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Local and global information in online stochastic shortest path problem and competitive analysis

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Abstract

In the online shortest path problem, the arc costs are the online parameters in a network and their values are not known for decision-makers in advance. So, the online decisions are made by arriving any node and with respect to some statistical information of the leaving nodes and not by the realization of the arc costs; however, the decisions for the traversed paths are not changed or rejected, and they are established permanently. The online decision criteria are defined as expected path lengths related to the available statistical information in the online manner. The online expected stochastic shortest path lengths are computed by two novel online adapted formulas. Then, the competitive analyses are presented for the obtained online stochastic optimal solution against the offline optimal solution. The expected competitive ratios are revealed differences between the application of some local and global information, so that it shows e–2 improvement for the local statistical information against the

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globalb statistical information, relatively. Numerical results and the average case analyses compared the obtained formulas as the online optimality indices. The formulas are applied in both acyclic and cyclic networks, and the obtained competitive ratios are verified by some numerical examples.

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Keywords: *Combinatorial optimization, Stochastic networks, Online routing, Online algorithm, Competitive ratio.*

1. Introduction

In the considered online stochastic shortest path problem, decisions are made based on some statistical information in an online manner; however, there is neither a function nor probability distribution to describe the unknown parameters of the network. The arc costs (and the path lengths) and the topology of the network are online parameters, so some online policies reveal the adjacency nodes by arrival any node. Thus, the online made decisions are not allowed to be changed or rejected (otherwise, there will be some penalties). The obtained online solutions are evaluated by the competitive analysis, and the obtained competitive ratio determines the optimality index of the obtained online solution against the optimal offline solution.

Local and global information interaction has been considered in some applications [4, 7, 12, 15, 18]. The advanced information play an important role in the traffic control for path planning problems [7, 10]. Chen et al. [10] compared the strategies to apply global local information, and determined a global algorithm who applies global information has an improved performance against a local one. Rout guidance The rout guidance strategies need to analyses the traffic data and provide a high qualified routing plan [2]. Information feedback has a critical role to avoid traffic congestion [11, 12, 21]. Chen et al. [11] studied that providing some partial information of a road to travelers could improve the performance of a routing system.

Kalaia and Vempala [14] determined a path between two nodes, and then the times on all arcs are revealed. Provan [17] considered the stochastic traversal lengths of the arcs, and they become known upon arrival at the tail nodes. Sever et al. [20] applied a discrete time finite horizon Markov decision process to model a dynamic stochastic shortest path problem. Ardakani and Sun [2] applied some advanced traveler information systems according to the stochastic realization of the arc costs. György et al. [13]

considered a weighted directed acyclic graph, and the arc weights are changed in an adversarial way. Awerbuch and Kleinberg [3] assumed unknown arc delays during routing sequential paths in a network. Balogh et al. [5] considered the online bin packing problem and they proposed an optimal competitive ratio.

In our considered model, the arc costs are the online parameters of the network; so, we apply some statistical information according to some local and global decision criteria. The online policies determine the permitted nodes, and they are generally defined as the adjacency nodes of the traversed nodes. Therefore, the topology of the network is revealed gradually where the arc costs are not known.

Provan [17] considered the model that the arc costs are taken from a given strictly increasing finite set of numbers with given probabilities. Kalaia and Vempala [14] wanted to ensure that their total cost is not much larger than the minimum total cost of any path. Sever et al. [20] applied dynamic programming to formulate and evaluate online and offline policies. Ardakani and Sun [2] tried to reduce optimization time by estimation of travel times. György et al. [13] designed a game where a decision maker had to choose a path between two distinguished vertices in each round, such that the loss of the selected path is as small as possible.

The competitive ratio presented by Awerbuch and Kleinberg [3] is bounded by the arc number (m) and the number of the most permitted arcs to use so that it could attain the bound $m^{\frac{8}{3}};$ in the best permutation of their presented method based on the online linear algorithm, the bound is m^2. Our obtained competitive ratios are bounded with m in both local and global formulations.

Here, the considered model is a routing planning model in the lack of information about either arc costs or path lengths. The contribution of this study are listed as the following highlights:

- An online shortest path routing model is developed, where there is not access to the information about the arc costs or the path lengths.
- The preference of the local information of the arc costs and the global information of the path lengths during routing process are analysed competitively by some theorems, for cyclic and acyclic networks.
- The expected competitive ratios are computed in the both cases locally and globally.

- It is shown that the application of local statistical information produces e^{-2}.

The paper is organized as follow: section 2 describes the routing model of the problem, section 3 produces the competitive analysis of the local and global routing policies, and section 4 presents the numerical results of the proposed online algorithm implementations on both cyclic and acyclic networks of different sizes.

