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Hesitant fuzzy sets in the meaning of infimum on UP-algebras

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Abstract

The idea of $\inf_t^\alpha$ -hesitant fuzzy UP-subalgebras ($\inf_t^\alpha$-HFUPSs) (resp., UP-filters (UPFs), UP-ideals (UPIs), strong UP-ideals (SUPIs)) is presented, various facts are proven, and the generalizations of these hesitant fuzzy sets (HFSs) are addressed. This concept is a generalization of interval-valued fuzzy UPSs (IvFUPSs) (resp., UPFs, UPIs, SUPIs) of UP-algebras. In addition, we analyze the relationships between the level subsets of the inf-hesitant fuzzy UPSs (inf-HFUPSs) (resp., UPFs, UPIs, SUPIs). Finally, characterizations of inf-HFUPSs (resp., UPFs, UPIs, SUPIs) are examined in terms of fuzzy sets (FSs), Pythagorean fuzzy sets (PFSs), HFSs, interval-valued fuzzy sets (IvFSs), and cubic sets (CSs).

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1. Introduction

In 2017, Iampan [2] presented the UP-algebra branch of logical algebra. The class of KU-algebras [9] is understood to be a proper subclass of the class of UP-algebras. It has been studied by a number of scholars, including Somjanta et al. [13], who created fuzzy UPSs (FUPSs), fuzzy UPFs (FUPFs), and fuzzy UPIs (FUPIs) of UP-algebras and researched some of its crucial aspects. Guntasow et al. [1] investigated linkages between UPSs, UPFs, UPIs, SUPIs, and studied the translations of FSs in UP-algebras. Kaijae et al. [5] developed ideas of anti-FUPSs and anti-FUPIs of UP-algebras, Cartesian product, and dot product of FSs, and then examined associated characteristics. Anti Q-FUPIs and anti Q-FUPSs were presented by Sripaeng, Tanamoon, and Iampan [11] and investigated. By applying the idea of PFSs to UP-algebras, Satirad, Chinram, and Iampan [10] created Pythagorean fuzzy UPSs (PFUPSs), Pythagorean fuzzy UPFs (PFUPFs), Pythagorean fuzzy UPIs (PFUPIs), and Pythagorean fuzzy SUPIs (PFSUPIs). Senapati, Jun, and Shum [12] extended the CSs concept to UP-algebras, established cubic UPSs (CUPSs) and cubic UPIs (CUPIs), and then studied and addressed the crucial characteristics of these structures. The research of CSs on UP-algebras was also continued by Taboon, Butsri, and Iampan [14], who also introduced cubic UPFs (CUPFs), cubic SUPIs (CSUPIs), anti-CUPSs, anti-CUPFs, anti-CUPIs, and anti-CSUPIs.

Following Torra and Narukawa's [16, 15] development of the notion of HFSs, Mosrijai et al. [8] established HFUPSs, HFUPFs, HFUPIs, and HFSUPIs by applying the concept of HFSs to UP-algebras. Following this, Mosrijai and Iampan [7] continued their research on the topic of HFSs on UP-algebras and introduced the concepts of anti-type of HFSs, referred to as anti-HFUPSs (resp., UPFs, UPIs, SUPIs), and discussed connections of these concepts. Mosrijai, Akarachai, and Iampan [6] examined HFSs, also known as sup-HFUPSs (resp., UPFs, UPIs, SUPIs), and looked at some of its key characteristics.

It inspired us to research HFSs on UP-algebras in the sense of the infimum of their images and to introduce the notion of $\inf_t^\alpha$-HFUPSs, which are generalizations of the notion of IvFUPSs, as well as UPFs, UPIs, and SUPIs. We next talk about how these notions may be used generally. In addition, relationships between HFSs of the inf-type and their level subsets are examined. Finally, using FSs, PFSs, HFSs, IvFSs, and CSs, we describe inf-HFUPSs (resp., UPFs, UPIs, SUPIs).

2. Preliminaries

We review basic definitions and information regarding UP-algebras before we begin our research.

If an algebra $\omega = (\omega, \cdot, 0)$ of type $(2, 0)$ meets the following statements, it is said to be a *UP-algebra* [2], where $\omega \neq \emptyset$, $\cdot$ is a binary operation on ω, and 0 is a constant in ω:

(UP-1) $(\forall p, q, r \in \omega)((q \cdot r) \cdot ((p \cdot q) \cdot (p \cdot r)) = 0)$,
(UP-2) $(\forall p \in \omega)(0 \cdot p = p)$,
(UP-3) $(\forall p \in \omega)(p \cdot 0 = 0)$,
(UP-4) $(\forall p, q \in \omega)(p \cdot q = 0, q \cdot p = 0 \Rightarrow p = q)$.

The following claims are true for the UP-algebra $\omega = (\omega, \cdot, 0)$ (see [2, 3]).

$(\forall p \in \omega)(p \cdot p = 0),$ (2.1)

$(\forall p, q, r \in \omega)(p \cdot q = 0, q \cdot r = 0 \Rightarrow p \cdot r = 0),$ (2.2)

$(\forall p, q, r \in \omega)(p \cdot q = 0 \Rightarrow (r \cdot p) \cdot (r \cdot q) = 0),$ (2.3)

$(\forall p, q, r \in \omega)(p \cdot q = 0 \Rightarrow (q \cdot r) \cdot (p \cdot r) = 0),$ (2.4)

$(\forall p, q \in \omega)(p \cdot (q \cdot p) = 0),$ (2.5)

$(\forall p, q \in \omega)((q \cdot p) \cdot p = 0 \Leftrightarrow p = q \cdot p),$ (2.6)

$(\forall p, q \in \omega)(p \cdot (q \cdot q) = 0),$ (2.7)

$(\forall p, q, r, a \in \omega)((p \cdot (q \cdot r)) \cdot (p \cdot ((a \cdot q) \cdot (a \cdot r))) = 0),$ (2.8)

$(\forall p, q, r, a \in \omega)((((a \cdot p) \cdot (a \cdot q)) \cdot r) \cdot ((p \cdot q) \cdot r) = 0),$ (2.9)

$(\forall p, q, r \in \omega)(((p \cdot q) \cdot r) \cdot (q \cdot r) = 0),$ (2.10)

$(\forall p, q, r \in \omega)(p \cdot q = 0 \Rightarrow p \cdot (r \cdot q) = 0),$ (2.11)

$(\forall p, q, r \in \omega)(((p \cdot q) \cdot r) \cdot (p \cdot (q \cdot r)) = 0),$ (2.12)

$(\forall p, q, r, a \in \omega)(((p \cdot q) \cdot r) \cdot (q \cdot (a \cdot r)) = 0).$ (2.13)

From this point forward, we assume that ω will always refer to the UP-algebra $(\omega, \cdot, 0)$.

Definition 2.1: [2] If the constant 0 of ω is in the subset S and $(S, \cdot, 0)$ is a UP-algebra, then $(S, \cdot, 0)$ is referred to as a *UPS* of ω.

A nonempty subset S of ω is a UPS of ω if and only if S is closed under the $\cdot$ multiplication on ω, as demonstrated by Iampan's helpful criterion in [2].

Definition 2.2: [13, 2, 1] A subset S of ω is called

(1) a *UPF* of ω if
  (i) $0 \in S$,
  (ii) $(\forall p, q \in \omega)(p \cdot q \in S, p \in S \Rightarrow q \in S)$,

(2) a *UPI* of ω if
  (i) $0 \in S$,
  (ii) $(\forall p, q, r \in \omega)(p \cdot (q \cdot r) \in S, q \in S \Rightarrow p \cdot r \in S)$,

(3) a *SUPI* of ω if
  (i) $0 \in S$,
  (ii) $(\forall p, q, r \in \omega)((r \cdot q) \cdot (r \cdot p) \in S, q \in S \Rightarrow p \in S)$.

It was established by Guntasow et al. in their paper [1] that the idea of a UPS is a generalization of the idea of a UPF, the idea of a UPF is a generalization of the idea of a UPI, and the idea of a UPI is a generalization of the idea of a SUPI. They also demonstrated that a UP-algebra ω is the only SUPI that exists by itself.

A *fuzzy subset* (FS) [19] of a set $\omega \neq \emptyset$ is a function from the set ω into $[0,1]$, where $[0,1] \subseteq R$. A *hesitant fuzzy set* (HFS) [15, 16] on ω is defined in term of a function $\hat{\varepsilon}$ that when applied to ω return a subset of $[0,1]$, that is, $\hat{\varepsilon}: \omega \to \wp([0,1])$, where $\wp([0,1])$ is the power set of $[0,1]$. Let $\hat{\varepsilon}$ be a HFS on

ω and p be a element of ω. In the case that $\hat{\varepsilon}(p)$ is empty, we denote $\sup\hat{\varepsilon}(p) = 0$. Mosrijai, Akarachai, and Iampan [6] studied the concept of HFSs of UP-algebras in the meaning of supremum of their images and introduced the concept of sup-HFUPSs, sup-HFUPFs, sup-HFUPIs, and sup-HFSUPIs of UP-algebras as follows:

Definition 2.3: [6] A HFS $\hat{\varepsilon}$ on ω is called

(1) a *sup-hesitant fuzzy UPS* (sup-HFUPS) of ω if $(\forall p, q \in \omega)(\sup\hat{\varepsilon}(p \cdot q) \geq \min\{\sup\hat{\varepsilon}(p), \sup\hat{\varepsilon}(q)\})$,

(2) a *sup-hesitant fuzzy UPF* (sup-HFUPF) of ω if
  (i) $(\forall p \in \omega)(\sup\hat{\varepsilon}(0) \geq \sup\hat{\varepsilon}(p))$,
  (ii) $(\forall p, q \in \omega)(\sup\hat{\varepsilon}(q) \geq \min\{\sup\hat{\varepsilon}(p \cdot q), \sup\hat{\varepsilon}(p)\})$,

(3) a *sup-hesitant fuzzy UPI* (sup-HFUPI) of ω if
  (i) $(\forall p \in \omega)(\sup\hat{\varepsilon}(0) \geq \sup\hat{\varepsilon}(p))$,
  (ii) $(\forall p, q, r \in \omega)(\sup\hat{\varepsilon}(p \cdot r) \geq \min\{\sup\hat{\varepsilon}(p \cdot (q \cdot r)), \sup\hat{\varepsilon}(q)\})$,

(4) a *sup-hesitant fuzzy SUPI* (sup-HFSUPI) of ω if
  (i) $(\forall p \in \omega)(\sup\hat{\varepsilon}(0) \geq \sup\hat{\varepsilon}(p))$,
  (ii) $(\forall p, q, r \in \omega)(\sup\hat{\varepsilon}(p) \geq \min\{\sup\hat{\varepsilon}((r \cdot q) \cdot (r \cdot p)), \sup\hat{\varepsilon}(q)\})$.

Mosrijai, Akarachai, and Iampan [6] also showed that the concept of a sup-HFUPS is a general concept of a sup-HFUPF, the concept of a sup-HFUPF is a general concept of a sup-HFUPI, and the concept of a sup-HFUPI is a general concept of a sup-HFSUPI. Moreover, they also proved that a HFS on a UP-algebra is a sup-HFSUPI if and only if it is a constant function.

3. $\inf_t^\alpha$-hesitant fuzzy UP-substructures

Assuming that α is an element of $\{-, +\}$ and t is an element of $[0,1]$, we will present notions of $\inf_t^\alpha$-HFUPSs, $\inf_t^\alpha$-HFUPFs, $\inf_t^\alpha$-HFUPIs, and $\inf_t^\alpha$-HFSUPIs of UP-algebras, demonstrate their generalizations, and look at some of their key properties in this section.

Let $x \in \omega$, $t \in [0,1]$, let $\hat{\varepsilon}$ be a HFS on ω. Define

$$\mathrm{INF}\hat{\varepsilon}(x) = \begin{cases} \inf\hat{\varepsilon}(x) & \text{if } \hat{\varepsilon}(x) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{INF}_t^-\hat{\varepsilon}(x) = \max\{\mathrm{INF}\hat{\varepsilon}(x) - t, 0\},$$

$$\mathrm{INF}_t^+\hat{\varepsilon}(x) = \min\{\mathrm{INF}\hat{\varepsilon}(x) + t, 1\}.$$

Clearly, $\mathrm{INF}_0^-\hat{\varepsilon}(x) = \mathrm{INF}_0^+\hat{\varepsilon}(x) = \mathrm{INF}\hat{\varepsilon}(x)$.

From now on throughout this paper, α is an element of $\{-, +\}$ and t is an element of $[0,1]$ unless otherwise. The concepts of $\inf_t^\alpha$-HFUPSs, $\inf_t^\alpha$-HFUPFs, $\inf_t^\alpha$-HFUPIs, and $\inf_t^\alpha$-HFSUPIs of UP-algebras are introduced in the following definition.

Definition 3.1: A HFS $\hat{\varepsilon}$ on ω is called

(1) an $\inf_t^\alpha$*-hesitant fuzzy UPS* ($\inf_t^\alpha$-HFUPS) of ω if $(\forall x, y \in \omega)(\mathrm{INF}_t^\alpha\hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}_t^\alpha\hat{\varepsilon}(x), \mathrm{INF}_t^\alpha\hat{\varepsilon}(y)\})$,

(2) an $\inf_t^\alpha$*-hesitant fuzzy UPF* ($\inf_t^\alpha$-HFUPF) of ω if
   (i) $(\forall x \in \omega)(\mathrm{INF}_t^\alpha \hat{\varepsilon}(0) \geq \mathrm{INF}_t^\alpha \hat{\varepsilon}(x))$,
   (ii) $(\forall x, y \in \omega)(\mathrm{INF}_t^\alpha \hat{\varepsilon}(y) \geq \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}(x \cdot y), \mathrm{INF}_t^\alpha \hat{\varepsilon}(x)\})$,
(3) an $\inf_t^\alpha$*-hesitant fuzzy UPI* ($\inf_t^\alpha$-HFUPI) of ω if
   (i) $(\forall x \in \omega)(\mathrm{INF}_t^\alpha \hat{\varepsilon}(0) \geq \mathrm{INF}_t^\alpha \hat{\varepsilon}(x))$,
   (ii) $(\forall x, y, z \in \omega)(\mathrm{INF}_t^\alpha \hat{\varepsilon}(x \cdot z) \geq \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}(x \cdot (y \cdot z)), \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)\})$,
(4) an $\inf_t^\alpha$*-hesitant fuzzy SUPI* ($\inf_t^\alpha$-HFSUPI) of ω if
   (i) $(\forall x \in \omega)(\mathrm{INF}_t^\alpha \hat{\varepsilon}(0) \geq \mathrm{INF}_t^\alpha \hat{\varepsilon}(x))$,
   (ii) $(\forall x, y, z \in \omega)(\mathrm{INF}_t^\alpha \hat{\varepsilon}(x) \geq \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}((z \cdot y) \cdot (z \cdot x)), \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)\})$.

