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# New bound of entropy with respect to refinement of the inequality of Jensen and application in graph theory

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**Abstract**

In this article, using the first development of a generalization of the Jensen's inequality, we give a newly limit for the Shannon entropy with respect to the uniformly convex functions. In other words, we obtained an accurate upper limit for this Entropy. So, it will be presented

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an example with utilization for structural complexity (SC) study for a computer network (CN) shaped using a connected graph (CG) and applications in probabilities.

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***Keywords:*** *Hermite-Hadamard inequality, s-convex functions, Holder's inequality.*

## 1. Preliminaries

The subject of entropy is used in many branches of mathematics such as statistics, code theory, information theory, dynamical systems, physics and thermodynamics. The Shannon entropy has been used by many researchers. The Shannon entropy has many applications in information theory especially for discrete-valued random variables [1,3,4,5,6,10,14], for example, in measure theory [2] having an important impact in data integration [7] and, also to probabilistic estimation [13]. In addition, many applications of Shannon entropy there are in the transmission, in the processing and also in storage of information. In [9], the authors give new generalizations of the inequality of Jensen and some applications. Recently was established new upper bounds of Shannon's entropy ([11]). In this paper, using the first development of a generalization of the inequality of Jensen, it was established a new bound for the Entropy of Shannon with respect to the uniformly convex (UC) functions. In other words, we found an accurate upper bound of the Shannon Entropy. So, we present an application for SC study for a CN, which it is shaped by a CG and applications in probabilities.

**Definition 1:** [4] $L: [c,d] \to \mathbb{R}$ is UC having the modulus $\phi_1: \mathbb{R}^+ \to [0,+\infty)$ if, in addition, $\phi_1$ is an increasing function, which vanishes only at $0$, satisfiyng:

$$L(\tau w + (1-\tau)v) \leq \tau L(w) + (1-\tau)L(v) - \tau(1-\tau)\phi_1(|v-w|),$$

for every $\tau \in [0,1]$, $w, v \in [c,d]$. Moreover, if $\phi_1(w) = c_1.w^2$ then $L$ will be named strongly convex.

**Theorem 1:** [12] *We consider $L: J \to \mathbb{R}$ as being a function which is UC having the modulus $\phi_1: \mathbb{R}^+ \to [0,+\infty]$ on $J$, $\{x_l\}_{l=1}^m \subseteq [c,d]$ being a sequence. If $\delta$ is a permutation on $\{1,\ldots,m\}$ with $x_{\delta(1)} \leq x_{\delta(2)} \leq \ldots \leq x_{\delta(m)}$, then we get*

$$L(\sum_{l=1}^m r_l x_l) \leq \sum_{l=1}^m r_l L(x_l) - \sum_{l=1}^{m-1} r_{\delta(l)} r_{\delta(l+1)} \phi_1(x_{\delta(l+1)} - x_{\delta(l)}) \qquad (1.1)$$

for convex combinations such as $\sum_{l=1}^m x_l r_l$ which contains points $x_l \in J$."

**Definition 2:** [8] The entropy of Shannon of a positive probability distribution $X = (r_1,\ldots,r_m)$ is $\mathcal{H}(X) := \sum_{i=1}^m r_i . \log\frac{1}{r_i}$."

## 2. Main results

Next theorem establishes a new generalization of the inequality of Jensen by using the functions which are uniformly convex. Considering $L$ a UC function, $\delta$ a perturbation on $\{1,2,\cdots,m\}$ which satisfy the condition

$$x_{\delta(m)} \geq \cdots \geq x_{\delta(2)} \geq x_{\delta(1)}$$

then it will be obtained:

$$
\begin{aligned}
L(\textstyle\sum_{l=1}^{m} r_l x_l) \le\ & r_{\delta(m)} L(x_{\delta(m)}) + r_{\delta(1)} L(x_{\delta(1)}) \\
& +(1 - r_{\delta(m)} - r_{\delta(1)}) L(\textstyle\sum_{l=1;l\neq 1,m}^{m} \frac{1}{1-r_{\delta(m)}-r_{\delta(1)}} x_{\delta(l)} r_{\delta(l)}) \\
& -r_{\delta(1)}(1 - r_{\delta(m)} - r_{\delta(1)}) \phi_1 (\textstyle\sum_{l=1;l\neq 1,m}^{m} \frac{1}{1-r_{\delta(m)}-r_{\delta(1)}} x_{\delta(l)} r_{\delta(l)} - x_{\delta(1)}) \\
& +r_{\delta(m)}(r_{\delta(m)} + r_{\delta(1)} - 1) \phi_1 (x_{\delta(m)} - \textstyle\sum_{l=1;l\neq 1,m}^{m} \frac{1}{1-r_{\delta(m)}-r_{\delta(1)}} x_{\delta(l)} r_{\delta(l)}) \\
& \le \textstyle\sum_{l=1}^{m} r_l L(x_l) - \sum_{l=2}^{m-2} \frac{r_{\delta(l)} r_{\delta(l+1)}}{1-r_{\delta(m)}-r_{\delta(1)}} \phi_1 (x_{\delta(l+1)} - x_{\delta(j)}) \\
& -r_{\delta(1)}(1 - r_{\delta(m)} - r_{\delta(1)}) \phi_1 (\textstyle\sum_{l\neq 1,m} \frac{x_{\delta(l)} r_{\delta(l)}}{1-r_{\delta(m)}-r_{\delta(1)}} - x_{\delta(1)}) \\
& +r_{\delta(m)}(r_{\delta(m)} + r_{\delta(1)} - 1) \phi_1 (x_{\delta(m)} - \textstyle\sum_{l=1;l\neq 1,m}^{m} \frac{1}{1-r_{\delta(m)}-r_{\delta(1)}} x_{\delta(l)} r_{\delta(l)})
\end{aligned}
$$

***Proof:*** Suppose that $x_{\delta(m)} \ge \cdots \ge x_{\delta(1)}$. We get,

$$
\begin{aligned}
L(\textstyle\sum_{l=1}^{m} x_l r_l) &= L\left(x_{\delta(m)} r_{\delta(m)} + x_{\delta(1)} r_{\delta(1)} + \sum_{l=1;l\neq 1,m}^{m} x_{\delta(l)} r_{\delta(l)}\right) \\
&= L\left(x_{\delta(m)} r_{\delta(m)} + x_{\delta(1)} r_{\delta(1)} + (1 - r_{\delta(m)} - r_{\delta(1)}) \sum_{l\neq 1,m} \frac{x_{\delta(l)} r_{\delta(l)}}{1-r_{\delta(m)}-r_{\delta(1)}}\right).
\end{aligned}
$$

From, $x_{\delta(1)} \le \cdots \le x_{\delta(m)}$ it will be obtained:

$$
x_{\delta(1)} \le \sum_{l=1,m} \frac{r_{\delta(l)} x_{\delta(l)}}{1-r_{\delta(1)}-r_{\delta(m)}} \le x_{\delta(m)}.
$$

In view [[12],Theorem 2.3] it will be obtained:

$$
\begin{aligned}
& L(\textstyle\sum_{l=1}^{m} r_l x_l) = L(r_{\delta(1)} x_{\delta(1)} + r_{\delta(m)} x_{\delta(m)} \\
& +(1 - r_{\delta(m)} - r_{\delta(1)}) \left(\sum_{l=1;l\neq 1,m}^{m} \frac{x_{\delta(l)} r_{\delta(l)}}{1-r_{\delta(m)}-r_{\delta(1)}}\right) \\
& \le L(x_{\delta(m)}) r_{\delta(m)} + r_{\delta(1)} l(x_{\delta(1)}) + (1 - r_{\delta(m)} - r_{\delta(1)}) l \left(\frac{1}{1-r_{\delta(m)}-r_{\delta(l)}} x_{\delta(l)} r_{\delta(l)}\right) \\
& +r_{\delta(1)}(r_{\delta(1)} + r_{\delta(m)} - 1) \phi_1 \left(\sum_{l=2}^{m-1} \frac{1}{1-r_{\delta(m)}-r_{\delta(1)}} x_{\delta(l)} r_{\delta(l)} - x_{\delta(1)}\right) \\
& -r_{\delta(m)} . (1 - r_{\delta(m)} - r_{\delta(1)}) \phi_1 \left(x_{\delta(m)} - \sum_{l=2}^{m-1} \frac{1}{1-r_{\delta(m)}-r_{\delta(1)}} x_{\delta(l)} r_{\delta(l)}\right) \qquad (2.1)
\end{aligned}
$$

Also, since

$$
\sum_{l=2}^{m-1} \frac{1}{1-r_{\delta(m)}-r_{\delta(1)}} r_{\delta(l)} = 1,
$$

according to Theorem 2.3 in [12] we have

$$
\begin{aligned}
L\left(\sum_{l=1;l\neq 1,m}^{m} \frac{x_{\delta(l)} r_{\delta(l)}}{1-r_{\delta(m)}-r_{\delta(1)}}\right) &\le \sum_{l=1;l\neq 1,m}^{m} \frac{r_{\delta(l)}}{1-r_{\delta(m)}-r_{\delta(1)}} L(x_{\delta(l)}) \\
& - \sum_{l=2}^{m-2} \frac{1}{(1-r_{\delta(m)}-r_{\delta(1)})^2} r_{\delta(l+1)} r_{\delta(l)} \phi_1 (x_{\delta(l+1)} - x_{\delta(l)}). \qquad (2.2)
\end{aligned}
$$

Finally, in view of relation (2.1) and (2.2) the proof of theorem will be complete.

**Corollary 1:** *When the map $l$ is strongly convex, when $\delta$ is a perturbation $\{1,2,\cdots,m\}$ such that*

$$x_{\delta(1)} \leq x_{\delta(2)} \leq \cdots \leq x_{\delta(m)}$$

*then we have*

$$\begin{aligned}
L(\textstyle\sum_{l=1}^{m} r_l x_l) \leq\ & r_{\delta(1)}L(x_{\delta(1)}) + r_{\delta(m)}L(x_{\delta(m)}) \\
&+(1-r_{\delta(m)}-r_{\delta(1)})L\left(\textstyle\sum_{l=2}^{m-1}\frac{1}{1-r_{\delta(m)}-r_{\delta(1)}}x_{\delta(l)}r_{\delta(l)}\right) \\
&+c_1 r_{\delta(1)}(r_{\delta(1)}+r_{\delta(m)}-1)\left(\textstyle\sum_{l=2}^{m-1}\frac{1}{1-r_{\delta(m)}-r_{\delta(1)}}x_{\delta(l)}r_{\delta(l)}-x_{\delta(1)}\right)^2 \\
&+c_1 r_{\delta(m)}(r_{\delta(1)}+r_{\delta(m)}-1)\left(x_{\delta(m)}-\textstyle\sum_{l=2}^{m-1}\frac{1}{1-r_{\delta(1)}-r_{\delta(m)}}x_{\delta(l)}r_{\delta(l)}\right)^2 \\
&\leq \textstyle\sum_{l=1}^{m} L(x_l)r_l - c_1\sum_{l=2}^{m-2}\frac{1}{1-r_{\delta(m)}-r_{\delta(1)}}.r_{\delta(l)}r_{\delta(l+1)}(x_{\delta(l)}-x_{\delta(l+1)})^2 \\
&+c_1 r_{\delta(1)}(r_{\delta(1)}+r_{\delta(m)}-1)\left(\textstyle\sum_{l=2}^{m-1}\frac{1}{1-r_{\delta(m)}-r_{\delta(1)}}x_{\delta(l)}r_{\delta(l)}-x_{\delta(1)}\right)^2 \\
&+c_1 r_{\delta(m)}(r_{\delta(1)}+r_{\delta(m)}-1)\left(x_{\delta(m)}-\textstyle\sum_{l=2}^{m-1}\frac{1}{1-r_{\delta(m)}-r_{\delta(1)}}x_{\delta(l)}r_{\delta(l)}\right)^2.
\end{aligned}$$