2. The online stochastic shortest path problem

There are some online routing policies and they show some nodes and arcs which are permitted to traverse at the current policy time. In the lack of the costs' values, we consider the advanced statistical information, and decisions are made to obtain the expected optimal value. So, the arc costs are not known during the online routing process and the next node is chosen by some available local and global statistical information. It is not considered a special distribution function for statistical retrieved information; however, the uniform distribution function is applied to produce some statistical values as the numerical instances. Thus, two formulas are considered and they are applied as the optimality indices for the online stochastic shortest path (O-SSP). It is not allowed to reject previously traversed nodes, although they could be traversed several times (in a cyclic network).

Awerbuch and Kleinberg [3] used some partial information of the arcs and paths in the network. Sever et al. [20] assumed traveler gets the real-time information about the state of the network before arriving at each node; in their model, the travel times can be predictable from historical data and retrieved from online data. In our studied model, the arc costs are online parameters and decisions are made according to the statistical information of the paths from the traversed nodes toward the destination node or the leaving arcs of the traversed nodes according to the global or local information, respectively. Kalaia and Vempala [14] defined some periods for the online made decisions and in each period they chose the decision which has done best so far; they used some realizations of the lengths of the arcs in each period.

Let $G=(N,A)$ be a directed network with the node set N and the arc set A and $s,d \in N$ are the original source and destination nodes, respectively; Π is the policy set and $\pi_t \in \Pi$ is the policy of time t and determines the permitted nodes, for $t=1,2,\ldots,|\Pi|$, in online manner. Let

A_t and N_t denote the arcs and nodes of the policy time t, π_t, and s_t and d_t are the source and destination nodes of π_t, respectively. For any $(i,j) \in A_t$, the decision variables of policy π_t are shown by $x_{ij}^{\pi_t} \in \{0,1\}$, and $c_{ij}^{\pi_t} > 0$ is the online cost of the arc (i,j) that is retrieved from the statistical information of the network; so, $u_i^{\pi_t}$ is the expected length of the online shortest path from node i given by the policy π_t toward the original destination node d; it is assumed that there is a path from any node toward node d. The initial policy π_1 and the last policy $\pi_{|\Pi|}$ consist of the original source node s and the original destination node d, respectively. The other policies π_t, $1 < t < |\Pi|$, are constructed of the leaving nodes of the traversed nodes in the previous policy π_{t-1}. The policies could be given dynamically [13, 20] or by a probability distribution function.

The online decision variables should satisfy the following local feasibility condition for any $1 \le t \le |\Pi|$

$$\forall i \in N_t : \sum_{j:(i,j)\in A_t} x_{ij}^{\pi_t} - \sum_{j:(j,i)\in A_t} x_{ji}^{\pi_t} = \begin{cases} 1 & i = s_t \\ 0 & \text{otherwise} \\ -1 & i = d_t \end{cases} \tag{1}$$

Inductively, one unit flow is sent from the original source node to the original destination node by intermediate source and destination nodes according to constraint (1). So, any feasible solution for the online problem is a feasible solution for the offline problem; finally, the following global feasibility conditions (2) are satisfied by time t

$$\forall i \in \bigcup_{r=1}^{t} N_r : \sum_{j:(i,j)\in \cup_{r=1}^{t} A_r} x_{ij}^{t} - \sum_{j:(j,i)\in \cup_{r=1}^{t} A_r} x_{ji}^{t} = \begin{cases} 1 & i = s \\ 0 & \text{otherwise} \\ -1 & i = d_t \end{cases} \tag{2}$$

where x_{ij}^t is the exact offline value of the decision variable related to arc (i,j) by time t. However, it is not allowed to return from the current traversed nodes and the solution process should be continued by policy π_t, but it could be traversed several times. Notice, according to the constraints (2), one unit flow is sent from the original source node to the current destination node and the final solution is feasible for the offline problem, inductively.

In the online model, the values of the decision variables are determined according to the policy set Π, such that they construct a shortest path from the original source node to the original destination node, finally. The expected length of the O-SSP is minimized with respect to the available statistical information and it is applied to determine which node will be the next node to traverse. So, contrary to the stochastic and real-time models [4, 17, 20], the realization of the arc costs is possible in the online problems.

Whenever one node is allowed to traverse by an online policy, it should satisfy constraint (1). So, if $x_{ij}^{\pi_t} = 1$ for $(i, j) \in A_t$, then there is a dual constraint that is in equal form because of the complementary slackness conditions (see [5, 9]). Consequently, the corresponding dual form of constraint (1) is $u_i^{\pi_t} - u_j^{\pi_t} \le c_{ij}^{\pi_t}$ for $(i, j) \in A_t$, if $x_{ij}^{\pi_t} = 1$ then $u_i^{\pi_t}$ is updated by $u_j^{\pi_t} + c_{ij}^{\pi_t}$. Then, in the optimal solution $c_{ij}^{\pi_t} = u_j^{\pi_t} - u_i^{\pi_t}$ is an estimation value of the online parameters $c_{ij}^{\pi_t}$ with respect to the expected values of $u_i^{\pi_t}$ and $u_j^{\pi_t}$.