If $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFUPS of ω, $x \in \omega$ and $0 < t$, then it follows from (2.1), we can obviously see that $\mathrm{INF}_t^\alpha \hat{\varepsilon}(0) \geq \mathrm{INF}_t^\alpha \hat{\varepsilon}(x)$ but $\mathrm{INF}\hat{\varepsilon}(0) \ngeq \mathrm{INF}\hat{\varepsilon}(x)$.

In this paper, an $\inf_0^\alpha$-HFUPS (resp., $\inf_0^\alpha$-HFUPF, $\inf_0^\alpha$-HFUPI, $\inf_0^\alpha$-HFSUPI) of ω is called an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω.

We shall explore the generalizations of these ideas according to Definition 3.1.

Proposition 3.2: Every inf -HFUPS (resp., inf -HFUPF, inf -HFUPI, inf -HFSUPI) of ω is an $\inf_t^+$-HFUPS (resp., $\inf_t^+$-HFUPF, $\inf_t^+$-HFUPI, $\inf_t^+$-HFSUPI) of ω.

Proof: Assume $\hat{\varepsilon}$ is an inf-HFUPS of ω and $x, y \in \omega$. Then $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\}$. Thus $\mathrm{INF}\ \hat{\varepsilon}(x \cdot y) \geq \mathrm{INF}\hat{\varepsilon}(x)$ or $\mathrm{INF}\ \hat{\varepsilon}(x \cdot y) \geq \mathrm{INF}\hat{\varepsilon}(y)$. Now, let's think about the next two cases.

Case 1, let $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq \mathrm{INF}\hat{\varepsilon}(x)$. Then $\mathrm{INF}\hat{\varepsilon}(x \cdot y) + t \geq \mathrm{INF}\hat{\varepsilon}(x) + t$ and thus $\mathrm{INF}_t^+ \hat{\varepsilon}(x \cdot y) = \min\{\mathrm{INF}\hat{\varepsilon}(x \cdot y) + t, 1\} \geq \min\{\mathrm{INF}\hat{\varepsilon}(x) + t, 1\} = \mathrm{INF}_t^+ \hat{\varepsilon}(x) \geq \min\{\mathrm{INF}_t^+ \hat{\varepsilon}(x), \mathrm{INF}_t^+ \hat{\varepsilon}(y)\}$. Hence, $\mathrm{INF}_t^+ \hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}_t^+ \hat{\varepsilon}(x), \mathrm{INF}_t^+ \hat{\varepsilon}(y)\}$.

Case 2, let $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq \mathrm{INF}\hat{\varepsilon}(y)$. We can prove similarly as Case 1 that $\mathrm{INF}_t^+ \hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}_t^+ \hat{\varepsilon}(x), \mathrm{INF}_t^+ \hat{\varepsilon}(y)\}$.

By the two cases, we get $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFUPS of ω.

The example below demonstrates the opposite of Proposition 3.2.

Example 3.3: Allow $\omega = \{0, u, v, w\}$ to be a UP-algebra with a constant 0 and a binary operation $\cdot$ specified by the table below:

$\cdot$	0	u	v	w
0	0	u	v	w
u	0	0	u	w
v	0	0	0	w
w	0	u	v	0

Table 1
Cayley Table for Example 3.3

We define a HFS $\hat{\varepsilon}$ on ω as follows: $\hat{\varepsilon}(0) = \{0.8\}, \hat{\varepsilon}(u) = [0.9,1], \hat{\varepsilon}(v) = (0.7,1], \hat{\varepsilon}(w) = \{0.7, 0.8, 0.9\}$. Then $\hat{\varepsilon}$ is an $\inf_t^+$-HFUPS (also, $\inf_t^+$-HFUPF, $\inf_t^+$-HFUPI, $\inf_t^+$-HFSUPI) of ω $\forall t \in [0.3,1]$ but $\hat{\varepsilon}$ is not an inf-HFUPS (also, inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω because INF $\hat{\varepsilon}(0) \ngeq$ INF$\hat{\varepsilon}(u)$.

Proposition 3.4: Every inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω is an $\inf_t^-$-HFUPS (resp., $\inf_t^-$-HFUPF, $\inf_t^-$-HFUPI, $\inf_t^-$-HFSUPI) of ω.

Proof: Assume $\hat{\varepsilon}$ is an inf-HFUPS of ω and $x, y \in \omega$. Then INF$\hat{\varepsilon}(x \cdot y) \geq$ INF$\hat{\varepsilon}(x)$ or INF$\hat{\varepsilon}(x \cdot y) \geq \hat{\varepsilon}(y)$. Now, let's think about the next two cases.

Case 1, let INF$\hat{\varepsilon}(x \cdot y) \geq$ INF$\hat{\varepsilon}(y)$. Then INF$\hat{\varepsilon}(x \cdot y) - t \geq$ INF$\hat{\varepsilon}(y) - t$ and so $\mathrm{INF}_t^-\hat{\varepsilon}(x \cdot y) = \max\{\mathrm{INF}\hat{\varepsilon}(x \cdot y) - t, 0\} \geq \max\{\mathrm{INF}\hat{\varepsilon}(y) - t, 0\} = \mathrm{INF}_t^-\hat{\varepsilon}(y) \geq \min\{\mathrm{INF}_t^-\hat{\varepsilon}(x), \mathrm{INF}_t^-\hat{\varepsilon}(y)\}$. Thus, $\mathrm{INF}_t^-\hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}_t^-\hat{\varepsilon}(x), \mathrm{INF}_t^-\hat{\varepsilon}(y)\}$.

Case 2, let INF$\hat{\varepsilon}(x \cdot y) \geq$ INF$\hat{\varepsilon}(x)$. We can prove similarly as Case 1 that $\mathrm{INF}_t^-\hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}_t^-\hat{\varepsilon}(x), \mathrm{INF}_t^-\hat{\varepsilon}(y)\}$.

By the two cases, we consequently have that $\hat{\varepsilon}$ is an $\inf_t^-$-HFUPS of ω.

The example below demonstrates the opposite of Proposition 3.4.

Example 3.5: Let $\omega = \{0, u, v, w\}$ be a UP-algebra defined in Example 3.3. We define a HFS $\hat{\varepsilon}$ on ω as follows: $\hat{\varepsilon}(0) = \emptyset, \hat{\varepsilon}(u) = \{0\}, \hat{\varepsilon}(v) = (0.5,0.9], \hat{\varepsilon}(w)$ $= [0.3,1]$. Then $\hat{\varepsilon}$ is an $\inf_t^-$-HFUPS (also, $\inf_t^-$-HFUPF, $\inf_t^-$-HFUPI, $\inf_t^-$-HFSUPI) of ω $\forall t \in [0.5,1]$. However, $\hat{\varepsilon}$ is not an inf-HFUPS (also, inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω because $\hat{\varepsilon}(0) \ngeq \hat{\varepsilon}(v)$.

By Propositions 3.2 and 3.4 and Examples 3.3 and 3.5, we consequently have that the concepts of an $\inf_t^-$-HFUPS (resp., $\inf_t^-$-HFUPF, $\inf_t^-$-HFUPI, $\inf_t^-$-HFSUPI) and an $\inf_t^+$-HFUPS (resp., $\inf_t^+$-HFUPF, $\inf_t^+$-HFUPI, $\inf_t^+$-HFSUPI) of a UP-algebra is a generalization of the concept of an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI).

Proposition 3.6: Let $\hat{\varepsilon}$ be a HFS on ω and $s \in (0,1]$. The following statements are true.

(1) If $\hat{\varepsilon}$ is an $\inf_t^-$-HFUPS (resp., $\inf_t^-$-HFUPF, $\inf_t^-$-HFUPI, $\inf_t^-$-HFSUPI) of ω $\forall t \in (0, s]$, then $\hat{\varepsilon}$ is an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω.

(2) If $\hat{\varepsilon}$ is an $\inf_t^+$-HFUPS (resp., $\inf_t^+$-HFUPF, $\inf_t^+$-HFUPI, $\inf_t^+$-HFSUPI) of ω $\forall t \in (0, s]$, then $\hat{\varepsilon}$ is an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω.

Proof:

(1) Suppose INF $\hat{\varepsilon}(x \cdot y) < \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\}$ for some $x, y \in \omega$. We can choose $t \in (0, s]$ such that $\min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} - t > 0$. Then $\mathrm{INF}_t^-\hat{\varepsilon}(x) = \max\{\mathrm{INF}\hat{\varepsilon}(x) - t, 0\} = \mathrm{INF}\hat{\varepsilon}(x) - t$ and $\mathrm{INF}_t^-\hat{\varepsilon}(y) = \max$ $\{\mathrm{INF}\hat{\varepsilon}(y) - t, 0\} = \mathrm{INF}\hat{\varepsilon}(y) - t$. Since $\hat{\varepsilon}$ is an $\inf_t^-$-HFUPS of ω, $\mathrm{INF}_t^-\hat{\varepsilon}$ $(x \cdot y) \geq \min\{\mathrm{INF}_t^-\hat{\varepsilon}(x), \mathrm{INF}_t^-\hat{\varepsilon}(y)\} = \min\{\mathrm{INF}\hat{\varepsilon}(x) - t, \mathrm{INF}\hat{\varepsilon}(y) - t\} =$

$\min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} - t > 0$. Thus $\mathrm{INF}_t^-\hat{\varepsilon}(x \cdot y) = \max\{\mathrm{INF}\hat{\varepsilon}(x \cdot y) - t, 0\} = \mathrm{INF}\hat{\varepsilon}(x \cdot y) - t < \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} - t = \min\{\mathrm{INF}_t^- \hat{\varepsilon}(x), \mathrm{INF}_t^-\hat{\varepsilon}(y)\}$, this is a contradiction. Hence, $\hat{\varepsilon}$ is an inf-HFUPS of ω.

(2) Suppose $\mathrm{INF}\hat{\varepsilon}(x \cdot y) < \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\}$ for some $x, y \in \omega$. We choose $t \in (0, s]$ such that $\mathrm{INF}\hat{\varepsilon}(x \cdot y) + t < 1$. Then $\mathrm{INF}\hat{\varepsilon}(x \cdot y) + t < \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} + t = \min\{\mathrm{INF}\hat{\varepsilon}(x) + t, \mathrm{INF}\hat{\varepsilon}(y) + t\}$. Thus $\mathrm{INF}_t^+ \hat{\varepsilon}(x \cdot y) = \min\{\mathrm{INF}\hat{\varepsilon}(x \cdot y) + t, 1\} < \min\{\mathrm{INF}\hat{\varepsilon}(x) + t, 1\} = \mathrm{INF}_t^+\hat{\varepsilon}(x)$ and similarly, we get $\mathrm{INF}_t^+\hat{\varepsilon}(x \cdot y) < \mathrm{INF}_t^+\hat{\varepsilon}(y)$. Since $\hat{\varepsilon}$ is an $\inf_t^+$-HFUPS of ω, $\mathrm{INF}_t^+\hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}_t^+\hat{\varepsilon}(x), \mathrm{INF}_t^+\hat{\varepsilon}(y)\} > \mathrm{INF}_t^+\hat{\varepsilon}(x \cdot y)$, this is a contradiction. Hence $\hat{\varepsilon}$ is an inf-HFUPS of ω.

Theorem 3.7: Every $\inf_t^\alpha$-HFUPF of ω is an $\inf_t^\alpha$-HFUPS.

Proof: Suppose $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFUPF of ω. Then $\forall x, y \in \omega$,

$$\begin{aligned}
\mathrm{INF}_t^\alpha\hat{\varepsilon}(x \cdot y) &\geq \min\{\mathrm{INF}_t^\alpha\hat{\varepsilon}(y \cdot (x \cdot y)), \mathrm{INF}_t^\alpha\hat{\varepsilon}(y)\} \\
&= \min\{\mathrm{INF}_t^\alpha\hat{\varepsilon}(0), \mathrm{INF}_t^\alpha\hat{\varepsilon}(y)\} \\
&= \mathrm{INF}_t^\alpha\hat{\varepsilon}(y) \qquad (2.5) \\
&\geq \min\{\mathrm{INF}_t^\alpha\hat{\varepsilon}(x), \mathrm{INF}_t^\alpha\hat{\varepsilon}(y)\}.
\end{aligned}$$

Hence, $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFUPS of ω.

The example below demonstrates the opposite of Proposition 3.7.

Example 3.8: Allow $\omega = \{0, a, b, c\}$ to be a UP-algebra with a constant 0 and a binary operation $\cdot$ specified by the table below:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	a	c
b	0	0	0	c
c	0	0	0	0

Table 2
Cayley Table for Example 3.8

We define a HFS $\hat{\varepsilon}$ on ω as follows: $\hat{\varepsilon}(0) = (0.6, 0.9], \hat{\varepsilon}(a) = \emptyset, \hat{\varepsilon}(b) = [0.4, 0.5) \cup \{0.8\}, \hat{\varepsilon}(c) = [0.5, 1]$. Then the following hold.

(1) $\hat{\varepsilon}$ is an $\inf_{0.1}^-$-HFUPS of ω but not an $\inf_{0.1}^-$-HFUPF of ω because
$$\mathrm{INF}_{0.1}^-\hat{\varepsilon}(a) = 0 \ngeq 0.3 = \min\{\mathrm{INF}_{0.1}^-\hat{\varepsilon}(b \cdot a), \mathrm{INF}_{0.1}^-\hat{\varepsilon}(b)\}.$$
(2) $\hat{\varepsilon}$ is an $\inf_{0.7}^+$-HFUPS of ω but not an $\inf_{0.7}^+$-HFUPF of ω because
$$\mathrm{INF}_{0.7}^+\hat{\varepsilon}(a) = 0.7 \ngeq 1 = \min\{\mathrm{INF}_{0.7}^+\hat{\varepsilon}(b \cdot a), \mathrm{INF}_{0.7}^+\hat{\varepsilon}(b)\}.$$

The following corollary is obtained from Theorem 3.7 by assuming $t = 0$.

Cortollary 3.9: *Every inf-HFUPF of ω is an inf-HFUPS.*

Theorem 3.10: *Every inf_t^α-HFUPI of ω is an inf_t^α-HFUPF.*

Proof: Assume $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFUPI of ω. Then $\forall x, y \in \omega$, $\mathrm{INF}_t^\alpha \hat{\varepsilon}(0) \geq \mathrm{INF}_t^\alpha \hat{\varepsilon}(x)$ and

$$\begin{aligned}
\mathrm{INF}_t^\alpha \hat{\varepsilon}(y) &= \mathrm{INF}_t^\alpha \hat{\varepsilon}(0 \cdot y) && \text{(UP-2)}\\
&\geq \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}(0 \cdot (x \cdot y)), \mathrm{INF}_t^\alpha \hat{\varepsilon}(x)\}\\
&= \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}(x \cdot y), \mathrm{INF}_t^\alpha \hat{\varepsilon}(x)\}. && \text{(UP-2)}
\end{aligned}$$

Therefore, $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFUPF of ω.

The example below demonstrates the opposite of Proposition 3.10.