***Proof:*** We take $\phi_1(x) = c_1 x^2$ in Theorem 2.

**Lemma 1:** *We consider the function $L:[c,d] \to \mathbb{R}$ given by $L(x) = log\frac{1}{x}$ also $c > 0$. The function $L$ will be strongly convex (SC).*

***Proof:*** Set modulus $\phi_1(r) = \frac{1}{2d^2}r^2$. The function $\phi_1$ will be increasing, and $\phi_1$ will vanish, but only at $0$. Furthermore it will be considered the fixed points $w, v$ in the interval $[c,d]$. Then it will be given the function,

$$\begin{aligned}
g(\tau) := (1-\tau).\log(v) + \tau.\log(w) & -\log((1-\tau).v+\tau.w) \\
&+\tau.\frac{(v-w)^2}{2d^2}.(1-\tau),
\end{aligned}$$

for every $\tau$ in the interval $[0,1]$. In this point, it will be shown that $0 \geq g(\tau)$, for every $\tau$ in $[0,1]$. By using that $g(0)=0, g(1)=0$, $0 \leq \frac{d^2 g}{d\tau^2} = \frac{(v-w)^2}{(\tau w+(1-\tau)v)^2} - \frac{(w-v)^2}{d^2}$, we have $0 \geq g(\tau)$, for every $w, v \in [c,d]$ when $\tau$ is in interval $[0,1]$. Therefore we get,

$$\tau.(1-\tau)\frac{(v-w)^2}{2d^2} + \log\left(\frac{1}{\tau w+(1-\tau)v}\right) \leq \tau\log\left(\frac{1}{w}\right) + (1-\tau)\log\left(\frac{1}{v}\right).$$

**Lemma 2:** *We will consider a function $\Phi_2 : I \to \mathbb{R}$. If this function is twice-differentiable with*

$$m := \inf\{\Phi_2''(x) | x \in I\} > 0,$$

*then the function $\Phi_2$ will be a UC function having the modulus $\varphi(u) = \frac{m}{2}u^2$.*

***Proof:*** The function $\varphi$ is increasing. Moreover, $\varphi$ will vanish only at $0$. It will be considered two fixed points $c_1, d_1 \in I$, and it will be defined the function,

$$\varphi(\lambda) := \lambda\Phi_2(c_1) + \Phi_2(d_1)(1-\lambda) - \Phi_2(\lambda.c_1 + d_1.(1-\lambda)) - \frac{\lambda.(1-\lambda)m}{2}(d_1 - c_1)^2$$

for all $\lambda \in [0,1]$. It will be shown that $\varphi(\lambda) \geq 0$, when $\lambda$ in $[0,1]$. Now using that $\varphi(0) = 0, \varphi(1) = 0$,

$$\frac{d^2\varphi}{d\lambda^2} = m(c_1 - d_1)^2 - (d_1 - c_1)^2\Phi_2''(\lambda c_1 + (1-\lambda)d_1) \leq 0,$$

$\Phi_2(\lambda) \geq 0$ for every $c_1, d_1 \in [\mu, \nu]$, $\lambda \in [0,1]$. Thus,

$$\lambda.\Phi_2(c_1) + (1-\lambda).\Phi_2(d_1) \geq \Phi_2(\lambda.c_1 + d_1.(1-\lambda)) + \frac{\lambda.(1-\lambda)m}{2}(d_1 - c_1)^2.$$

Therefore, the proof is complete.