For two networks with the same number of nodes, the policy sets in the acyclic network contains more distinct nodes than the cyclic network. Node i is a reached node if there is a path from the original source node that traverses the node t by time $t-1$ according to constraint (2). In an online manner, there are two kinds of update states for arc $(i, j) \in A_t$ and $i, j \in N_t$:

(a) $i \in \bigcup_{r=1}^{t-1} N_r$ is a reached node but $j \notin \bigcup_{r=1}^{t-1} N_r$ is released by π_t as seen in Figure 1. State (a) shows a node could be reached from a previously reached node.

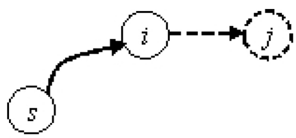

Figure 1

State (a) for online released nodes and arcs.

b) $i, j \in \bigcup_{r=1}^{t-1} N_r$ both are released by π_t as seen in Figure 2. If there is a cycle in the network then it may cause the states (b1) and (b2). In the state (b1) arc (i, j) is traversed for the first time, but in the state (b2)

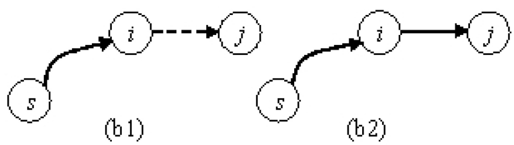

Figure 2

States (b1) and (b2) for online released nodes and arcs.

arcs and nodes are traversed several times. However, there is not any negative cycle (a cycle with negative cost) in the network that causes to infinity optimal solution.

So, the states in the cyclic networks contain some repeated nodes those are traversed previously, and then the policy sets of the acyclic networks expand the search area to find a better solution.

Two formulations of the O-SSP are presented to compute the expected length. In the first one, it uses the statistical information of the arc costs; then, the expected length is computed as follows

$$\begin{aligned} E\left(\sum_{t=1}^{|\Pi|}\sum_{(i,j)\in A_t} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right) &= \sum_{t=1}^{|\Pi|}\sum_{(i,j)\in A_t} E(c_{ij}^{\pi_t}) x_{ij}^{\pi_t} \\ &= \sum_{t=1}^{|\Pi|}\sum_{(i,j)\in A_t} E(u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t} \\ &= \sum_{t=1}^{|\Pi|}\sum_{(i,j)\in A_t} P_c(u_i^{\pi_t} - u_j^{\pi_t}) \times (u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t} \end{aligned} \tag{3}$$

P_c is the probability distribution of the arc cost for any $(i,j)\in A_t$, and it is retrieved from the local statistical information. For the second formulation, the O-SSP length from the departure nodes is independent of the current arc cost, then we apply the conditional probability (see [19])

$$P(u_i^{\pi_t} \mid c_{ij}^{\pi_t} = u_i^{\pi_t} - u_j^{\pi_t}) = P_c(u_i^{\pi_t} - u_j^{\pi_t}) P_u(u_i^{\pi_t})$$

P_u is the probability distribution of the O-SSP length from the departure node i to the destination node d, and it is retrieved from the global statistical information. So, the expected length is computed as follows

$$\begin{aligned} E\left(\sum_{t=1}^{|\mathrm{P}|}\sum_{(i,j)\in A_t} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right) &= \sum_{t=1}^{|\mathrm{P}|}\sum_{(i,j)\in A_t} E(c_{ij}^{\pi_t}) x_{ij}^{\pi_t} \\ &= \sum_{t=1}^{|\mathrm{P}|}\sum_{(i,j)\in A_t} E(u_i^{\pi_t} \mid c_{ij}^{\pi_t} = u_i^{\pi_t} - u_j^{\pi_t})(u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t} \\ &= \sum_{t,i,j} P_c(u_i^{\pi_t} - u_j^{\pi_t}) P_u(u_i^{\pi_t})(u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t} \end{aligned} \tag{4}$$

The first formulation (3) applies the local information of the arc costs, but in the second one (4) the global information of the O-SSP length from the current nodes toward the destination node is applied. In both formulations, decisions are made according to the minimum expected length.