Example 3.11: Allow $\omega = \{0, x, y, z\}$ to be a UP-algebra with a constant 0 and a binary operation $\cdot$ specified by the table below:

$\cdot$	0	x	y	z
0	0	x	y	z
x	0	0	z	z
y	0	x	0	0
z	0	x	y	0

Table 3
Cayley Table for Example 3.11

We define a HFS $\hat{\varepsilon}$ on ω as follows: $\hat{\varepsilon}(0) = (0.7,0.9), \hat{\varepsilon}(x) = [0.5,0.6]$, $\hat{\varepsilon}(y) = \{0.3\}, \hat{\varepsilon}(z) = (0.3,0.5]$. Then the following hold.

(1) $\hat{\varepsilon}$ is an $\inf_{0.4}^-$-HFUPF of ω but not an $\inf_{0.4}^-$-HFUPI of ω because
$$\mathrm{INF}_{0.4}^- \hat{\varepsilon}(z \cdot y) = 0 \ngeq 0.1 = \min\{\mathrm{INF}_{0.4}^- \hat{\varepsilon}(z \cdot (x \cdot y)), \mathrm{INF}_{0.4}^- \hat{\varepsilon}(x)\}.$$

(2) $\hat{\varepsilon}$ is an $\inf_{0.2}^+$-HFUPF of ω but not an $\inf_{0.2}^+$-HFUPI of ω because
$$\mathrm{INF}_{0.2}^+ \hat{\varepsilon}(z \cdot y) = 0.5 \ngeq 0.7 = \min\{\mathrm{INF}_{0.2}^+ \hat{\varepsilon}(z \cdot (x \cdot y)), \mathrm{INF}_{0.2}^+ \hat{\varepsilon}(x)\}.$$

The following corollary is obtained from Theorem 3.10 by assuming $t = 0$.

Corollary 3.12: Every inf-HFUPI of ω is an inf-HFUPF.

Theorem 3.13: Every $\inf_t^\alpha$-HFSUPI of ω is an $\inf_t^\alpha$-HFUPI.

Proof: Assume $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFSUPI of ω. Then $\forall x, y, z \in \omega$, $\mathrm{INF}_t^\alpha \hat{\varepsilon}(0) \geq \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)$ and

$$\begin{aligned}
\mathrm{INF}_t^\alpha \hat{\varepsilon}(x \cdot z) &\geq \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}((z \cdot y) \cdot (z \cdot (x \cdot z))), \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)\}\\
&= \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}((z \cdot y) \cdot 0), \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)\} && (2.5)\\
&= \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}(0), \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)\} && \text{(UP-3)}\\
&= \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)\\
&\geq \min\{\mathrm{INF}_t^\alpha \hat{\varepsilon}(x \cdot (y \cdot z)), \mathrm{INF}_t^\alpha \hat{\varepsilon}(y)\}.
\end{aligned}$$

Therefore, $\hat{\varepsilon}$ is an $\inf_t^\alpha$-HFUPI of ω.

The example below demonstrates the opposite of Theorem 313.

Example 3.14: Allow $\omega = \{0,1,2,3\}$ to be a UP-algebra with a constant 0 and a binary operation $\cdot$ specified by the table below:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	0
3	0	0	2	0

Table 4
Cayley Table for Example 3.14

We define a HFS h on ω as follows: $\hat{\varepsilon}(0) = \{0.8\}, \hat{\varepsilon}(1) = [0.6,0.8]$, $\hat{\varepsilon}(2) = (0.4,0.5], \hat{\varepsilon}(3) = [0.6,0.8] \cup \{0.9\}$. Then the following hold.

(1) $\hat{\varepsilon}$ is an $\inf_{0.2}^{-}$-HFUPI of ω but not an $\inf_{0.2}^{-}$-HFSUPI of ω because

$$\mathrm{INF}_{0.2}^{-}\hat{\varepsilon}(1) = 0.4 \ngeq 0.6 = \min\{\max\{\mathrm{INF}\hat{\varepsilon}((3\cdot 0)\cdot(3\cdot 1)) - 0.2,0\}, \max\{\mathrm{INF}\hat{\varepsilon}(0) - 0.2,0\}\}.$$

(2) $\hat{\varepsilon}$ is an $\inf_{0.3}^{+}$-HFUPI of ω but not an $\inf_{0.3}^{+}$-HFSUPI of ω because

$$\mathrm{INF}_{0.3}^{+}\hat{\varepsilon}(1) = 0.9 \ngeq 1 = \min\{\mathrm{INF}_{0.3}^{+}\hat{\varepsilon}((3\cdot 0)\cdot(3\cdot 1)), \mathrm{INF}_{0.3}^{+}\hat{\varepsilon}(0)\}.$$

The following corollary is obtained from Theorem 3.13 by assuming $t = 0$.

Corollary 3.15: *Every inf-HFSUPI of ω is an inf-HFUPI.*

By Theorems 3.7, 3.10, and 3.13 and Examples 3.8, 3.11, and 3.14, we get that the concept of an $\inf_t^{\alpha}$-HFUPS is a generalization of the concept of an $\inf_t^{\alpha}$-HFUPF, the concept of an $\inf_t^{\alpha}$-HFUPF is a generalization of the concept of an $\inf_t^{\alpha}$-HFUPI, and the concept of an $\inf_t^{\alpha}$-HFUPI is a generalization of the concept of an $\inf_t^{\alpha}$-HFSUPI.

Lemma 3.16: A HFS $\hat{\varepsilon}$ on ω is an $\inf_t^{\alpha}$-HFSUPI of ω if and only if $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(a) = \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0)\ \forall a \in \omega$.

Proof: Assume $\hat{\varepsilon}$ is an $\inf_t^{\alpha}$-HFSUPI of ω. Then $\forall a \in \omega$, $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0) \geq \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(a)$ and $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(a) \geq \min\{\mathrm{INF}_t^{\alpha}\hat{\varepsilon}((c\cdot b)\cdot(c\cdot a)), \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(b)\}$. Thus $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(a) \geq \min\{\mathrm{INF}_t^{\alpha}\hat{\varepsilon}((a\cdot 0)\cdot(a\cdot a)), \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0)\} = \min\{\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0\cdot 0), \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0)\} = \min\{\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0), \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0)\} = \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0) \geq \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(a)\ \forall a,b,c \in \omega$. Hence, $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0) = \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(a)\ \forall a \in \omega$.

Conversely, given that $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0) = \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(x)\ \forall x \in \omega$. Obviously, $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(0) \geq \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(x)$ and $\mathrm{INF}_t^{\alpha}\hat{\varepsilon}(x) \geq \min\{\mathrm{INF}_t^{\alpha}\hat{\varepsilon}((z\cdot y)\cdot(z\cdot x)), \mathrm{INF}_t^{\alpha}\hat{\varepsilon}(y)\}$ $\forall x,y,z \in \omega$. Hence, $\hat{\varepsilon}$ is an $\inf_t^{\alpha}$-HFSUPI of ω.

Theorem 3.17: HFS $\hat{\varepsilon}$ on ω is an $\inf_t^{-}$-HFSUPI of ω if and only if the infimum of all images of $\hat{\varepsilon}$ is equal or $\sup\{\mathrm{INF}\hat{\varepsilon}(a) | a \in \omega\} \leq t$.

Proof: Assume $\hat{\varepsilon}$ is an $\inf_t^{-}$-HFSUPI of ω. By Lemma 3.16, we obtain $\mathrm{INF}_t^{-}\hat{\varepsilon}(0) = \mathrm{INF}_t^{-}\hat{\varepsilon}(a)\ \forall a \in \omega$. Now, let's think about the next two cases.

Case 1, let $\mathrm{INF}\hat{\varepsilon}(0) > t$. Then $\mathrm{INF}\hat{\varepsilon}(0) - t > 0$ and so $\forall a \in \omega$, $\max\{\mathrm{INF}\hat{\varepsilon}(a) - t, 0\} = \mathrm{INF}_t^{-}\hat{\varepsilon}(a) = \mathrm{INF}_t^{-}\hat{\varepsilon}(0) = \max\{\mathrm{INF}\hat{\varepsilon}(0) - t, 0\} = \mathrm{INF}\hat{\varepsilon}(0) - t > 0$. Thus $\mathrm{INF}\hat{\varepsilon}(0) - t = \mathrm{INF}\hat{\varepsilon}(a) - t\quad \forall a \in \omega$, which signifies that $\mathrm{INF}\hat{\varepsilon}(0) = \mathrm{INF}\hat{\varepsilon}(a)\ \forall a \in \omega$. Therefore the infimum of all images of $\hat{\varepsilon}$ is equal.

Case 2, let $\mathrm{INF}\hat{\varepsilon}(0) \leq t$. Then $\mathrm{INF}\hat{\varepsilon}(0) - t \leq 0$ and so $\mathrm{INF}_t^- \hat{\varepsilon}(0) = \max\{\mathrm{INF}\hat{\varepsilon}(0) - t, 0\} = 0$. Thus $\forall a \in \omega$, $\max\{\mathrm{INF}\hat{\varepsilon}(a) - t, 0\} = \mathrm{INF}_t^- \hat{\varepsilon}(a) = \mathrm{INF}_t^- \hat{\varepsilon}(0) = 0$. Hence, $\mathrm{INF}\hat{\varepsilon}(a) - t \leq 0 \quad \forall a \in \omega$, which signifies that $\mathrm{INF}\hat{\varepsilon}(a) \leq t \ \forall a \in \omega$. Therefore, $\sup\{\mathrm{INF}\hat{\varepsilon}(x) | x \in \omega\} \leq t$.

Assume that the infimum of all images of $\hat{\varepsilon}$ is equal or $\sup\{\mathrm{INF}\hat{\varepsilon}(a) | a \in \omega\} \leq t$. If the infimum of all images of $\hat{\varepsilon}$ is equal, then it is clear that $\hat{\varepsilon}$ is an $\inf_t^-$-HFSUPI of ω. On the other hand, let $\sup\{\mathrm{INF}\hat{\varepsilon}(a) | a \in \omega\} \leq t$. Then $\mathrm{INF}_t^- \hat{\varepsilon}(a) = \max\{\mathrm{INF}\hat{\varepsilon}(a) - t, 0\} = 0 \quad \forall a \in \omega$. Obviously, $\mathrm{INF}_t^- \hat{\varepsilon}(0) \geq \mathrm{INF}_t^- \hat{\varepsilon}(a)$ and $\mathrm{INF}_t^- \hat{\varepsilon}(a) \geq \min\{\mathrm{INF}_t^- \hat{\varepsilon}((c \cdot b) \cdot (c \cdot a)), \mathrm{INF}_t^- \hat{\varepsilon}(b)\} \ \forall a, b, c \in \omega$. Hence, $\hat{\varepsilon}$ is an $\inf_t^-$-HFSUPI of ω.

Theorem 3.18: A HFS $\hat{\varepsilon}$ on ω is an $\inf_t^+$-HFSUPI of ω if and only if the infimum of all images of $\hat{\varepsilon}$ is equal or $\inf\{\mathrm{INF}\hat{\varepsilon}(x) | x \in \omega\} + t \geq 1$.

Proof: Assume $\hat{\varepsilon}$ is an $\inf_t^+$-HFSUPI of ω. By Lemma 3.16, we obtain $\mathrm{INF}_t^+ \hat{\varepsilon}(0) = \mathrm{INF}_t^+ \hat{\varepsilon}(a) \ \forall a \in \omega$. Now, let's think about the next two cases.

Case 1, suppose $\mathrm{INF}\hat{\varepsilon}(0) + t < 1$. Let $a \in S$. Then $\mathrm{INF}_t^+ \hat{\varepsilon}(a) = \mathrm{INF}_t^+ \hat{\varepsilon}(0) = \min\{\mathrm{INF}\hat{\varepsilon}(0) + t, 1\} = \mathrm{INF}\hat{\varepsilon}(0) + t < 1$, which signifies that $\mathrm{INF}_t^+ \hat{\varepsilon}(a) = \mathrm{INF}\hat{\varepsilon}(a) + t$. Thus $\mathrm{INF}\hat{\varepsilon}(a) + t = \mathrm{INF}\hat{\varepsilon}(0) + t$ and so $\mathrm{INF}\hat{\varepsilon}(a) = \mathrm{INF}\hat{\varepsilon}(0)$. Hence the infimum of all images of $\hat{\varepsilon}$ is equal.

Case 2, suppose $\mathrm{INF}\hat{\varepsilon}(0) + t \geq 1$. Let $a \in \omega$. Then $\mathrm{INF}_t^+ \hat{\varepsilon}(a) = \mathrm{INF}_t^+ \hat{\varepsilon}(0) = \min\{\mathrm{INF}\hat{\varepsilon}(0) + t, 1\} = 1$. Thus $\mathrm{INF}\hat{\varepsilon}(a) + t \geq 1$ and so $\mathrm{INF}\hat{\varepsilon}(a) \geq 1 - t$. Hence, $\inf\{\mathrm{INF}\hat{\varepsilon}(a) | a \in \omega\} \geq 1 - t$ which implies that $\inf\{\mathrm{INF}\hat{\varepsilon}(a) | a \in \omega\} + t \geq 1$.

By the two cases, we can conclude that the infimum of all images of $\hat{\varepsilon}$ is equal or $\inf\{\mathrm{INF}\hat{\varepsilon}(x) | x \in \omega\} + t \geq 1$.

Assume that the infimum of all images of $\hat{\varepsilon}$ is equal or $\inf\{\mathrm{INF}\hat{\varepsilon}(x) | x \in \omega\} + t \geq 1$. It is clear that $\hat{\varepsilon}$ is an $\inf_t^-$-HFSUPI of ω if the infimum of all images of $\hat{\varepsilon}$ is equal. On the other hand, let $\inf\{\mathrm{INF}\hat{\varepsilon}(x) | x \in \omega\} + t \geq 1$. Then $\mathrm{INF}_t^+ \hat{\varepsilon}(a) = \min\{\mathrm{INF}\hat{\varepsilon}(a) + t, 1\} = 1 \quad \forall a \in \omega$. Obviously, $\mathrm{INF}_t^+ \hat{\varepsilon}(0) \geq \mathrm{INF}_t^+ \hat{\varepsilon}(a)$ and $\mathrm{INF}_t^+ \hat{\varepsilon}(a) \geq \min\{\mathrm{INF}_t^+ \hat{\varepsilon}((c \cdot b) \cdot (c \cdot a)), \mathrm{INF}_t^+ \hat{\varepsilon}(b)\} \ \forall a, b, c \in \omega$. Therefore, $\hat{\varepsilon}$ is an $\inf_t^+$-HFSUPI of ω.

The following corollary is obtained from Theorem 3.17 (or Theorem 3.18) by assuming $t = 0$.

Corollary 3.19: A HFS $\hat{\varepsilon}$ on ω is an inf-HFSUPI of ω if and only if the infimum of all images of $\hat{\varepsilon}$ is equal.

4. t-Level subsets of a HFS

There are two sections in this paragraph. We examine inf-upper t-level subsets in the first part, and inf-lower t-level subsets in the second.

