - 7\boldsymbol{e_7})]$$
$$-(OLe_{m+1} - OLe_m)\hat{\boldsymbol{e}} - \hat{\boldsymbol{e}}(OLe_n - OLe_{n+1}).$$

Proof: From (2.4) and (3.3), we obtain

$$OLe_{m+1}OLe_n - OLe_m OLe_{n+1}$$
$$= (2OF_{m+2} - \hat{\boldsymbol{e}})(2OF_{n+1} - \hat{\boldsymbol{e}}) - (2OF_{m+1} - \hat{\boldsymbol{e}})(2OF_{n+2} - \hat{\boldsymbol{e}})$$
$$= 4(OF_{m+2}OF_{n+1} - OF_{m+1}OF_{n+2}) - (2OF_{m+2} - 2OF_{m+1})\hat{\boldsymbol{e}}$$
$$-\hat{\boldsymbol{e}}(2OF_{n+1} - 2OF_{n+2})$$
$$= 4(-1)^{m+1}[F_{n-m}OL_0 + L_{n-m}(OF_0 - 14\boldsymbol{e_5} - 14\boldsymbol{e_6} - 7\boldsymbol{e_7})]$$
$$-(OLe_{m+1} - OLe_m)\hat{\boldsymbol{e}} - \hat{\boldsymbol{e}}(OLe_n - OLe_{n+1}).$$

4. Conclusions

Catarino and Borges [3] took into account the Leonardo sequence and present various characteristics relating this sequence. The authors [5] defined the Fibonacci Octonions OF_n and Lucas Octonions OL_n. In this article, we describe the Leonardo Octonions OLe_n. This family is a new family of Octonions. We provide some relationships among the Fibonacci, Lucas and Leonardo Octonions. Then, we demonstrate generating function, Binet's formula, exponential generating function, some summation formulas and some fundamental identities of the Leonardo Octonions OLe_n.

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Received July, 2022