Consider the example network in Figure 3; it is a cyclic network with 6 nodes and 10 arcs. The policy set of the example network (Figure 3) is shown in Table 1 and the nodes of policy π_{t+1} are released by arrival any nodes of policy π_t, for $t = 1, 2, 3, \ldots, 5$. Node 1 is the original source node and node 6 is the original destination node, so $\pi_1 = \{1\}$ and $\pi_6 = \{6\}$; the other policies π_2, π_3, π_4, π_5 are represented in Table 1. The policies determine the available information of the arrival nodes. By arrival any node in the policy time *t*, it is allowed to access some statistical local and global information of the arc costs and the path lengths, respectively; so, the expected length of the O-SSP from the source node 1 into the destination node 6 is computed by the equations (3) for the case local arc costs released information, and (4) for the case global path lengths released information.

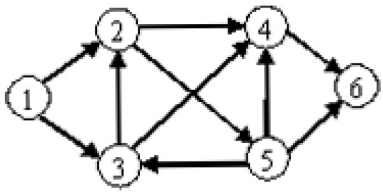

Figure 3

The example network

Table 1

The policy set of the example network.

Policies	π_1	π_2	π_3	π_4	π_5	π_6
Nodes	{1}	{2,3}	{2,4,5}	{3,4,5,6}	{2,3,4,6}	{6}

3. The online algorithm and the Competitive analysis

The online algorithms are applied to solve the online problems and their performances are measured by the competitive analysis. Obviously, the online solution is not better than the optimal offline solution, so the optimality index is presented by the competitive ratio. The approximate complementary slackness theorem (Theorem 1) provided the relationship between a feasible solution and the optimal solution [6].

Theorem 1 (Bazaraa et al. [6]): *Let* $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *are feasible solutions for the primal and the dual of the linear optimization problems, respectively:*

- Primal complementary slackness: there is a $\alpha > 1$ that for any i, $1 \le i \le n$, if $x_i > 0$ then $\frac{c_i}{\alpha} \le \sum_{j=1}^{m} a_{ij} y_j \le c_i$
- Dual complementary slackness: there is a $\beta > 1$ that for any j, $1 \le j \le m$, if $y_j > 0$ then $b_j \le \sum_{i=1}^{n} a_{ij} x_i \le \beta b_j$.

Therefore, there is ratio $\alpha\beta$ and by weak duality the following inequalities are obtained

$$\sum_{i=1}^{n} c_i x_i \le \alpha\beta \sum_{j=1}^{m} b_j y_j \le \alpha\beta \sum_{i=1}^{n} c_i x_i^*$$

where x_i and y_j are the feasible primal and dual variables, c_i and b_j are the primal and dual objective function coefficients, respectively, and x_i^* are the optimal primal solutions. Notice, the adjacency matrix of the shortest path problem is unimodular, so the integer constraint could be relaxed [1]. Then, by Theorem 1, there is a ratio for any feasible online made path and the optimal shortest path which is called the competitive ratio.

Let ALG be an online algorithm and OPT is the optimal corresponding offline algorithm for a minimization problem P, then ALG is σ-competitive with respect to OPT if there is an independent constant ε that for all input I of P, $ALGP(I) \le \sigma.OPTP(I) + \varepsilon$ [8].

For the online stochastic model, the competitive analysis is done according to the expected value of $ALG_P(I)$ which is executed on the stochastic problem P, $E[ALG_P(I)] \le \sigma.OPT_P(I) + \varepsilon$ [8].

In the general case, if $C = \max_{(i,j)\in\cup_{t=1}^{|\Pi|} A_t} c_{ij}^{\pi_t}$ and $|A| = m$ then the competitive ratio is mC, because it is possible to exchange almost m arcs with possibly C improvement. Let network be acyclic then the competitive ratio is $(n-1)C$, where $n = |N|$ and it needs to exchange almost $n-1$ arcs.

The algorithm establishes a path from the current source node s_t to the current destination node d_t with respect to the policy π_t, for any time t (see Algorithm O-SSP) by minimizing the expected path length. Then, the selection procedure is implemented according to the statistical information of the network due to minimize the expected length of the O-SSP. In each iteration, some values of variables $x_{ij}^{\pi_t}$ are set 1 or 0, for $(i, j) \in A_t$. Porvan [17] obtained the shortest path length as the decision variable values, in every iteration. The competitive analysis of the proposed methods according to the equations (3) and (4) are presented in the following Theorems 2 and 3, based on some local and global information, respectively.

Algorithm O-SSP

Input $t=0, d_0=s$

While $d_t \neq d$ **do**

$t=t+1, s_t=d_{t-1}$

SELECTION PROCEDURE

Select d_t = argmin {expected path length from s_t to v for all $v \in N_t$} according to $\pi_t=(N_t, A_t)$

if for any $i \in N_t$, $(i,j) \in A_t$ is in the minimum expected path from s_t to d_t **then**

set $x_{ij}^{\pi_t}=1$

end if

end while

The Hoeffding's inequality provides an exponential upper bound for a set of independent random variables [16]. So, let $0 \leq X \leq 1$, $j=1,2,\ldots,n$, $\overline{X}=\frac{1}{n}\sum_{j=1}^{n} X_j$, and $\mu=E(\overline{X})$, then for $0<\delta<1-\mu$ we have [16]

$$p(\overline{X}-\mu \geq \delta) \leq e^{-2n\delta^2}$$

Consequently, the proves of the following theorems apply the Hoeffding's inequality on the single random variable $c_{ij}^{\pi_t}$, where its expected value is $\overline{c}_{ij}$.