## 3. New Entropy Bound

In this section, we apply Lemma 1 obtaining new strong bounds of Shannon's entropy.

**Theorem 3:** *If* $r_1 \leq \cdots \leq r_m$ *with* $\sum_{l=1}^{m} r_l = 1$ *then*

$$\begin{aligned}\mathcal{H}(X) + \frac{r_1^2}{2(1-r_1-r_m)}\sum_{l=2}^{m-2}\frac{(r_{l+1}-r_l)^2}{r_l r_{l+1}} &\leq \log\left(\frac{m-2}{1-r_1-r_m}\right)^{1-r_1-r_2}\left(\frac{1}{r_1}\right)^{r_1}\left(\frac{1}{r_m}\right)^{r_m} \\ &\leq \log m - \frac{r_1(1-r_m-mr_1+r_1)^2}{2(1-r_1-r_m)} \\ &\quad - \frac{r_1^2(mr_m-r_m+r_1-1)^2}{2r_m(1-r_1-r_m)}\end{aligned}$$

***Proof:*** Assume that $r_1 \leq \cdots \leq r_m$. Consider the function $f(x) = -\log x$ on $\left[\frac{1}{r_m}, \frac{1}{r_1}\right]$. According to Lemma 1, $f$ is SC on $\left[\frac{1}{r_m}, \frac{1}{r_1}\right]$ with modulus $c_1 = \frac{r_1^2}{2}$. Now, in Corollary 1 set $f(x) = -\log x, c_1 = \frac{r_1^2}{2}, x_l = \frac{1}{r_l}$ for $1 \leq l \leq m$. Hence, we have

$$\frac{1}{r_m} \leq \cdots \leq \frac{1}{r_1}.$$

So, perturbation $\delta$ on $\{1,2,\cdots,m\}$ could be $\delta(k) = m-k+1$. So,

$$\begin{aligned}-\log m \leq\ & -r_m\log\left(\frac{1}{r_m}\right) - r_1\log\left(\frac{1}{r_1}\right) \\ & -(1-r_m-r_1)\log\left(\frac{m-2}{1-r_m-r_1}\right) \\ & -\frac{r_1^2 r_m}{2}(1-r_m-r_1)\left(\frac{m-2}{1-r_m-r_1} - \frac{1}{r_m}\right)^2 \\ & -\frac{r_1^2}{2}r_1(1-r_m-r_1)\left(-\frac{m-2}{1-r_m-r_1} + \frac{1}{r_1}\right)^2 \\ & \leq -\sum_{l=1}^{m} r_l\log\left(\frac{1}{r_l}\right) - \frac{r_1^2}{2}\sum_{l=2}^{m-2}\frac{r_{m+1-l}r_{m-l}}{1-r_m-r_1}\left(\frac{1}{r_{m+1-l}} - \frac{1}{r_{m-l}}\right)^2\end{aligned}$$

by some calculus we obtain the result.

In proof we use the following equality:

$$\sum_{l=2}^{m-2} \frac{r_{m+1-l}r_{m-l}}{1-r_m-r_1}\left(\frac{1}{r_{m+1-l}}-\frac{1}{r_{m-l}}\right)^2 = \frac{1}{1-r_1-r_m}\sum_{l=2}^{m-2} \frac{(r_l-r_{l+1})^2}{r_l r_{l+1}}.$$

**Lemma 3:** *Assume that* $d > c \geq 0$, $d \geq x_l \geq c, l = 1, \cdots, m$, *and also* $L$ *is SC having the modulus* $c_1$. *Then we obtain*

$$\begin{aligned} L\left(\frac{\sum_{l=0}^{m+1} x_l}{2+m}\right) &+ \frac{c_1 m}{(2+m)^2}\left[\left(\frac{\sum_{l=1}^{m} x_l}{m} - c\right)^2 + \left(d - \frac{\sum_{l=1}^{m} x_l}{m}\right)^2\right] \\ &\leq \frac{L(c)+L(d)}{2+m} + \frac{m}{2+m} L\left(\frac{\sum_{l=1}^{m} x_l}{m}\right) \\ &\leq \frac{1}{2+m}\sum_{l=0}^{m+1} L(x_l). \end{aligned}$$