Definition 4.1: Let $\hat{\varepsilon}$ be a HFS on ω and $t \in [0,1]$, the sets $U_{\mathrm{INF}}(\hat{\varepsilon}; t) = \{x \in \omega | \mathrm{INF}\hat{\varepsilon}(x) \geq t\}$ and $L_{\mathrm{INF}}(\hat{\varepsilon}; t) = \{x \in \omega | \mathrm{INF}\hat{\varepsilon}(x) \leq t\}$ are called an *inf-upper t-level subset* and an *inf-lower t-level subset* of $\hat{\varepsilon}$, respectively.

4.1 *inf-Upper t-level subsets*

The relationships between inf-HFUPSs (resp., inf-HFUPFs, inf-HFUPIs, inf-HFSUPIs) and their inf-upper t-level subsets are covered in this section.

Theorem 4.2: A HFS $\hat{\varepsilon}$ on ω is an inf-HFUPS of ω if and only if $\forall t \in [0,1]$, $\emptyset \neq U_{\mathrm{INF}}(\hat{\varepsilon}; t)$ is a UPS of ω.

Proof: Assume $\hat{\varepsilon}$ is an inf-HFUPS of ω. Let $t \in [0,1]$ be such that $U_{\mathrm{INF}}(\hat{\varepsilon}; t) \neq \emptyset$ and let $x, y \in U_{\mathrm{INF}}(\hat{\varepsilon}; t)$. Then $\min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} \geq t$. By assumption, $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq t$ and so $x \cdot y \in U_{\mathrm{INF}}(\hat{\varepsilon}; t)$. Therefore, $U_{\mathrm{INF}}(\hat{\varepsilon}; t)$ is a UPS of ω.

Conversely, assume $\forall t \in [0,1]$, a subset $\emptyset \neq U_{\mathrm{INF}}(\hat{\varepsilon}; t)$ of ω is a UPS of ω. Let $x, y \in \omega$. We choose $t = \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\}$. Thus $\mathrm{INF}\hat{\varepsilon}(x) \geq t$ and $\mathrm{INF}\hat{\varepsilon}(y) \geq t$ which implies $x, y \in U_{\mathrm{INF}}(\hat{\varepsilon}; t)$. By assumption, $U_{\mathrm{INF}}(\hat{\varepsilon}; t)$ is a UPS of ω and so $x \cdot y \in U_{\mathrm{INF}}(\hat{\varepsilon}; t)$. Hence, $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq t = \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\}$.Therefore, $\hat{\varepsilon}$ is an inf-HFUPS of ω.

The argument used to prove Theorem 4.2 may also be used to prove Theorems 4.3, 4.4, and 4.5.

Theorem 4.3: *A HFS $\hat{\varepsilon}$ on ω is an inf-HFUPF of ω if and only if* $\forall t \in [0,1]$, $\emptyset \neq U_{INF}(\hat{\varepsilon}; t)$ *is a UPF of ω.*

Theorem 4.4: *A HFS $\hat{\varepsilon}$ on ω is an inf-HFUPI of ω if and only if* $\forall t \in [0,1]$, $\emptyset \neq U_{INF}(\hat{\varepsilon}; t)$ *is a UPI of ω.*

Theorem 4.5: *A HFS $\hat{\varepsilon}$ on ω is an inf-HFSUPI of ω if and only if* $\forall t \in [0,1]$, $\emptyset \neq U_{INF}(\hat{\varepsilon}; t)$ *is a SUPI of ω.*

4.2 *inf-Lower t-level subsets*

In this part, we discuss the relations between inf-HFUPSs (resp., inf-HFUPFs, inf-HFUPIs, inf-HFSUPIs) and their inf-lower t-level subsets.

For a HFS $\hat{\varepsilon}$ on ω, the HFS $\hat{\varepsilon}^*$ defined by $\hat{\varepsilon}^*(x) = \{1 - \mathrm{INF}\hat{\varepsilon}(x)\}$ $\forall x \in \omega$ is said to be the *infimum complement* of $\hat{\varepsilon}$. Then $\mathrm{INF}\hat{\varepsilon}^*(x) = 1 - \mathrm{INF}\hat{\varepsilon}(x)$ $\forall x \in \omega$. As we can see $(\hat{\varepsilon}^*)^*(x) = \{\mathrm{INF}\hat{\varepsilon}(x)\}$ $\forall x \in \omega$, and then $\mathrm{INF}(\hat{\varepsilon}^*)^*(x) = \mathrm{INF}\hat{\varepsilon}(x)$ $\forall x \in \omega$.

Theorem 4.6: *Let $\hat{\varepsilon}$ be a HFS on ω. Then $\hat{\varepsilon}^*$ is an inf-HFUPS of ω if and only if* $\forall t \in [0,1]$, $\emptyset \neq L_{INF}(\hat{\varepsilon}; t)$ *is a UPS of ω.*

Proof: Assume $\hat{\varepsilon}^*$ is an inf-HFUPS of ω. Let $t \in [0,1]$ be such that $L_{\text{INF}}(\hat{\varepsilon}; t) \neq \emptyset$ and let $x, y \in L_{\text{INF}}(\hat{\varepsilon}; t)$. Then $\max\{\text{INF}\hat{\varepsilon}(x), \text{INF}\hat{\varepsilon}(y)\} \leq t$. By assumption, $1 - \text{INF}\hat{\varepsilon}(x \cdot y) = \text{INF}\hat{\varepsilon}^*(x \cdot y) \geq \min\{\text{INF}\hat{\varepsilon}^*(x), \text{INF}\hat{\varepsilon}^*(y)\} = \min\{1 - \text{INF}\hat{\varepsilon}(x), 1 - \text{INF}\hat{\varepsilon}(y)\} = 1 - \max\{\text{INF}\hat{\varepsilon}(x), \text{INF}\hat{\varepsilon}(y)\}$. Hence, $\text{INF}\hat{\varepsilon}(x \cdot y) \leq \max\{\text{INF}\hat{\varepsilon}(x), \text{INF}\hat{\varepsilon}(y)\} \leq t$, and so $x \cdot y \in L_{\text{INF}}(\hat{\varepsilon}; t)$. Therefore, $L_{\text{INF}}(\hat{\varepsilon}; t)$ is a UPS of ω.

Conversely, assume $\forall t \in [0,1]$, a subset $\emptyset \neq L_{\text{INF}}(\hat{\varepsilon}; t)$ of ω is a UPS of ω. Let $x, y \in \omega$. Choose $t = \max\{\text{INF}\hat{\varepsilon}(x), \text{INF}\hat{\varepsilon}(y)\}$. Then $\text{INF}\hat{\varepsilon}(x) \leq t$ and $\text{INF}\hat{\varepsilon}(y) \leq t$ which implies $x, y \in L_{\text{INF}}(\hat{\varepsilon}; t)$. By assumption, $L_{\text{INF}}(\hat{\varepsilon}; t)$ is a UPS of ω and so $x \cdot y \in L_{\text{INF}}(\hat{\varepsilon}; t)$. Thus $\text{INF}\hat{\varepsilon}(x \cdot y) \leq t = \max\{\text{INF}\hat{\varepsilon}(x), \text{INF}\hat{\varepsilon}(y)\}$. Hence, $\text{INF}\hat{\varepsilon}^*(x \cdot y) = 1 - \text{INF}\hat{\varepsilon}(x \cdot y) \geq 1 - \max\{\text{INF}\hat{\varepsilon}(x), \text{INF}\hat{\varepsilon}(y)\} = \min\{1 - \text{INF}\hat{\varepsilon}(x), 1 - \text{INF}\hat{\varepsilon}(y)\} = \min\{\text{INF}\hat{\varepsilon}^*(x), \text{INF}\hat{\varepsilon}^*(y)\}$.
Therefore, $\hat{\varepsilon}^*$ is an inf-HFUPS of ω.

The argument used to prove Theorem 4.6 may also be used to prove Theorems 4.7, 4.8, and 4.9.

Theorem 4.7: *Let $\hat{\varepsilon}$ be a HFS on ω. Then $\hat{\varepsilon}^*$ is an inf-HFUPF of ω if and only if* $\forall t \in [0,1]$, $\emptyset \neq L_{INF}(\hat{\varepsilon}; t)$ *is a UPF of ω.*

Theorem 4.8: *Let $\hat{\varepsilon}$ be a HFS on ω. Then $\hat{\varepsilon}^*$ is an inf-HFUPI of ω if and only if* $\forall t \in [0,1]$, $\emptyset \neq L_{INF}(\hat{\varepsilon}; t)$ *is a UPI of ω.*

Theorem 4.9: *Let $\hat{\varepsilon}$ be a HFS on ω. Then $\hat{\varepsilon}^*$ is an inf-HFSUPI of ω if and only if* $\forall t \in [0,1]$, $\emptyset \neq L_{INF}(\hat{\varepsilon}; t)$ *is a SUPI of ω.*

5. Characterizing inf-types of HFSs

We divide this section into three parts. We study inf-types of HFSs by using fuzzy sets and PFSs in the first part. In the second part, we study inf-types of HFSs by using HFSs. In the final part, we study inf-types of HFSs by using interval-valued fuzzy sets and cubic sets.

5.1 *FSs and PFSs*

In this part, we study HFSs: inf-HFUPS, inf-HFUPF, inf-HFUPI and inf-HFSUPI by using FSs and PFSs.

Definition 5.1: [13] A FS ϕ in ω is called

(1) a *fuzzy UPS* (FUPS) of ω if $(\forall x, y \in \omega)(\phi(x \cdot y) \geq \min\{\phi(x), \phi(y)\})$,
(2) a *fuzzy UPF* (FUPF) of ω if
 (i) $(\forall x \in \omega)(\phi(0) \geq \phi(x))$,
 (ii) $(\forall x, y \in \omega)(\phi(y) \geq \min\{\phi(x \cdot y), \phi(x)\})$,
(3) a *fuzzy UPI* (FUPI) of ω if
 (i) $(\forall x \in \omega)(\phi(0) \geq f\phi(x))$,
 (ii) $(\forall x, y, z \in \omega)(\phi(x \cdot z) \geq \min\{\phi(x \cdot (y \cdot z)), \phi(y)\})$.

Definition 5.2: [1] A FS ϕ in ω is called a *fuzzy SUPI* (FSUPI) of ω if

(i) $(\forall x \in \omega)(\phi(0) \geq \phi(x))$,

(ii) $(\forall x, y, z \in \omega)(\phi(x) \geq \min\{\phi((z \cdot y) \cdot (z \cdot x)), \phi(y)\})$.

Definition 5.3: [5] A FS ϕ in ω is called

(1) an *anti-fuzzy UPS* (AFUPS) of ω if $(\forall x, y \in \omega)(\phi(x \cdot y) \leq \max\{\phi(x), \phi(y)\})$,

(2) an *anti-fuzzy UPI* (AFUPI) of ω if
   (i) $(\forall x \in \omega)(\phi(0) \leq \phi(x))$,
   (ii) $(\forall x, y, z \in \omega)(\phi(x \cdot z) \leq \max\{\phi(x \cdot (y \cdot z)), \phi(y)\})$.

Definition 5.4: A FS ϕ in ω is called

(1) an *anti-fuzzy UPF* (AFUPF) of ω if
   (i) $(\forall x \in \omega)(\phi(0) \leq \phi(x))$,
   (ii) $(\forall x, y \in \omega)(\phi(y) \leq \max\{\phi(x \cdot y), \phi(x)\})$,

(2) an *anti-fuzzy SUPI* (AFSUPI) of ω if
   (i) $(\forall x \in \omega)(\phi(0) \leq \phi(x))$,
   (ii) $(\forall x, y, z \in \omega)(\phi(x) \leq \max\{\phi((z \cdot y) \cdot (z \cdot x)), \phi(y)\})$.

If ϕ is a FS of ω, we define the HFS $\mathcal{H}_\phi$ on ω by $\mathcal{H}_\phi(x) = \{\phi(x)\}$ $\forall x \in \omega$, and then $\mathrm{INF}\mathcal{H}_\phi(x) = \phi(x) = \sup\mathcal{H}_\phi$ $\forall x \in \omega$. For every HFS $\hat{\varepsilon}$ on ω, we define the FSs $\mathcal{F}_{\hat{\varepsilon}}$ and $\mathcal{F}^{\hat{\varepsilon}}$ of ω by $\mathcal{F}_{\hat{\varepsilon}}(x) = \mathrm{INF}\hat{\varepsilon}(x)$ and $\mathcal{F}^{\hat{\varepsilon}}(x) = \sup\hat{\varepsilon}(x)$ $\forall x \in \omega$.

Theorem 5.5: *Let ϕ be a FS of ω. Then the following are equivalent.*

(1) ϕ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω.

(2) $\mathcal{H}_\phi$ is an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω.

(3) $\mathcal{H}_\phi$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of ω.

Proof: It is straightforward.

Corollary 5.6: *If ϕ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω, then $\mathcal{H}_\phi$ is an inf_t^α-HFUPS (resp., inf_t^α-HFUPF, inf_t^α-HFUPI, inf_t^α-HFSUPI) of ω.*

Proof: It is implied by Propositions 3.2 and 3.4 and Theorem 5.5.

Corollary 5.7: *Let ϕ be a FS of ω and $s \in (0,1]$. The following are true.*

(1) *If $\mathcal{H}_\phi$ is an inf_t^--HFUPS (resp., inf_t^--HFUPF, inf_t^--HFUPI, inf_t^--HFSUPI) of ω $\forall t \in (0, s]$, then ϕ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω.*

(2) *If $\mathcal{H}_\phi$ is an inf_t^+-HFUPS (resp., inf_t^+-HFUPF, inf_t^+-HFUPI, inf_t^+-HFSUPI) of ω $\forall t \in (0, s]$, then ϕ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω.*

Proof: It is implied by Proposition 3.6 and Theorem 5.5.

Let ϕ be a FS of ω, we denote $\dagger = 1 - \sup\{\phi(x) | x \in \omega\}$ and $\ddagger = \inf\{\phi(x) | x \in \omega\}$. Guntasow et al. [1] applied the concept of the translations of a fuzzy set to UP-algebras seen in the following definition.

Definition 5.8: [1] Let ϕ be a FS of ω, $t \in [0, \dagger]$ and $s \in [0, \ddagger]$.

(1) The FS $\phi_t^\dagger$ of ω defined by $\phi_t^\dagger: \omega \to [0,1], x \mapsto \phi(x) + t$ is said to be a *fuzzy t-translation* of ϕ of type I.
(2) The FS $\phi_s^\ddagger$ of ω defined by $\phi_s^\ddagger: \omega \to [0,1], x \mapsto \phi(x) - s$ is said to be a *fuzzy s-translation* of ϕ of type II.

We introduce general concepts of the fuzzy t-translation of ϕ of type I and the fuzzy s-translation of ϕ of type II in the following definition.

Definition 5.9: Let ϕ be a FS of ω and $t \in [0,1]$.