Theorem 2: *If the selection procedure of O-SSP algorithm applies the expected value as defined in equation (3), then the competitive ratio is at most $e^{-2}mC$.*

Proof: According to equation (3) the optimal offline solution is

$$E\left(\sum_{t,i,j} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right)=\sum_{t,i,j} P_c(u_i^{\pi_t}-u_j^{\pi_t}) \times (u_i^{\pi_t}-u_j^{\pi_t}) x_{ij}^{\pi_t}$$

Since $P_c(u_i^{\pi_t}-u_j^{\pi_t})=P(c_{ij}^{\pi_t}=u_i^{\pi_t}-u_j^{\pi_t})$, and if $u_i^{\pi_t}-u_j^{\pi_t} \geq \overline{c}_{ij}$ then

$$P(c_{ij}^{\pi_t}=u_i^{\pi_t}-u_j^{\pi_t})=P(c_{ij}^{\pi_t}-\overline{c}_{ij}=u_i^{\pi_t}-u_j^{\pi_t}-\overline{c}_{ij})$$

where, $\overline{c}_{ij}=E(c_{ij}^{\pi_t})$, and put $\overline{\Delta}_{ij}^{\pi_t}=u_i^{\pi_t}-u_j^{\pi_t}-\overline{c}_{ij}$, so

$$P(c_{ij}^{\pi_t}-\overline{c}_{ij}=\overline{\Delta}_{ij}^{\pi_t}) \leq P(c_{ij}^{\pi_t}-\overline{c}_{ij} \geq \overline{\Delta}_{ij}^{\pi_t}) \leq e^{-\frac{2(\overline{\Delta}_{ij}^{\pi_t})^2}{C^2}} \tag{5}$$

If $u_i^{\pi_t}-u_j^{\pi_t} \leq \overline{c}_{ij}$ then

$$P(c_{ij}^{\pi_t} = u_i^{\pi_t} - u_j^{\pi_t}) = P(\overline{c}_{ij} - c_{ij}^{\pi_t} = \overline{c}_{ij} - u_i^{\pi_t} + u_j^{\pi_t})$$

and put $\overline{\Delta}_{ij}^{\pi_t} = \overline{c}_{ij} - u_i^{\pi_t} + u_j^{\pi_t}$, so

$$P(\overline{c}_{ij} - c_{ij}^{\pi_t} = \overline{\Delta}_{ij}^{\pi_t}) \leq P(\overline{c}_{ij} - c_{ij}^{\pi_t} \geq \overline{\Delta}_{ij}^{\pi_t}) \leq e^{-\frac{2(\overline{\Delta}_{ij}^{\pi_t})^2}{C^2}} \tag{6}$$

The last inequalities (5) and (6) are held by the Hoeffding's inequality [16], where $P(0 \leq c_{ij}^{\pi_t} \leq C) = 1.$ However, the inequalities (5) and (6) imply

$$P_c(u_i^{\pi_t} - u_j^{\pi_t}) \leq e^{-\frac{2(\overline{\Delta}_{ij}^{\pi_t})^2}{C^2}}.$$

Then, we have

$$E\left(\sum_{t,i,j} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right) \leq \sum_{t,i,j} e^{-\frac{2(\overline{\Delta}_{ij}^{\pi_t})^2}{C^2}} (u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t}.$$

In the worst case, $\overline{\Delta}_{ij}^{\pi_t} = C,$ where $\inf_{\overline{\Delta}_{ij}^{\pi_t}} e^{-2(\overline{\Delta}_{ij}^{\pi_t})^2 C^2} = e^{-2}.$ Further, the values of the decision variables $x_{ij}^{\pi_t}$ determine a path from node s to node d by time $|\Pi|$

$$E\left(\sum_{t,i,j} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right) \leq e^{-2} \sum_{t,i,j} (u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t} = e^{-2}(u_s^{|\Pi|} - u_d^{\Pi|})$$

Initially $c_{ij}^{\pi_t} = u_i^{\pi_t} - u_j^{\pi_t}$, and then by updating procedure, the feasibility is held for the dual variables. If there is $(i, j) \in A$ that $c_{ij}^{|\Pi|} > c_{ij}$, then it needs almost mC improving replacement of arcs, so

$$E\left(\sum_{t,i,j} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right) \leq e^{-2} mC \sum_{i,j} c_{ij} x_{ij}^*.$$

Corollary 1: *If the network is acyclic then the competitive ratio is* $e^{-2}(n-1)C$, *because any cycle is determined by at most* $n-1$ *numbers of the arcs.*

Theorem 3: *If the selection procedure of O-SSP algorithm applies the expected value as defined in equation (4), then the competitive ratio is at most* $e^{-4}mC$.