***Proof:*** We suppose that

$$c = x_0 \leq x_{\delta(1)} \leq \cdots \leq x_{\delta(m)} \leq x_{m+1} = d,$$

without to loss of generality. Put $\delta(0) = 0$, $\delta(m+1) = m+1$ we have

$$x_{\delta(0)} \leq x_{\delta(1)} \leq \cdots \leq x_{\delta(m)} \leq x_{\delta(m+1)}.$$

Also, put $r_l = \frac{1}{2+m}$ for $m+1 \geq l \geq 0$ in Corollary 1 we get

$$\begin{aligned} L\left(\frac{1}{2+m}\sum_{l=0}^{m+1} x_l\right) &\leq \frac{L(c)+L(d)}{2+m} + \frac{m}{2+m} L\left(\frac{1}{m}\sum_{l=1}^{m} x_l\right) \\ &- \frac{c_1.m}{(2+m)^2}\left[\left(\frac{1}{m}\sum_{l=1}^{m} x_l - c\right)^2 + \left(d - \frac{1}{m}\sum_{l=1}^{m} x_l\right)^2\right] \\ &\leq \frac{1}{2+m}\sum_{l=0}^{m+1} L(x_l) - \frac{1}{(m+2)m}\sum_{l=1}^{m-1} \left(x_{\delta(l+1)} - x_{\delta(l)}\right)\big)^2 \\ &- \frac{c_1.m}{(m+2)^2}\left[\left(\frac{1}{m}\sum_{l=1}^{m} x_l - c\right)^2 + \left(d - \frac{1}{m}\sum_{l=1}^{m} x_i\right)^2\right] \\ &\leq \frac{1}{2+m}\sum_{l=0}^{m+1} L(x_l) - \frac{c_1.m}{(2+m)^2}\left[\left(\frac{1}{m}\sum_{l=1}^{m} x_l - c\right)^2 + \left(d - \frac{1}{m}\sum_{l=1}^{m} x_l\right)^2\right], \end{aligned}$$

which end the proof.

## 4. Application in statistical theory

It will be given here several application with respect to SC functions. Also, we use the following notation in the next: Assume that $x_l \in [c, d], 1 \leq l \leq m$, $0 \leq c < d$. Let $X = \{x_1, \cdots, x_m\}$, $\tilde{X} = \{c, x_1, \cdots, x_m, d\}$, $\sqrt{X} := \{\sqrt{x_1}, \cdots, \sqrt{x_m}\}$ and $\mathcal{E}(L(X)) := \frac{1}{m}\sum_{l=1}^{m} L(x_l)$.

**Proposition 1:** Let $L$ be SC having modulus $c_1$. We get,

$$\begin{aligned} L(\mathcal{E}(\tilde{X})) &+ \frac{c_1 m}{(2+m)^2}\left[(\mathcal{E}(X) - a)^2 + (b - \mathcal{E}(X))^2\right] \\ &\leq \frac{1}{m+2}\left[L(c) + L(d) + mL(\mathcal{E}(X))\right] \\ &\leq \mathcal{E}(L(\tilde{X})). \end{aligned}$$

***Proof:*** A direct utilization of Lemma 3 will lead us to conclusions.

**Lemma 4:** *It will be considered* $L:[c,d]\to\mathbb{R}$ *given by* $L(x)=-x^{\frac{1}{2}}$. *The function* $L$ *is SC having the modulus* $\frac{1}{8}d^{-\frac{3}{2}}x^2$.