(1) The FS ϕ_t^+ of ω defined by $\phi_t^+: \omega \to [0,1], x \mapsto \min\{\phi(x) + t, 1\}$.
(2) The FS ϕ_t^- of ω defined by $\phi_t^-: \omega \to [0,1], x \mapsto \max\{\phi(x) - t, 0\}$.

Form Definitions 5.8 and 5.9, we get that $\phi_t^+ = \phi_t^\dagger \ \ \forall t \in [0, \dagger]$ and $\phi_s^- = \phi_s^\ddagger \ \ \forall s \in [0, \ddagger]$. For FSs ϕ and τ of ω, define $\phi \leq \tau$ if $\phi(x) \leq \tau(x)$ $\forall x \in \omega$. For a FS ϕ of ω and $s, t \in [0,1]$, we have the following. [label=($*$)]

(1) $\phi_s^- \leq \phi \leq \phi_t^+$.
(2) If $s \geq t$, then $\phi_s^- \leq \phi_t^-$.
(3) If $s \leq t$, then $\phi_s^+ \leq \phi_t^+$.

Next, we use the FSs ϕ_t^+ and ϕ_t^- to study HFSs.

Theorem 5.10: *Let ϕ be a FS of ω. The following are true.*

(1) *If $\mathcal{H}_\phi$ is an inf_t^--HFUPS (resp., inf_t^--HFUPF, inf_t^--HFUPI, inf_t^--HFSUPI) of ω, then ϕ_t^- is a FUPS (resp., FUPF, FUPI, FSUPI) of ω.*
(2) *If $\mathcal{H}_\phi$ is an inf_t^+-HFUPS (resp., inf_t^+-HFUPF, inf_t^+-HFUPI, and inf_t^+-HFSUPI) of ω, then ϕ_t^+ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω.*

Proof:

(1) Assume $\mathcal{H}_\phi$ is an inf_t^--HFUPS of ω.. Then $\forall a, b \in \omega$, $\phi_t^-(a \cdot b) = \max\{\phi(a \cdot b) - t, 0\} = \max\{\mathrm{INF}\mathcal{H}_\phi(a \cdot b) - t, 0\} = \mathrm{INF}_t^-\mathcal{H}_\phi(a \cdot b) \geq \min\{\mathrm{INF}_t^-\mathcal{H}_\phi(a), \ \mathrm{INF}_t^-\mathcal{H}_\phi(b)\} = \min\{\max\{\mathrm{INF}\mathcal{H}_\phi(a) - t, 0\}, \ \max\{\mathrm{INF}\mathcal{H}_\phi(b) - t, 0\}\} = \min\{\max\{\phi(a) - t, 0\}, \ \max\{\phi(b) - t, 0\}\} = \min\{\phi_t^-(a), \phi_t^-(b)\}$. Thus ϕ_t^- is a FUPS of ω. We are also able to demonstrate the other results.

(2) Assume $\mathcal{H}_\phi$ is an $\inf_t^+$-HFUPS of ω. Then $\forall a,b \in \omega$, $\phi_t^+(a \cdot b) = \min\{\phi(a \cdot b) + t, 1\} = \min\{\mathrm{INF}\mathcal{H}_\phi(a \cdot b) + t, 1\} = \mathrm{INF}_t^+\mathcal{H}_\phi(a \cdot b) \geq \min\{\mathrm{INF}_t^+\mathcal{H}_\phi(a), \mathrm{INF}_t^+\mathcal{H}_\phi(b)\} = \min\{\min\{\mathrm{INF}\mathcal{H}_\phi(a) + t, 1\}, \min\{\mathrm{INF}\mathcal{H}_\phi(b) + t, 1\}\} = \min\{\min\{\phi(a) + t, 1\}, \min\{\phi(b) + t, 1\}\} = \min\{\phi_t^+(a), \phi_t^+(b)\}$. Hence, ϕ_t^+ is a FUPS of ω. We are also able to demonstrate the other results.

Proposition 5.11: *Let $\hat{\varepsilon}$ be a HFS on ω. Then $INF_t^\alpha\hat{\varepsilon}(x) = (\mathcal{F}_{\hat{\varepsilon}})_t^\alpha(x)$ $\forall x \in \omega$.*

Proof: Let $x \in \omega$. Then $\mathrm{INF}_t^-\hat{\varepsilon}(x) = \max\{\mathrm{INF}\hat{\varepsilon}(x) - t, 0\} = \max\{\mathcal{F}_{\hat{\varepsilon}}(x) - t, 0\} = (\mathcal{F}_{\hat{\varepsilon}})_t^-(x)$ and $\mathrm{INF}_t^+\hat{\varepsilon}(x) = \min\{\mathrm{INF}\hat{\varepsilon}(x) + t, 1\} = \min\{\mathcal{F}_{\hat{\varepsilon}}(x) + t, 1\} = (\mathcal{F}_{\hat{\varepsilon}})_t^+(x)$. Therefore, $\mathrm{INF}_t^\alpha\hat{\varepsilon}(x) = (\mathcal{F}_{\hat{\varepsilon}})_t^\alpha(x)$ $\forall x \in \omega$.

Theorem 5.12: *Let $\hat{\varepsilon}$ be a HFS on ω. The following are equivalent.*

(1) *$\hat{\varepsilon}$ is an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω.*
(2) *$\mathcal{F}_{\hat{\varepsilon}}$ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω.*
(3) *$\mathcal{F}_{\hat{\varepsilon}^*}$ is an AFUPS (resp., AFUPF, AFUPI, AFSUPI) of ω.*
(4) *$(\mathcal{F}_{\hat{\varepsilon}})_t^\alpha$ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω $\forall t \in [0,1]$ and $\alpha \in \{-,+\}$.*
(5) *$(\mathcal{F}_{\hat{\varepsilon}^*})_t^\alpha$ is an AFUPS (resp., AFUPF, AFUPI, AFSUPI) of ω $\forall t \in [0,1]$ and $\alpha \in \{-,+\}$.*

Proof: $(1) \Rightarrow (4)$. It is implied by Propositions 3.2, 3.4, and 5.11.

$(4) \Rightarrow (2) \Rightarrow (1)$ and $(5) \Rightarrow (3)$. They are clear.

$(1) \Rightarrow (3)$. Assume (1) holds. Let $x, y \in \omega$. Then $\mathcal{F}_{\hat{\varepsilon}^*}(x \cdot y) = \mathrm{INF}\hat{\varepsilon}^*(x \cdot y) = 1 - \mathrm{INF}\hat{\varepsilon}(x \cdot y) \leq 1 - \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} = \max\{1 - \mathrm{INF}\hat{\varepsilon}(x), 1 - \mathrm{INF}\hat{\varepsilon}(y)\} = \max\{\mathcal{F}_{\hat{\varepsilon}^*}(x), \mathcal{F}_{\hat{\varepsilon}^*}(y)\}$. Hence $\mathcal{F}_{\hat{\varepsilon}^*}$ is an AFUPS of ω. We are also able to demonstrate the other results.

$(3) \Rightarrow (5)$. Assume (3) holds. Let $x, y \in \omega$ and $t \in [0,1]$. Then $(\mathcal{F}_{\hat{\varepsilon}^*})_t^-(x \cdot y) = \max\{\mathcal{F}_{\hat{\varepsilon}^*}(x \cdot y) - t, 0\} \leq \max\{\max\{\mathcal{F}_{\hat{\varepsilon}^*}(x), \mathcal{F}_{\hat{\varepsilon}^*}(y)\} - t, 0\} = \max\{\max\{\mathcal{F}_{\hat{\varepsilon}^*}(x) - t, \mathcal{F}_{\hat{\varepsilon}^*}(y) - t\}, 0\} = \max\{\max\{\mathcal{F}_{\hat{\varepsilon}^*}(x) - t, 0\}, \max\{\mathcal{F}_{\hat{\varepsilon}^*}(y) - t, 0\}\} = \max\{(\mathcal{F}_{\hat{\varepsilon}^*})_t^-(x), (\mathcal{F}_{\hat{\varepsilon}^*})_t^-(y)\}$ and $(\mathcal{F}_{\hat{\varepsilon}^*})_t^+(x \cdot y) = \min\{\mathcal{F}_{\hat{\varepsilon}^*}(x \cdot y) + t, 1\} \leq \min\{\max\{\mathcal{F}_{\hat{\varepsilon}^*}(x), \mathcal{F}_{\hat{\varepsilon}^*}(y)\} + t, 1\} = \min\{\max\{\mathcal{F}_{\hat{\varepsilon}^*}(x) + t, \mathcal{F}_{\hat{\varepsilon}^*}(y) + t\}, 1\} = \max\{\min\{\mathcal{F}_{\hat{\varepsilon}^*}(x) + t, 1\}, \min\{\mathcal{F}_{\hat{\varepsilon}^*}(y) + t, 1\}\} = \max\{(\mathcal{F}_{\hat{\varepsilon}^*})_t^+(x), (\mathcal{F}_{\hat{\varepsilon}^*})_t^+(y)\}$. Hence, $(\mathcal{F}_{\hat{\varepsilon}^*})_t^\alpha$ is an AFUPS of ω $\forall t \in [0,1]$ and $\alpha \in \{-,+\}$. We are also able to demonstrate the other results.

Corollary 5.13: *Let $\hat{\varepsilon}$ be a HFS on ω. If $\mathcal{F}_{\hat{\varepsilon}}$ is a FUPS (resp., FUPF, FUPI, FSUPI) of ω, then $\hat{\varepsilon}$ is an inf_t^α-HFUPS (resp., inf_t^α-HFUPF, inf_t^α-HFUPI, inf_t^α-HFSUPI) of ω.*

Proof: It is implied by Propositions 3.2 and 3.4 and Theorem 5.12.

Corollary 5.14: *Let* $\hat{\varepsilon}$ *be a HFS on* ω *and* $s \in (0,1]$. *The following are true.*

(1) *If* $\hat{\varepsilon}$ *is an* inf_t^-*-HFUPS (resp.,* inf_t^-*-HFUPF,* inf_t^-*-HFUPI,* inf_t^-*-HFSUPI) of* ω $\forall t \in (0,s]$, *then* $\mathcal{F}_{\hat{\varepsilon}}$ *is a FUPS (resp., FUPF, FUPI, FSUPI) of* ω.

(2) *If* $\hat{\varepsilon}$ *is an* inf_t^+*-HFUPS (resp.,* inf_t^+*-HFUPF,* inf_t^+*-HFUPI,* inf_t^+*-HFSUPI) of* ω $\forall t \in (0,s]$, *then* $\mathcal{F}_{\hat{\varepsilon}}$ *is a FUPS (resp., FUPF, FUPI, FSUPI) of* ω.

Proof: It is implied by Proposition 3.6 and Theorem 5.12.

Theorem 5.15: *Let* $\hat{\varepsilon}$ *be a HFS on* ω. *Then* $\mathcal{F}^{\hat{\varepsilon}}$ *is a FUPS (resp., FUPF, FUPI, FSUPI) of* ω *if and only if* $\hat{\varepsilon}$ *is a* sup*-HFUPS (resp.,* sup*-HFUPF,* sup*-HFUPI,* sup*-HFSUPI) of* ω.

Proof: It is straightforward.

PFSs were first defined as follows by Yager [17] and Yager and Abbasov [18] in 2013.

Definition 5.16: [17, 18] A *Pythagorean fuzzy set* (PFS) $\mathcal{P}$ in ω is an object having the form $\mathcal{P} = \{(x, \phi(x), \tau(x)) | x \in \omega\}$, where $\phi: A \to [0,1]$ and $\tau: A \to [0,1]$, and $0 \leq (\phi(x))^2 + (\tau(x))^2 \leq 1$ $\forall x \in \omega$.

We will utilize the symbol (ϕ, τ) from the PFS $\{(x, \phi(x), \tau(x)) | x \in \omega\}$ for the sake of simplicity. For a FS ϕ of ω, we define a FS $\frac{\phi}{2}$ by $\frac{\phi}{2}(x) = \frac{\phi(x)}{2}$ $\forall x \in \omega$. Then $(\frac{\phi}{2}, \frac{\tau}{2})$ is a PFS in ω for all FSs ϕ and τ of ω.

Proposition 5.17: Let $\hat{\varepsilon}$ be a HFS on ω. Then the following hold.

(1) $(\mathcal{F}_{\hat{\varepsilon}}, \mathcal{F}_{\hat{\varepsilon}^*})$ is a PFS in ω.

(2) $((\mathcal{F}_{\hat{\varepsilon}})_t^-, (\mathcal{F}_{\hat{\varepsilon}^*})_s^-)$, $((\mathcal{F}_{\hat{\varepsilon}})_t^+, (\mathcal{F}_{\hat{\varepsilon}^*})_t^-)$, $((\mathcal{F}_{\hat{\varepsilon}})_t^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+)$ and $(\frac{(\mathcal{F}_{\hat{\varepsilon}})_t^\alpha}{2}, \frac{(\mathcal{F}_{\hat{\varepsilon}^*})_s^\beta}{2})$ are PFSs in ω $\forall s, t \in [0,1]$ and $\alpha, \beta \in \{-, +\}$.

(3) There exist $s, t \in [0,1]$ and $\alpha, \beta \in \{-, +\}$ such that $((\mathcal{F}_{\hat{\varepsilon}})_t^\alpha, (\mathcal{F}_{\hat{\varepsilon}^*})_s^\beta)$ is a PFS in ω.

Definition 5.18: [10] A PFS $\mathcal{P} = (\phi, \tau)$ in ω is called

(1) a *Pythagorean fuzzy UPS* (PFUPS) of ω whether it complies with the following:
$$(\forall x, y \in \omega)(\phi(x \cdot y) \geq \min\{\phi(x), \phi(y)\}), \tag{5.1}$$
$$(\forall x, y \in \omega)(\tau(x \cdot y) \leq \max\{\tau(x), \tau(y)\}), \tag{5.2}$$

(2) a *Pythagorean fuzzy UPF* (PFUPF) of ω whether it complies with the following:
$$(\forall x \in \omega)(\phi(0) \geq \phi(x)), \tag{5.3}$$
$$(\forall x \in \omega)(\tau(0) \leq \tau(x)), \tag{5.4}$$

$$(\forall x, y \in \omega)(\phi(y) \geq \min\{\phi(x \cdot y), \phi(x)\}), \tag{5.5}$$

$$(\forall x, y \in \omega)(\tau(y) \leq \max\{\tau(x \cdot y), \tau(x)\}), \tag{5.6}$$

(3) a *Pythagorean fuzzy UPI* (PFUPI) of ω if it fulfills the assertions (5.3), (5.4), and

$$(\forall x, y, z \in \omega)(\phi(x \cdot z) \geq \min\{\phi(x \cdot (y \cdot z)), \phi(y)\}), \tag{5.7}$$

$$(\forall x, y, z \in \omega)(\tau(x \cdot z) \leq \max\{\tau(x \cdot (y \cdot z)), \tau(y)\}), \tag{5.8}$$

(4) a *Pythagorean fuzzy SUPI* (PFSUPI) of ω if it fulfills the assertions (5.3), (5.4), and

$$(\forall x, y, z \in \omega)(\phi(x) \geq \min\{\phi((z \cdot y) \cdot (z \cdot x)), \phi(y)\}), \tag{5.9}$$

$$(\forall x, y, z \in \omega)(\tau(x) \leq \max\{\tau((z \cdot y) \cdot (z \cdot x)), \tau(y)\}). \tag{5.10}$$

From Definition 5.18, Proposition 5.17, and Theorem 5.12, we obtain characterizations of an inf-HFUPS, an inf-HFUPF, an inf-HFUPI, and an inf-HFSUPI of a UP-algebra in terms of PFSs seen in Theorem 5.19.