Proof: The expected length of the O-SSP is obtained by equation (4) and it is depended on both the distribution probabilities of the arc cost and the expected length of the O-SSP from the current nodes,

$$E\left(\sum_{t,i,j} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right) = \sum_{t,i,j} P_c(u_i^{\pi_t} - u_j^{\pi_t}) P_u(u_i^{\pi_t})(u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t}$$

Since $0 \le u_i^{\pi_t} \le mC$, suppose $E(u_i^{\pi_t}) = \overline{u}_i^{\pi_t}$ then by Hoeffding's inequality [16] (as the proof of theorem 2 we have

$$P_u(u_i^{\pi_t}) \le e^{-\frac{2(\overline{\Delta}_i^{\pi_t})^2}{m^2C^2}},$$

where $\overline{\Delta}_j^{\pi_t} = |u_i^{\pi_t} - \overline{u}_i^{\pi_t}|$. So,

$$E\left(\sum_{t,i,j} c_{ij}^{\pi_t} x_{ij}^{\pi_t}\right) \le \sum_{t,i,j} e^{-\frac{2(\overline{\Delta}_{ij}^{\pi_t})^2}{C^2}} \times e^{-\frac{2(\overline{\Delta}_i^{\pi_t})^2}{m^2C^2}} (u_i^{\pi_t} - u_j^{\pi_t}) x_{ij}^{\pi_t}.$$

On the other hand,

$$\inf_{(\overline{\Delta}_{ij}^{\pi_t}, \overline{\Delta}_i^{\pi_t})} e^{-\frac{2(\overline{\Delta}_{ij}^{\pi_t})^2}{C^2}} \times e^{-\frac{2(\overline{\Delta}_i^{\pi_t})^2}{m^2C^2}} = e^{-2} \times e^{-2} = e^{-4},$$

when $\overline{\Delta}_{ij}^{\pi_t} = C$ and $\overline{\Delta}_i^{\pi_t} = mC$. Finally $E(\sum_{t,i,j} c_{ij}^{\pi_t} x_{ij}^{\pi_t}) \le e^{-4} mC \sum_{i,j} c_{ij} x_{ij}^*$.

Corollary 2: *If the network is acyclic then the competitive ratio is* $e^{-4}(n-1)C$.

Consequently, the improvement of the competitive ratio is obtained at least $e^{-2} / e^{-4} = e^2 \approx 7.3891$ relatively, where the global information is applied according to equation (4) against the local information according to equation (3).

4. Numerical results

The networks are created randomly with different sizes of nodes and arcs and also topologies. However, there is a path from any node toward the original destination node and there is a path form the original source node toward any node, and this property is satisfied in both cyclic and acyclic networks (property I). Property I helps us to avoid trivial solutions. Also, another property is satisfied for acyclic networks where there is not any directed cycle in the acyclic networks (property II). Property II is not satisfied by the cyclic networks, however there is no negative cycle in the network.

The costs of the arcs are positive integer values and they are generated according to the uniform distribution; it is considered $C = 7$ in the generated networks. The online routing policies are revealed by arriving of the traversed node including all of the departure nodes of the current policy nodes in both cyclic and acyclic networks.

The numerical results of O-SSP algorithm implementation are shown in Table 2. So, if the local routing information is applied, then O-SSP algorithm considers the selection operator (3) (Opt. online1); and if the global routing information is applied, then O-SSP algorithm considers the selection operator (4) (Opt. online2). The random networks are generated as cyclic (Cyc.) and acyclic (Acyc.) with thousands of arcs and different topologies. The optimal values in Table 2 are obtained as the average value for ten times of implementations for both online and offline versions of the stochastic shortest path problem.

The policy times in Table 2 determine the online routing released information policies those are applied in the online routing process.

Table 2

The numerical results for the instance networks.