***Proof:*** We suppose that $d\geq\beta>\alpha\geq c$ are fixed. We take $h:[0,1]\to\mathbb{R}$ by

$$h(s)=(\alpha.s+(1-s).\beta)^{\frac{1}{2}}-s.\alpha^{\frac{1}{2}}-(1-s).\beta^{\frac{1}{2}}-\frac{1}{8}.s.(1-s).\frac{(\beta-\alpha)^2}{d^{\frac{3}{2}}}.$$

Here we get $h(0)=0$, and $h(1)=0$,

$$h''(s)=\frac{1}{4}(\beta-\alpha)^2(d^{-\frac{3}{2}}-(\alpha s+\beta(1-s))^{-\frac{3}{2}})<0,$$

for all $1>s>0$. Hence, the result is obtained.

**Proposition 2:** If $d>c\geq 0$, $d\geq x_l\geq c, l=1,\cdots,m$ then it will be obtained:

$$\mathcal{E}\left(\sqrt{\tilde{X}}\right)\leq\frac{\sqrt{c}+\sqrt{d}+m\mathcal{E}^{\frac{1}{2}}(X)}{2+m}$$
$$\leq\mathcal{E}^{\frac{1}{2}}(\tilde{X})-\frac{md^{-\frac{3}{2}}}{m(m+2)^2}[(\mathcal{E}(X)-c)^2+(d-\mathcal{E}(X))^2]$$

***Proof:*** According to Lemma 4, the function $f(x)=-\sqrt{x}$ is a SC with modulus $c_1=\frac{1}{8}d^{-\frac{3}{2}}$. Now, in Lemma 3 put $f(x)=-\sqrt{x}$ and $c_1=\frac{1}{8}d^{-\frac{3}{2}}$ the results obtained.

## 5. Application in graph theory

Here, we improved bound of Shannon Entropy network which modeled by Popescu et al. in [9]. We have a graph as follows: $G=(\mathcal{V},\mathcal{E})$, so that $\mathcal{V}$ - the set of nodes as $|\mathcal{V}|=n$, and $\mathcal{E}$ - the set of edges as ($|\mathcal{E}|=m$). If the edge $(n_j,n_k)\in\mathcal{E}$, then two nodes $n_j$ and $n_k$ are in a relation. The edges $(n_k,n_j)$ and $(n_j,n_k)$ are equivalent. The number of edges which are incident to the node $n_i$: $d(n_i)=\sum_{j=1}^{n}\delta(n_i,n_j)$, determines the degree of $n_i$. If $(n_i,n_j)\in\mathcal{E}$, then $\delta(n_i,n_j)=1$, if else, $\delta(n_i,n_j)=0$.

The probability that a node be “in relation” is defined by:

$$r_i=r(n_i)=\frac{d(n_i)}{\sum_{n_j\in\mathcal{V}}d(n_j)}.$$

In the case of connected graphs we have $\sum_{n_j\in\mathcal{V}}d(n_j)=2m$, so $r_i=\frac{1}{2m}d(n_i)$. The structural stability of the graph is shown by graph entropy. The random node re-connections increase the entropy. Therefore this decreases the stability. We have the Shannon entropy as:

$$\mathcal{H}(G):=\sum_{i=1,n_i\in\mathcal{V}}r(n_i)\log\frac{1}{r(n_i)}.$$

Using Theorem 3 we obtain next bound for $\mathcal{H}(G)$:

$$\mathcal{H}(G) \leq \log n - \frac{r_1(1-r_n-nr_1+r_1)^2}{2(1-r_1-r_n)} - \frac{r_1^2(nr_n-r_n+r_1-1)^2}{2r_n(1-r_1-r_n)}$$
$$-\frac{2^{-1}r_1^2}{1-r_1-r_n}\sum_{l=2}^{n-2}\frac{(r_{l+1}-r_l)^2}{r_l r_{l+1}}.$$