Theorem 5.19: *Let $\hat{\varepsilon}$ a HFS on ω. Then the following are equivalent.*

(1) *$\hat{\varepsilon}$ is an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω.*

(2) *$(\mathcal{F}_{\hat{\varepsilon}}, \mathcal{F}_{\hat{\varepsilon}^*})$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of ω.*

(3) *$((\mathcal{F}_{\hat{\varepsilon}})_t^-, (\mathcal{F}_{\hat{\varepsilon}^*})_s^-)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of ω $\forall s, t \in [0,1]$.*

(4) *$((\mathcal{F}_{\hat{\varepsilon}})_t^+, (\mathcal{F}_{\hat{\varepsilon}^*})_t^-)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of ω $\forall t \in [0,1]$.*

(5) *$((\mathcal{F}_{\hat{\varepsilon}})_t^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of ω $\forall t \in [0,1]$.*

(6) *$\left(\frac{(\mathcal{F}_{\hat{\varepsilon}})_t^\alpha}{2}, \frac{(\mathcal{F}_{\hat{\varepsilon}^*})_s^\beta}{2}\right)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of ω $\forall s, t \in [0,1]$ and $\alpha, \beta \in \{-, +\}$.*

(7) For all $s, t \in [0,1]$ and $\alpha, \beta \in \{-, +\}$ if $((\mathcal{F}_{\hat{\varepsilon}})_t^\alpha, (\mathcal{F}_{\hat{\varepsilon}^*})_s^\beta)$ is a PFS in ω, then $((\mathcal{F}_{\hat{\varepsilon}})_t^\alpha, (\mathcal{F}_{\hat{\varepsilon}^*})_s^\beta)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of ω.

5.2 *HFSs*

We study inf-types of HFSs by HFSs.

Definition 5.20: [8] A HFS $\hat{\varepsilon}$ on ω is called

(1) a *hesitant fuzzy UPS* (HFUPS) of ω if $(\forall x, y \in \omega)(\hat{\varepsilon}(x \cdot y) \supseteq \hat{\varepsilon}(x) \cap \hat{\varepsilon}(y))$,

(2) a *hesitant fuzzy UPF* (HFUPF) of ω if
 (i) $(\forall x \in \omega)(\hat{\varepsilon}(0) \supseteq \hat{\varepsilon}(x))$,
 (ii) $(\forall x, y \in \omega)(\hat{\varepsilon}(y) \supseteq \hat{\varepsilon}(x \cdot y) \cap \hat{\varepsilon}(x))$,

(3) a *hesitant fuzzy UPI* (HFUPI) of ω if
 (i) $(\forall x \in \omega)(\hat{\varepsilon}(0) \supseteq \hat{\varepsilon}(x))$,
 (ii) $(\forall x, y, z \in \omega)(\hat{\varepsilon}(x \cdot z) \supseteq \hat{\varepsilon}(x \cdot (y \cdot z)) \cap \hat{\varepsilon}(y))$,

(4) a *hesitant fuzzy SUPI* (HFSUPI) of ω if [label=()]
   (i) $(\forall x \in \omega)(\hat{\varepsilon}(0) \supseteq \hat{\varepsilon}(x))$,
   (ii) $(\forall x, y, z \in \omega)(\hat{\varepsilon}(x) \supseteq \hat{\varepsilon}((z \cdot y) \cdot (z \cdot x)) \cap \hat{\varepsilon}(y))$.

For a HFS $\hat{\varepsilon}$ on ω and an element $\Bbbk$ of $\wp([0,1])$, we define the HFS $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)$ on ω by $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x) = \{t \in \Bbbk | \mathrm{INF}\hat{\varepsilon}(x) \geq t\}$ for all $x \in \omega$. We denote $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; [0,1])$ by $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}$.

Theorem 5.21: *For a HFS $\hat{\varepsilon}$ on A, the following are equivalent.*
(1) $\hat{\varepsilon}$ *is an* inf*-HFUPS of* ω.
(2) $\mathcal{H}_{INF}^{\hat{\varepsilon}}$ *is a HFUPS of* ω.
(3) $\mathcal{H}_{INF}(\hat{\varepsilon}; \Bbbk)$ *is a HFUPS of* ω $\forall \Bbbk \in \wp([0,1])$.

Proof: $(1) \Rightarrow (3)$. Let $x, y \in \omega$ and $\Bbbk \in \wp([0,1])$. If $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x) \cap \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(y)$ is empty, then $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x \cdot y) \supseteq \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x) \cap \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(y)$. On the other hand, let $t \in \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x) \cap \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(y)$. Then $t \in \Bbbk$, $\mathrm{INF}\hat{\varepsilon}(x) \geq t$ and $\mathrm{INF}\hat{\varepsilon}(y) \geq t$. By the assumption (1), $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} \geq t$. Thus $t \in \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x \cdot y)$. Hence, we get $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x \cdot y) \supseteq \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(x) \cap \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)(y)$. Therefore, we obtain $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}; \Bbbk)$ is a HFUPS of ω.

$(3) \Rightarrow (2)$. It is clear.

$(2) \Rightarrow (1)$. Let $x, y \in \omega$. Then $\mathrm{INF}\hat{\varepsilon}(x) \in \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x)$, $\mathrm{INF}\hat{\varepsilon}(y) \in \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(y)$ and $\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y) \in [0,1]$. Thus $\min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} \in \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x) \cap \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(y)$. By the assumption (2), $\min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} \in \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x \cdot y)$. Hence, $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\}$. Therefore, $\hat{\varepsilon}$ is an inf-HFUPS of ω.

The argument used to prove Theorem 5.21 may also be used to prove Theorems 5.22, 5.23, and 5.24.

Theorem 5.22: *For a HFS $\hat{\varepsilon}$ on A, the following are equivalent.*
(1) $\hat{\varepsilon}$ *is an* inf*-HFUPF of* ω.
(2) $\mathcal{H}_{INF}^{\hat{\varepsilon}}$ *is a HFUPF of* ω.
(3) $\mathcal{H}_{INF}(\hat{\varepsilon}; \Bbbk)$ *is a HFUPF of* ω $\forall \Bbbk \in \wp([0,1])$.

Theorem 5.23: *For a HFS $\hat{\varepsilon}$ on A, the following are equivalent.*
(1) $\hat{\varepsilon}$ *is an* inf*-HFUPI of* ω.
(2) $\mathcal{H}_{INF}^{\hat{\varepsilon}}$ *is a HFUPI of* ω.
(3) $\mathcal{H}_{INF}(\hat{\varepsilon}; \Bbbk)$ *is a HFUPI of* ω $\forall \Bbbk \in \wp([0,1])$.

Theorem 5.24: *For a HFS $\hat{\varepsilon}$ on A, the following are equivalent.*
(1) $\hat{\varepsilon}$ *is an* inf*-HFSUPI of* ω.
(2) $\mathcal{H}_{INF}^{\hat{\varepsilon}}$ *is a HFSUPI of* ω.
(3) $\mathcal{H}_{INF}(\hat{\varepsilon}; \Bbbk)$ *is a HFSUPI of* ω $\forall \Bbbk \in \wp([0,1])$.

Definition 5.25: [7] A HFS $\hat{\varepsilon}$ on ω is called

(1) an anti-hesitant fuzzy UPS (AHFUPS) of ω if

$$(\forall x, y \in \omega)(\hat{\varepsilon}(x \cdot y) \subseteq \hat{\varepsilon}(x) \cup \hat{\varepsilon}(y)).$$

(2) an anti-hesitant fuzzy UPF (AHFUPF) of ω if

(i) $(\forall x \in \omega)(\hat{\varepsilon}(0) \subseteq \hat{\varepsilon}(x))$,

(ii) $(\forall x, y \in \omega)(\hat{\varepsilon}(y) \subseteq \hat{\varepsilon}(x \cdot y) \cup \hat{\varepsilon}(x))$.

(3) an anti-hesitant fuzzy UPI (AHFUPI) of ω if

(i) $(\forall x \in \omega)(\hat{\varepsilon}(0) \subseteq \hat{\varepsilon}(x))$,

(ii) $(\forall x, y, z \in \omega)(\hat{\varepsilon}(x \cdot z) \subseteq \hat{\varepsilon}(x \cdot (y \cdot z)) \cup \hat{\varepsilon}(y))$.

(4) an anti-hesitant fuzzy SUPI (AHFSUPI) of ω if

(i) $(\forall x \in \omega)(\hat{\varepsilon}(0) \subseteq \hat{\varepsilon}(x))$,

(ii) $(\forall x, y, z \in \omega)(\hat{\varepsilon}(x) \subseteq \hat{\varepsilon}((z \cdot y) \cdot (z \cdot x)) \cup \hat{\varepsilon}(y))$.

Theorem 5.26: *For a HFS $\hat{\varepsilon}$ on A, the following are equivalent.*

(1) *$\hat{\varepsilon}$ is an inf-HFUPS of ω.*

(2) *$\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ is an AHFUPS of ω.*

(3) *$\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)$ is a AHFUPS of ω $\forall \Bbbk \in \wp([0,1])$.*

Proof: $(1) \Rightarrow (3)$. Let $x, y \in \omega$ and $\Bbbk \in \wp([0,1])$. If $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(x \cdot y)$ is empty, then $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(x \cdot y) \subseteq \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(x) \cup \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(y)$. On the other hand, let $t \in \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(x \cdot y)$. Then $\mathrm{INF}\hat{\varepsilon}(x \cdot y) \geq t \in \Bbbk$. By the assumption (1) and Theorem 5.12, we obtain $\mathcal{F}_{\hat{\varepsilon}^*}$ is an AFUPS of ω and then $t \leq \mathrm{INF}\hat{\varepsilon}^*(x \cdot y) = \mathcal{F}_{\hat{\varepsilon}^*}(x \cdot y) \leq \max\{\mathcal{F}_{\hat{\varepsilon}^*}(x), \mathcal{F}_{\hat{\varepsilon}^*}(y)\} = \max\{\mathrm{INF}\hat{\varepsilon}^*(x), \mathrm{INF}\hat{\varepsilon}^*(y)\}$, which implies that $t \leq \mathrm{INF}\hat{\varepsilon}^*(x)$ or $t \leq \mathrm{INF}\hat{\varepsilon}^*(y)$. Hence $t \in \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(x)$ or $t \in \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(y)$. Therefore, $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(x \cdot y) \subseteq \mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)(x) \cup \mathcal{H}_{\mathrm{INF}}$ $(\hat{\varepsilon}^*; \Bbbk)(y)$. This is shown that $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)$ is an AHFUPS of ω.

$(3) \Rightarrow (2)$. It is clear.

$(2) \Rightarrow (1)$. Let $x, y \in \omega$. Then $\mathrm{INF}\hat{\varepsilon}^*(x \cdot y) \in \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x \cdot y)$. By the assumption (3), we get $\mathrm{INF}\hat{\varepsilon}^*(x \cdot y) \in \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x) \cup \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(y)$. Thus $\mathrm{INF}\hat{\varepsilon}^*(x \cdot y) \leq \mathrm{INF}\hat{\varepsilon}^*(x)$ or $\mathrm{INF}\hat{\varepsilon}^*(x \cdot y) \leq \mathrm{INF}\hat{\varepsilon}^*(y)$, which implies that $\mathcal{F}_{\hat{\varepsilon}^*}(x \cdot y) = \mathrm{INF}\hat{\varepsilon}^*(x \cdot y) \leq \max\{\mathrm{INF}\hat{\varepsilon}^*(x), \mathrm{INF}\hat{\varepsilon}^*(y)\} = \max\{\mathcal{F}_{\hat{\varepsilon}^*}(x), \mathcal{F}_{\hat{\varepsilon}^*}(y)\}$. Hence, we obtain $\mathcal{F}_{\hat{\varepsilon}^*}$ is an AFUPS of ω. Therefore, it follows from Theorem 5.12 that $\hat{\varepsilon}$ is an inf-HFUPS of ω.

The argument used to prove Theorem 5.26 may also be used to prove Theorems 5.27, 5.28, and 5.29.

Theorem 5.27: *For a HFS $\hat{\varepsilon}$ on A, the following are equivalent.*

(1) *$\hat{\varepsilon}$ is an inf-HFUPF of ω.*

(2) *$\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ is an AHFUPF of ω.*

(3) *$\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*; \Bbbk)$ is a AHFUPF of ω $\forall \Bbbk \in \wp([0,1])$.*

Theorem 5.28: *For a HFS $\hat{\varepsilon}$ on A, the following are equivalent.*

(1) *$\hat{\varepsilon}$ is an inf-HFUPI of ω.*

(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ *is an AHFUPI of* ω.
(3) $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*;\mathbb{k})$ is a AHFUPI of ω $\forall\mathbb{k}\in\wp([0,1])$.

Theorem 5.29: *For a HFS* $\hat{\varepsilon}$ *on* A, *the following are equivalent.*
(1) $\hat{\varepsilon}$ *is an* inf*-HFSUPI of* ω.
(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ *is an AHFSUPI of* ω.
(3) $\mathcal{H}_{\mathrm{INF}}(\hat{\varepsilon}^*;\mathbb{k})$ *is a AHFSUPI of* ω $\forall\mathbb{k}\in\wp([0,1])$.

5.3 *IvFSs and CSs*

We study inf-types of HFSs by interval-valued fuzzy sets and cubic sets.

By an interval number $\bar{a}$ we mean an interval $[a^-,a^+]$, where $0\le a^-\le a^+\le 1$. The set of all interval numbers is denoted by $\mathcal{D}[0,1]$. For $\bar{a}=[a^-,a^+],\bar{b}=[b^-,b^+]\in\mathcal{D}[0,1]$, define the symbols $\preccurlyeq,=,\prec$, rmax, and rmin in case of two elements in $\mathcal{D}[0,1]$ as follows:
(1) $\bar{a}\preccurlyeq\bar{b}\Leftrightarrow a^-\le b^-,a^+\le b^+$,
(2) $\bar{a}=\bar{b}\Leftrightarrow a^-=b^-,a^+=b^+$,
(3) $\bar{a}\prec\bar{b}\Leftrightarrow \bar{a}\circ\bar{b},\bar{a}\ne\bar{b}$,
(4) $\mathrm{rmax}\{\bar{a},\bar{b}\}=[\max\{a^-,b^-\},\max\{a^+,b^+\}]$,
(5) $\mathrm{rmin}\{\bar{a},\bar{b}\}=[\min\{a^-,b^-\},\min\{a^+,b^+\}]$.