Node	Arc		Opt. online1		Opt. online2		Opt. offline		Policy time	
	Cyc.	Acyc.	Cyc.	Acyc.	Cyc.	Acyc.	Cyc.	Acyc.	Cyc.	Acyc.
100	4929	2483	6.9	3.8	6.9	3.6	3.6	2.5	54	58
150	11077	5590	11.4	4.1	11.4	4.3	3.1	1.7	67	86
200	19763	9899	11.3	2.8	12.0	3.3	2.6	1.3	93	117
250	31028	15496	13.2	4.3	12.5	4.1	2.6	1.7	127	149
300	44937	22440	7.8	3.8	8.1	3.7	2.3	1.9	147	174
350	60558	30460	12.4	4.4	11.9	4.3	2.3	1.7	180	212
400	79689	39890	13.3	4.9	13.1	4.8	2.2	1.7	202	222
450	101173	50554	13.6	4.4	13.6	3.5	2.4	1.3	242	272
500	124464	62178	12.1	5.3	12.1	5.3	2.0	1.7	257	269
550	150959	75537	12.8	4.6	13.7	4.6	2.0	1.5	288	329
600	179297	89660	11.8	5.0	11.8	4.2	2.2	1.5	298	346
650	210632	105624	10.0	4.5	10.0	4.5	1.9	1.5	339	404
700	244150	122116	13.9	4.2	13.9	3.9	1.9	1.4	333	420
750	280177	140530	16.7	4.8	16.7	4.1	1.7	1.4	360	440
800	319558	159645	11.6	4.7	11.6	4.0	1.9	1.5	400	468
850	360525	180680	12.5	4.5	12.5	4.0	1.7	1.3	416	490
900	403198	202550	16.0	4.8	16.0	4.3	2.0	1.6	432	517
950	450705	225830	10.7	5.7	10.7	5.7	1.7	1.4	484	544
1000	499064	249671	11.5	4.6	11.5	4.4	1.8	1.7	465	577
Average			12.1053	4.4842	12.0789	4.2421	2.2053	1.5947		

Actually, the online policies in the acyclic networks are produced by adding the new adjacency nodes, so the numbers show the obtained sets of the new adjacency nodes where they could be traversed exactly once. In the cyclic networks, the policies could contain some repeated nodes, so the numbers show the most reached nodes by the online routing process, while they could be traversed several times. Therefore, the search area in the acyclic networks is relatively more expanded than the cyclic networks (see policy time in Table 2).

The competitive analyses of the proposed methods are done on both cyclic and acyclic networks. The average cases of the competitive ratio with respect to the theorems 2 and 3 are shown in Figures 4 and 5 for cyclic and acyclic networks, respectively.

In the cyclic networks, the average case analyses show the length of the online obtained path does not almost differ whether the local information or the global information is applied, as well as the competitive ratio; since, the cycles cause the repeated traversed nodes and consequently

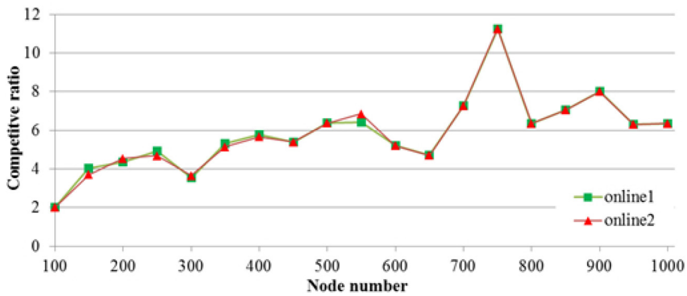

Figure 4

The competitive ratio of cyclic networks in average case

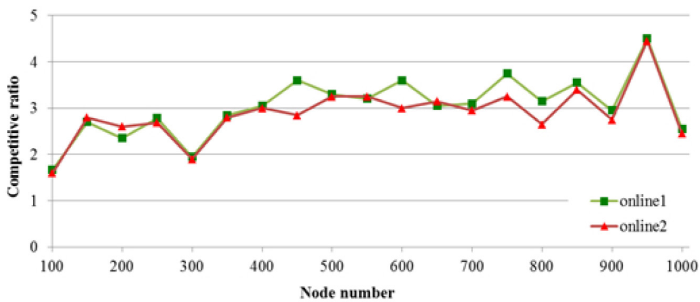

Figure 5

The competitive ratio of acyclic networks in average case

the policies contained some repeated nodes (see policy time in Table 2), and then there are some paths as cycles those could not make differences between the global information and the local information.

However, the results of the acyclic networks reveal the differences of the equations (3) and (4), where the global information (4) (online2) acts better than the local information (3) (online1). So, in the acyclic networks, the obtained paths contain some useful global information of the network and it affects the competitive ratio, relatively.

The overall average of the competitive ratios when decisions are done with respect to the equations (3) and (4) are 5.4892 and 5.4772 for the cyclic networks; also, they are obtained 2.8119 and 2.6601 for the acyclic networks. So, the obtained relative improvements are 0.9978 and 0.946 for the cyclic and the acyclic networks, respectively.