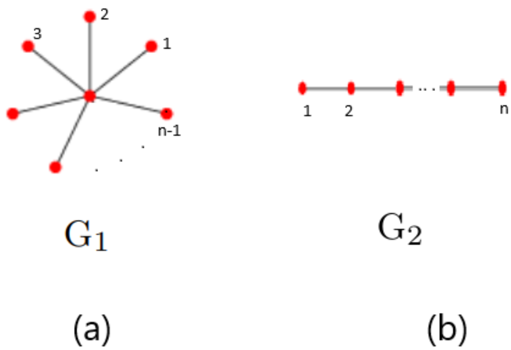


**Figure 1**

**The graphs for $G_1$ and $G_2$.**

**Example 1:** Two graphs $G_1$ and $G_2$ with given vertices are assumed. Now we check the $\mathcal{H}(G_i)$ and $I(G_i)$, $i = 1,2$ bounds for these two graphs with the table of values and the graph.

For the graphs and tables of $G_1, G_2, \mathcal{H}(G_1)$ and $I(G_1)$, see Table 1 then Table 2, and then Figure 1, and Figure 2.

**Table 1**

**Table of $G_1$**

| $n$ | Entropy $\mathcal{H}(G_1)$ | Entropy bounds $I(G_1)$ |
|---|---|---|
| 3 | 0.451 | 0.461 |
| 5 | 0.602 | 0.681 |
| 10 | 0.778 | 0.989 |
| 15 | 0.874 | 1.168 |
| 20 | 0.94 | 1.295 |
| 25 | 0.991 | 1.393 |
| 30 | 1.032 | 1.473 |
| 35 | 1.066 | 1.54 |
| 40 | 1.096 | 1.599 |

**Table 2**

**Table of $G_2$**

| $n$ | Entropy $\mathcal{H}(G_2)$ | Entropy bounds $I(G_2)$ |
|---|---|---|
| 3 | 0.451 | 0.398 |
| 5 | 0.677 | 0.686 |
| 10 | 0.987 | 0.994 |
| 15 | 1.167 | 1.172 |
| 20 | 1.294 | 1.297 |
| 25 | 1.392 | 1.395 |
| 30 | 1.472 | 1.475 |
| 35 | 1.54 | 1.542 |
| 40 | 1.598 | 1.6 |

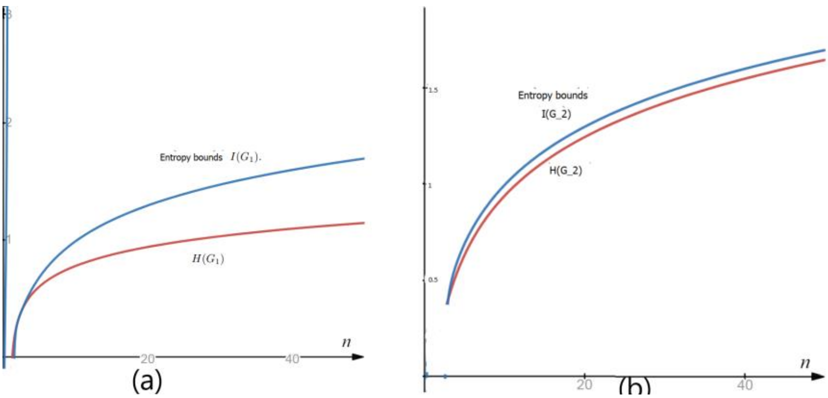


**Figure 2**
**The graphs for $\mathcal{H}(G_1)$ and $I(G_1)$.**

## 8. Conclusion

Using the first extension of Jensen's inequality modification, we give a new bound of the Shannon entropy with respect to uniformly convex functions. In other words, we find an exact upper bound in the case of the entropy of Shannon. For better understanding, two examples of different graphs are provided with the table of results along with the comparison of the limits, which shows the differentiation of the limits.

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