Definition 5.30: [20] A mapping $\tilde{\lambda}:\omega\to\mathcal{D}[0,1]$ is called an *interval-valued fuzzy set* (IvFS) on ω, where $\tilde{\lambda}(x)=[\lambda^-(x),\lambda^+(x)]$ $\forall x\in\omega$, λ^- and λ^+ are FSs of ω such that $\lambda^-\le\lambda^+$.

Note that every IvFS of ω is a HFS.

Definition 5.31: [4] A *cubic set* (CS) $\mathcal{C}$ in ω is a structure of the form $\mathcal{C}=\{(x,\tilde{\lambda}(x),\phi(x))|x\in\omega\}$ when $\tilde{\lambda}$ is an IvFS on ω and ϕ is a FS of ω. We will use the symbol $\langle\tilde{\lambda},\phi\rangle$ of the CS $\{(x,\tilde{\lambda}(x),\phi(x))|x\in\omega\}$ for the sake of simplicity.

Definition 5.32: [12] A CS $\langle\tilde{\lambda},\tau\rangle$ in ω is called
(1) a *cubic UPS* (CUPS) of ω whether it complies with the following:
$$(\forall x,y\in\omega)(\mathrm{rmin}\{\tilde{\lambda}(x),\tilde{\lambda}(y)\}\preccurlyeq\tilde{\lambda}(x\cdot y)),\tag{5.11}$$
$$(\forall x,y\in\omega)(\tau(x\cdot y)\le\max\{\tau(x),\tau(y)\}),\tag{5.12}$$
(2) a *cubic UPI* (CUPI) of ω whether it complies with the following:
$$(\forall x\in\omega)(\tilde{\lambda}(x)\preccurlyeq\tilde{\lambda}(0)),\tag{5.13}$$
$$(\forall x\in\omega)(\tau(0)\le\tau(x)),\tag{5.14}$$
$$(\forall x,y,z\in\omega)(\mathrm{rmin}\{\tilde{\lambda}(x\cdot(y\cdot z)),\tilde{\lambda}(y)\}\preccurlyeq\tilde{\lambda}(x\cdot z)),\tag{5.15}$$
$$(\forall x,y,z\in\omega)(\tau(x\cdot z)\le\max\{\tau(x\cdot(y\cdot z)),\tau(y)\}).\tag{5.16}$$

Definition 5.33: [14] A CS $\langle\tilde{\lambda},\tau\rangle$ in ω is called
(1) a *cubic UPF* (CUPF) of ω if it fulfills the assertions (5.13), (5.14), and

$$(\forall x, y \in \omega)(\text{rmin}\{\tilde{\lambda}(x \cdot y), \tilde{\lambda}(x)\} \preccurlyeq \tilde{\lambda}(y)), \tag{5.17}$$

$$(\forall x, y \in \omega)(\tau(y) \leq \max\{\tau(x \cdot y), \tau(x)\}), \tag{5.18}$$

(2) a *cubic SUPI* (CSUPI) of ω if it fulfills the assertions (5.13), (5.14), and

$$(\forall x, y, z \in \omega)(\text{rmin}\{\tilde{\lambda}((z \cdot y) \cdot (z \cdot x)), \tilde{\lambda}(y)\} \preccurlyeq \tilde{\lambda}(x)), \tag{5.19}$$

$$(\forall x, y, z \in \omega)(\tau(x) \leq \max\{\tau((z \cdot y) \cdot (z \cdot x)), \tau(y)\}). \tag{5.20}$$

Definition 5.34: An IvFS $\tilde{\lambda}$ on ω is called

(1) an *interval-valued fuzzy UPS* (IvFUPS) of ω if it fulfills the assertion (5.11),

(2) an *interval-valued fuzzy UPF* (IvFUPF) of ω if it fulfills the assertions (5.13) and (5.17),

(3) an *interval-valued fuzzy UPI* (IvFUPI) of ω if it fulfills the assertions (5.13) and (5.15),

(4) an *interval-valued fuzzy SUPI* (IvFSUPI) of ω if it fulfills the assertions (5.13) and (5.19).

Lemma 5.35: *Every IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of ω is an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω.*

Proof: Assume $\tilde{\lambda}$ is an IvFUPS of ω. Then $\forall x, y \in \omega$, $\text{rmin}\{\tilde{\lambda}(x), \tilde{\lambda}(y)\}^{\circ}\tilde{\lambda}(x \cdot y)$. Thus $\text{INF}\tilde{\lambda}(x \cdot y) = \lambda^{-}(x \cdot y) \geq \min\{\lambda^{-}(x), \lambda^{-}(y)\} = \min\{\text{INF}\tilde{\lambda}(x),$ $\text{INF}\tilde{\lambda}(y)\}$ $\forall x, y \in \omega$. Therefore, $\tilde{\lambda}$ is an inf-HFUPS of ω.

The following example shows that the converses of Lemma 5.35 is not true in general.

Example 5.36: Allow $\omega = \{0, a, b, c, d\}$ to be a UP-algebra with a constant 0 and a binary operation $\cdot$ specified by the table below:

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	c	0
b	0	b	0	c	0
c	0	b	b	0	0
d	0	b	b	c	0

Table 5

Cayley Table for Example 5.36

(1) Define an IvFS $\tilde{\lambda}$ on ω as follows: $\tilde{\lambda}(0) = [0.4, 0.9], \tilde{\lambda}(a) = [0.4, 1]$, $\tilde{\lambda}(b) = \tilde{\lambda}(c) = [0.4, 0.6], \tilde{\lambda}(d) = [0.4, 0.7]$, we have that $\tilde{\lambda}$ is an inf-HFUPS (also, inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω but not an IvFUPS (also, IvFUPF, IvFUPI, IvFSUPI) of ω because $\tilde{\lambda}(0) \prec \tilde{\lambda}(a)$.

(2) Define a HFS $\hat{\varepsilon}$ on ω as follows: $\hat{\varepsilon}(0) = (0.8, 1], \hat{\varepsilon}(a) = (0.8, 1), \hat{\varepsilon}(b) = \{0.8, 0.9, 1\}, \hat{\varepsilon}(c) = \{0.8\}, \hat{\varepsilon}(d) = [0.8, 0.9]$. Then $\hat{\varepsilon}$ is an inf-HFUPS (also, inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω and since $\hat{\varepsilon}$ is not an IvFS of ω, we see that it is not an IvFUPS (also, IvFUPF, IvFUPI, IvFSUPI) of ω.

Similar to the proof of Lemma 5.35, the evidence of Lemma 5.37 may be established.

Lemma 5.37: *Every IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of ω is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of ω.*

The following example shows that the converses of Lemma 5.37 is not true in general.

Example 5.38: Consider a UP-algebra $\omega = \{0, a, b, c, d\}$ defined in Example 5.36, we define an IvFS $\tilde{\lambda}$ as follows: $\tilde{\lambda}(0) = [0.4,0.8], \tilde{\lambda}(a) = [0.3,0.8]$, $\tilde{\lambda}(b) = \tilde{\lambda}(c) = [0.6,0.8], \tilde{\lambda}(d) = [0.4,0.8]$. Then $\tilde{\lambda}$ is a sup-HFUPS (also, sup-HFUPF, sup-HFUPI, sup-HFSUPI) of ω but not an IvFUPS (also, IvFUPF, IvFUPI, IvFSUPI) of ω because $\tilde{\lambda}(0) \prec \tilde{\lambda}(c)$.

By Lemmas 5.35 and 5.37 and Examples 5.36 and 5.38, we get that the concept of an inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) of ω is a general concept of an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) and the concept of a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of ω is a general concept of an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI).

Theorem 5.39: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) *$\tilde{\lambda}$ is an IvFUPS of ω.*
(2) *$\langle \tilde{\lambda}, \mathcal{F}_{\tilde{\lambda}^*} \rangle$ is a CUPS of ω.*
(3) *$\tilde{\lambda}$ is both an inf-HFUPS and a sup-HFUPS of ω.*
(4) *λ^- and λ^+ are FUPSs of ω.*

Proof: The proof that (1) implies (2) follows from Theorem 5.12. The proof that (2) implies (1) is clear. The proof that (1) implies (3) follows from Lemmas 5.35 and 5.37. Since $\lambda^-(x) = \mathrm{INF}\tilde{\lambda}(x)$ and $\lambda^+ = \sup\tilde{\lambda}(x)$ $\forall x \in \omega$, we get that (3) and (4) are equivalent.

Next, assume (3) holds. Then for all $x, y \in \omega$, $\mathrm{rmin}\{\tilde{\lambda}(x), \tilde{\lambda}(y)\} = [\min\{\lambda^-(x), \lambda^-(y)\}, \min\{\lambda^+(x), \lambda^+(y)\}] = [\min\{\mathrm{INF}\tilde{\lambda}(x), \mathrm{INF}\tilde{\lambda}(y)\}, \min\{\sup\tilde{\lambda}(x), \sup\tilde{\lambda}(y)\}] \preccurlyeq [\mathrm{INF}\tilde{\lambda}(x \cdot y), \sup\tilde{\lambda}(x \cdot y)] = \tilde{\lambda}(x \cdot y)$. Therefore, $\tilde{\lambda}$ is an IvFUPS of ω. The proof that (3) implies (1) is complete.

The argument used to prove Theorem 5.39 may also be used to prove Theorems 5.40, 5.41, and 5.42.

Theorem 5.40: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) *$\tilde{\lambda}$ is an IvFUPF of ω.*
(2) *$\langle \tilde{\lambda}, \mathcal{F}_{\tilde{\lambda}^*} \rangle$ is a CUPF of ω.*
(3) *$\tilde{\lambda}$ is both an inf-HFUPF and a sup-HFUPF of ω.*
(4) *λ^- and λ^+ are FUPFs of ω.*

Theorem 5.41: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) *$\tilde{\lambda}$ is an IvFUPI of ω.*
(2) *$\langle \tilde{\lambda}, \mathcal{F}_{\tilde{\lambda}^*} \rangle$ is a CUPI of ω.*
(3) *$\tilde{\lambda}$ is both an inf-HFUPI and a sup-HFUPI of ω.*
(4) *λ^- and λ^+ are FUPIs of ω.*

Theorem 5.42: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) *$\tilde{\lambda}$ is an IvFSUPI of ω.*
(2) *$\langle \tilde{\lambda}, \mathcal{F}_{\tilde{\lambda}^*} \rangle$ is a CSUPI of ω.*
(3) *$\tilde{\lambda}$ is both an inf-HFSUPI and a sup-HFSUPI of ω.*
(4) *λ^- and λ^+ are FSUPIs of ω.*

For FSs ϕ and τ of ω with $\phi \leq \tau$, define the IvFS $[\phi, \tau]$ on ω by $[\phi, \tau]: \omega \to \mathcal{D}[0,1], x \mapsto [\phi(x), \tau(x)]$. Let ϕ and τ be FSs of ω such that $\phi \leq \tau$. Then the following hold.

(1) $[\phi_t^-, \tau]$ is an IvFS on ω $\forall t \in [0,1]$.
(2) $[\phi, \tau_t^+]$ is an IvFS on ω $\forall t \in [0,1]$.
(3) $[\phi_s^-, \tau_t^+]$ is an IvFS on ω $\forall s, t \in [0,1]$.
(4) $[\phi_s^-, \tau_t^-]$ is an IvFS on ω $\forall s, t \in [0,1]$ with $s \geq t$.
(5) $[\phi_s^+, \tau_t^+]$ is an IvFS on ω $\forall s, t \in [0,1]$ with $s \leq t$.

For a HFS $\hat{\varepsilon}$ on ω, we see that $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}$ is an IvFS on ω and $0 = \mathrm{INF}\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x) \leq \mathrm{INF}\hat{\varepsilon}(x) = \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x)$ $\forall x \in \omega$. Now, we characterize inf-HFUPS of a UP-algebra in terms of IvFSs.

Theorem 5.43: *For a HFS $\hat{\varepsilon}$ on A, then the following are equivalent.*

(1) *$\hat{\varepsilon}$ is an inf-HFUPS of ω.*
(2) *$\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}$ is an IvFUPS of ω.*
(3) *$[(\mathcal{F}_{\hat{\varepsilon}})_t^-, \mathcal{F}_{\hat{\varepsilon}}]$ is an IvFUPS of ω $\forall t \in [0,1]$.*
(4) *$[\mathcal{F}_{\hat{\varepsilon}}, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ is an IvFUPS of ω $\forall t \in [0,1]$.*
(5) *$[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ is an IvFUPS of ω $\forall s, t \in [0,1]$.*
(6) *$[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^-]$ is an IvFUPS of ω $\forall s, t \in [0,1]$ with $s \geq t$.*
(7) *$[(\mathcal{F}_{\hat{\varepsilon}})_s^+, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ is an IvFUPS of ω $\forall s, t \in [0,1]$ with $s \leq t$.*

Proof: By using Theorems 5.12 and 5.39, we can conclude that the conditions (1) and $(3) - (7)$ are equivalent. Assume (1) holds. Let $x, y \in \omega$. Then $\min\{\sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x), \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(y)\} = \min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\} \leq \mathrm{INF}\hat{\varepsilon}(x \cdot y) = \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x \cdot y)$ and so $\mathrm{rmin}\{\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x), \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(y)\} = [0, \min\{\sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x), \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(y)\}] \preccurlyeq [0, \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x \cdot y)] = \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x \cdot y)$. Hence, $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}$ is an IvFUPS of ω. The proof that (1) implies (2) is complete.

Finally, assume (2) holds. Let $x, y \in \omega$. By using the assumption and Lemma 5.37, $\mathrm{INF}\hat{\varepsilon}(x \cdot y) = \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x \cdot y) \geq \min\{\sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(x), \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}(y)\} =$

$\min\{\mathrm{INF}\hat{\varepsilon}(x), \mathrm{INF}\hat{\varepsilon}(y)\}$. Therefore, $\hat{\varepsilon}$ is an inf-HFUPS of ω. This shows that (2) implies (1).

The argument used to prove Theorem 5.43 may also be used to prove Theorems 5.44, 5.45, and 5.46.

Theorem 5.44: *For a HFS $\hat{\varepsilon}$ on A, then the following are equivalent.*

(1) $\hat{\varepsilon}$ *is an* inf*-HFUPF of* ω.
(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}$ *is an IvFUPF of* ω.
(3) $[(\mathcal{F}_{\hat{\varepsilon}})_t^-, \mathcal{F}_{\hat{\varepsilon}}]$ *is an IvFUPF of* ω $\forall t \in [0,1]$.
(4) $[\mathcal{F}_{\hat{\varepsilon}}, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFUPF of* ω $\forall t \in [0,1]$.
(5) $[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFUPF of* ω $\forall s, t \in [0,1]$.
(6) $[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^-]$ *is an IvFUPF of* ω $\forall s, t \in [0,1]$ *with* $s \geq t$.
(7) $[(\mathcal{F}_{\hat{\varepsilon}})_s^+, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFUPF of* ω $\forall s, t \in [0,1]$ *with* $s \leq t$.