5. Conclusion

The online stochastic shortest path problem in the cyclic and the acyclic networks were studied, in this paper. The competitive analyses of the presented online algorithm were done according to the application of the local and global information in the network. Two formulations were obtained for the O-SSP length with respect to the available local or global statistical information. The numerical results showed that the global statistical information of the paths could lead to a better competitive ratio. However, cycles in the network caused a smaller number of distinct nodes, but in the acyclic networks decisions were made in more expanded search area, relatively. Also, the obtained competitive ratios were improved by the bound of the number of arcs against the previous works. Other stochastic models of the O-SSP problem could be studied for future works. Markov decision process is one of the stochastic models which can be used to improve the online made decisions. Another direction is the decision criterion that was an average case for our online made decisions.

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Network Flows: Theory, Algorithms, and Applications. Englewood Cliffs, NJ: Prentice-Hall (1993).

[2] M. K. Ardakani and L. Sun, "Decremental algorithm for adaptive routing incorporating traveler information," *Comput. Oper. Res.*, vol. 39, no. 12, pp. 3012–3020 (2012).

[3] B. Awerbuch and R. Kleinberg, "Online linear optimization and adaptive routing," *J. Comput. Syst. Sci.*, vol. 74, no. 1, pp. 97–114 (2008).

[4] J. Balogh, J. Békési, G. Dósa, L. Epstein, and A. Levin, "Online bin packing with cardinality constraints resolved," *J. Comput. Syst. Sci.,* vol. 112, pp. 34–49 (2020).

[5] J. Balogh, J. Békési, G. Dósa, J. Sgall, and R. van Stee, "The optimal absolute ratio for online bin packing," *J. Comput. Syst. Sci.*, vol. 102, pp. 1–17 (2019).

[6] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali, Linear Programming and Network Flows. Hoboken, NJ: John Wiley & Sons (2008).

[7] G. N. Bifulco, G. E. Cantarella, F. Simonelli, and P. Veloná, "Advanced traveller information systems under recurrent traffic conditions: Network equilibrium and stability," Transp. Res. Part B: Methodol., vol. 92, pp. 73–87 (2016).

[8] A. Borodin and R. El-Yaniv, Online Computation and Competitive Analysis. Cambridge, U.K.: Cambridge University Press (2005).

[9] S. D. Boyles, "An exact label-correcting algorithm for the online shortest path problem in cyclic transportation networks," Tech. Report (2012).

[10] B. Chen, Z. Ding, Y. Wu, J. Zhou, and Y. Chen, "An optimal global algorithm for route guidance in advanced traveler information systems," *Inf. Sci.,* vol. 555, pp. 33–45 (2021).

[11] J. Chen, M. Li, R. Jiang, and M.-B. Hu, "Effects of the amount of feedback information on urban traffic with advanced traveler information system," *Phys. Lett*. A, vol. 381, pp. 2934–2938 (2017).

[12] Z. He, W. Guan, and S. Ma, "A traffic-condition-based route guidance strategy for a single destination road network," Transp. Res. Part C: Emerg., vol. 32, pp. 89–102 (2013).

[13] A. György, T. Linder, G. Lugosi, and G. Ottucsák, "The on-line shortest path problem under partial monitoring," *J. Mach. Learn. Res.,* vol. 8, no. 10, pp. 2369–2403 (2007).

[14] A. Kalai and S. Vempala, "Efficient algorithms for online decision problems," *J. Comput. Syst. Sci.*, vol. 71, no. 3, pp. 291–307 (2005).

[15] X. Kong and C. D. Schunn, "Global vs. local information processing in visual/spatial problem solving: The case of traveling salesman problem," Cogn. Syst. Res., vol. 8, pp. 192–207 (2007).

[16] Z. Lin and Z. Bai, Probability Inequalities. Berlin, Germany: Springer Science & Business Media (2011).

[17] J. S. Provan, "A polynomial-time algorithm to find shortest paths with recourse," *Networks*, vol. 41, no. 2, pp. 115–125 (2003).

[18] M. R. C. Raj and R. Sukumaran, "On applying stochastic network calculus for Gilbert-Elliot fading channel in underwater wireless communication networks," *J. Inf. Optim. Sci.*, vol. 39, pp. 1591–1605 (2018).

[19] S. M. Ross, Introduction to Probability Models. 10th ed. Waltham, MA: Academic Press (2006).

[20] D. Sever, L. Zhao, N. Dellaert, E. Demir, T. Van Woensel, and T. De Kok, "The dynamic shortest path problem with time-dependent stochastic disruptions," *Transp. Res. Part C: Emerg. Technol.*, vol. 92, pp. 42–57 (2018).

[21] M. Yildirimoglu, I. I. Sirmatel, and N. Geroliminis, "Hierarchical control of heterogeneous large-scale urban road networks via path assignment and regional route guidance," *Transp. Res. Part B: Methodol.*, vol. 118, pp. 106–123 (2018).

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