Theorem 5.45: *For a HFS $\hat{\varepsilon}$ on A, then the following are equivalent.*

(1) $\hat{\varepsilon}$ *is an* inf*-HFUPI of* ω.
(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}$ *is an IvFUPI of* ω.
(3) $[(\mathcal{F}_{\hat{\varepsilon}})_t^-, \mathcal{F}_{\hat{\varepsilon}}]$ *is an IvFUPI of* ω $\forall t \in [0,1]$.
(4) $[\mathcal{F}_{\hat{\varepsilon}}, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFUPI of* ω $\forall t \in [0,1]$.
(5) $[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFUPI of* ω $\forall s, t \in [0,1]$.
(6) $[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^-]$ *is an IvFUPI of* ω $\forall s, t \in [0,1]$ *with* $s \geq t$.
(7) $[(\mathcal{F}_{\hat{\varepsilon}})_s^+, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFUPI of* ω $\forall s, t \in [0,1]$ *with* $s \leq t$.

Theorem 5.46: *For a HFS $\hat{\varepsilon}$ on A, then the following are equivalent.*

(1) $\hat{\varepsilon}$ *is an* inf*-HFSUPI of* ω.
(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}}$ *is an IvFSUPI of* ω.
(3) $[(\mathcal{F}_{\hat{\varepsilon}})_t^-, \mathcal{F}_{\hat{\varepsilon}}]$ *is an IvFSUPI of* ω $\forall t \in [0,1]$.
(4) $[\mathcal{F}_{\hat{\varepsilon}}, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFSUPI of* ω $\forall t \in [0,1]$.
(5) $[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFSUPI of* ω $\forall s, t \in [0,1]$.
(6) $[(\mathcal{F}_{\hat{\varepsilon}})_s^-, (\mathcal{F}_{\hat{\varepsilon}})_t^-]$ *is an IvFSUPI of* ω $\forall s, t \in [0,1]$ *with* $s \geq t$.
(7) $[(\mathcal{F}_{\hat{\varepsilon}})_s^+, (\mathcal{F}_{\hat{\varepsilon}})_t^+]$ *is an IvFSUPI of* ω $\forall s, t \in [0,1]$ *with* $s \leq t$.

In Theorems 5.47, 5.48, 5.49, and 5.50, we study inf-types of HFSs of UP-algebras in terms of CSs. The results of these theorems are a direct result of Theorems 5.12, 5.43, 5.44, 5.45, and 5.46.

Theorem 5.47: *A HFS $\hat{\varepsilon}$ on A is an* inf*-HFUPS of ω if and only if* $< \mathcal{H}_{INF}^{\hat{\varepsilon}}, \mathcal{F}_{\hat{\varepsilon}^*} >$ *is a CUPS of* ω.

Theorem 5.48: *HFS $\hat{\varepsilon}$ on A is an* inf*-HFUPF of ω if and only if* $< \mathcal{H}_{INF}^{\hat{\varepsilon}}, \mathcal{F}_{\hat{\varepsilon}^*} >$ *is a CUPF of* ω.

Theorem 5.49: *A HFS $\hat{\varepsilon}$ on A is an* inf*-HFUPI of ω if and only if* $< \mathcal{H}_{INF}^{\hat{\varepsilon}}, \mathcal{F}_{\hat{\varepsilon}^*} >$ *is a CUPI of* ω.

Theorem 5.50: *A HFS $\hat{\varepsilon}$ on A is an inf-HFSUPI of ω if and only if $<\mathcal{H}_{INF}^{\hat{\varepsilon}}, \mathcal{F}_{\hat{\varepsilon}^*}>$ is a CSUPI of ω.*

Definition 5.51: [14] A CS $\langle\tilde{\lambda}, \tau\rangle$ in ω is called

(1) an *anti-cubic UPS* (ACUPS) of ω whether it complies with the following:

$$(\forall x, y \in \omega)(\tilde{\lambda}(x \cdot y)) \preccurlyeq \text{rmax}\{\tilde{\lambda}(x), \tilde{\lambda}(y)\}, \tag{5.21}$$

$$(\forall x, y \in \omega)(\tau(x \cdot y) \geq \min\{\tau(x), \tau(y)\}), \tag{5.22}$$

(2) an *anti-cubic UPF* (ACUPF) of ω whether it complies with the following:

$$(\forall x \in \omega)(\tilde{\lambda}(0) \preccurlyeq \tilde{\lambda}(x)), \tag{5.23}$$

$$(\forall x \in \omega)(\tau(x) \leq \tau(0)), \tag{5.24}$$

$$(\forall x, y \in \omega)(\tilde{\lambda}(y) \preccurlyeq \text{rmax}\{\tilde{\lambda}(x \cdot y), \tilde{\lambda}(x)\}), \tag{5.25}$$

$$(\forall x, y \in \omega)(\tau(y) \geq \min\{\tau(x \cdot y), \tau(x)\}), \tag{5.26}$$

(3) an *anti-cubic UPI* (ACUPI) of ω if it fulfills the assertions (5.23), (5.25), and

$$(\forall x, y, z \in \omega)(\tilde{\lambda}(x \cdot z) \preccurlyeq \text{rmax}\{\tilde{\lambda}(x \cdot (y \cdot z)), \tilde{\lambda}(y)\}), \tag{5.27}$$

$$(\forall x, y, z \in \omega)(\tau(x \cdot z) \geq \min\{\tau(x \cdot (y \cdot z)), \tau(y)\}), \tag{5.28}$$

(4) an *anti-cubic SUPI* (ACSUPI) of ω if it fulfills the assertions (5.23), (5.24), and

$$(\forall x, y, z \in \omega)(\tilde{\lambda}(x) \preccurlyeq \text{rmax}\{\tilde{\lambda}((z \cdot y) \cdot (z \cdot x)), \tilde{\lambda}(y)\}), \tag{5.29}$$

$$(\forall x, y, z \in \omega)(\tau(x) \geq \min\{\tau((z \cdot y) \cdot (z \cdot x)), \tau(y)\}). \tag{5.30}$$

Definition 5.52: An IvFS $\tilde{\lambda}$ on ω is called

(1) an *anti-interval-valued fuzzy UPS* (AIvFUPS) of ω if it fulfills the assertion (5.21),
(2) an *anti-interval-valued fuzzy UPF* (AIvFUPF) of ω if it fulfills the assertions (5.23) and (5.25),
(3) an *anti-interval-valued fuzzy UPI* (AIvFUPI) of ω if it fulfills the assertions (5.23) and (5.27),
(4) an *anti-interval-valued fuzzy SUPI* (AIvFSUPI) of ω if it fulfills the assertions (5.23) and (5.29).

Lemma 5.53: *The infimum complement of an AIvFUPS of ω is an inf-HFUPS of ω.*

Proof: Assume $\tilde{\lambda}$ is an AIvFUPS of ω. Since $\text{INF}\tilde{\lambda}^*(x) = 1 - \lambda^-(x)\ \forall x \in \omega$, $\text{INF}\tilde{\lambda}^*(x \cdot y) = 1 - \lambda^-(x \cdot y) \geq 1 - \max\{\lambda^-(x), \lambda^-(y)\} = \min\{1 - \lambda^-(x), 1 - \lambda^-(y)\} = \min\{\text{INF}\tilde{\lambda}^*(x), \text{INF}\tilde{\lambda}^*(y)\}\ \forall x \in \omega$. Therefore, it follows from Theorem 5.12 that $\tilde{\lambda}^*$ is an inf-HFUPS of ω.

The argument used to prove Lemma 5.53 may also be used to prove Lemmas 5.54, 5.55, and 5.56.

Lemma 5.54: *The infimum complement of an AIvFUPF of ω is an inf-HFUPF of ω.*

Lemma 5.55: *The infimum complement of an AIvFUPI of ω is an inf-HFUPI of ω.*

Lemma 5.56: *The infimum complement of an AIvFSUPI of ω is an inf-HFSUPI of ω.*

By definitions of an AIvFUPS, an ACUPS, and an AFUPS, Lemma 5.53, and Theorem 5.12, we have the following theorem.

Theorem 5.57: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) $\tilde{\lambda}$ *is an AIvFUPS of* ω.
(2) $\langle\tilde{\lambda},\mathcal{F}_{\tilde{\lambda}^*}\rangle$ *is an ACUPS of* ω.
(3) λ^- *and* λ^+ *are AFUPSs of* ω.

By definitions of an AIvFUPF, an ACUPF, and an AFUPF, Lemma 5.54, and Theorem 5.12, we have the following theorem.

Theorem 5.58: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) $\tilde{\lambda}$ *is an AIvFUPF of* ω.
(2) $\langle\tilde{\lambda},\mathcal{F}_{\tilde{\lambda}^*}\rangle$ *is an ACUPF of* ω.
(3) λ^- *and* λ^+ *are AFUPFs of* ω.

By definitions of an AIvFUPI, an ACUPI and an AFUPI of a UP-algebra, Lemma 5.55, and Theorem 5.12, the following theorem.

Theorem 5.59: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) $\tilde{\lambda}$ *is an AIvFUPI of* ω.
(2) $\langle\tilde{\lambda},\mathcal{F}_{\tilde{\lambda}^*}\rangle$ *is an ACUPI of* ω.
(3) λ^- *and* λ^+ *are AFUPIs of* ω.

By definitions of an AIvFUPI, an ACUPI, and an AFUPI, Lemma 5.56, and Theorem 5.12, we have the following theorem.

Theorem 5.60: *Let $\tilde{\lambda}$ be an IvFS on ω. Then the following are equivalent.*

(1) $\tilde{\lambda}$ *is an AIvFSUPI of* ω.
(2) $\langle\tilde{\lambda},\mathcal{F}_{\tilde{\lambda}^*}\rangle$ *is an ACSUPI of* ω.
(3) λ^- *and* λ^+ *are AFSUPIs of* ω.

Now, we characterize an inf-HFUPS of a UP-algebra in terms of IvFUPSs.

Theorem 5.61: *For a HFS $\hat{\varepsilon}$ on A, then the following are equivalent.*

(1) $\hat{\varepsilon}$ *is an* inf*-HFUPS of* ω.

(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ *is an AIvFUPS of* ω.
(3) $[(\mathcal{F}_{\hat{\varepsilon}^*})_t^-, \mathcal{F}_{\hat{\varepsilon}^*}]$ *is an AIvFUPS of* ω $\forall t \in [0,1]$.
(4) $[\mathcal{F}_{\hat{\varepsilon}^*}, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPS of* ω $\forall t \in [0,1]$.
(5) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPS of* ω $\forall s, t \in [0,1]$.
(6) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^-]$ *is an AIvFUPS of* ω $\forall s, t \in [0,1]$ *with* $s \geq t$.
(7) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^+, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPS of* ω $\forall s, t \in [0,1]$ *with* $s \leq t$.

Proof: By using Theorems 5.12 and 5.57, we can conclude that the conditions (1) and (3) $-$ (7) are equivalent. Next, assume (1) holds. Let $x, y \in \omega$. By Theorem 5.12, we get $\mathrm{INF}\hat{\varepsilon}^*(x \cdot y) \leq \max\{\mathrm{INF}\hat{\varepsilon}^*(x), \mathrm{INF}\hat{\varepsilon}^*(y)\}$ and so $\sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x \cdot y) \leq \max\{\sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x), \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(y)\}$. Thus $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x \cdot y) = [0, \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x \cdot y)] \preccurlyeq [0, \max\{\sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x), \sup\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(y)\}] = \mathrm{rmax}\{\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(x), \mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}(y)\}$. Hence, $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ is an AIvFUPS of ω. The proof that (1) implies (2) is complete.

Finally, assume (2) holds. Let $x, y \in \omega$. Then $(\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*})^+(x \cdot y) \leq \max\{(\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*})^+(y), (\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*})^+(y)\}$. Since $(\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*})^+(a) = \mathrm{INF}\hat{\varepsilon}^*(a)$ $\forall a \in \omega$ and by using Theorem 5.57, $\mathcal{F}_{\hat{\varepsilon}^*}(x \cdot y) = \mathrm{INF}\hat{\varepsilon}^*(x \cdot y) \leq \max\{\mathrm{INF}\hat{\varepsilon}^*(x), \mathrm{INF}\hat{\varepsilon}^*(y)\} = \max\{\mathcal{F}_{\hat{\varepsilon}^*}(x), \mathcal{F}_{\hat{\varepsilon}^*}(y)\}$. Hence, $\mathcal{F}_{\hat{\varepsilon}^*}$ is an AFUPS of ω. Therefore, it follows from Theorem 5.12 that $\hat{\varepsilon}$ is an inf-HFUPS of ω. This shows that (2) implies (1).

The argument used to prove Theorem 5.61 may also be used to prove Theorems 5.62, 5.63, and 5.64.

Theorem 5.62: *For a HFS* $\hat{\varepsilon}$ *on* A*, then the following are equivalent.*
(1) $\hat{\varepsilon}$ is an inf-HFUPF of ω.
(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ is an AIvFUPF of ω.
(3) $[(\mathcal{F}_{\hat{\varepsilon}^*})_t^-, \mathcal{F}_{\hat{\varepsilon}^*}]$ is an AIvFUPF of ω $\forall t \in [0,1]$.
(4) $[\mathcal{F}_{\hat{\varepsilon}^*}, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ is an AIvFUPF of ω $\forall t \in [0,1]$.
(5) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ is an AIvFUPF of ω $\forall s, t \in [0,1]$.
(6) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^-]$ is an AIvFUPF of ω $\forall s, t \in [0,1]$ with $s \geq t$.
(7) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^+, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ is an AIvFUPF of ω $\forall s, t \in [0,1]$ with $s \leq t$.

Theorem 5.63: *For a HFS* $\hat{\varepsilon}$ *on* A*, then the following are equivalent.*
(1) $\hat{\varepsilon}$ *is an* inf*-HFUPI of* ω.
(2) $\mathcal{H}_{\mathrm{INF}}^{\hat{\varepsilon}^*}$ *is an AIvFUPI of* ω.
(3) $[(\mathcal{F}_{\hat{\varepsilon}^*})_t^-, \mathcal{F}_{\hat{\varepsilon}^*}]$ *is an AIvFUPI of* ω $\forall t \in [0,1]$.
(4) $[\mathcal{F}_{\hat{\varepsilon}^*}, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPI of* ω $\forall t \in [0,1]$.
(5) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPI of* ω $\forall s, t \in [0,1]$.
(6) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^-]$ *is an AIvFUPI of* ω $\forall s, t \in [0,1]$ *with* $s \geq t$.
(7) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^+, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPI of* ω $\forall s, t \in [0,1]$ *with* $s \leq t$.

Theorem 5.64: *For a HFS* $\hat{\varepsilon}$ *on* A*, then the following are equivalent.*
(1) $\hat{\varepsilon}$ *is an* inf*-HFUPI of* ω.

(2) $\mathcal{H}_{\text{INF}}^{\hat{\varepsilon}^*}$ *is an AIvFUPI of* ω.
(3) $[(\mathcal{F}_{\hat{\varepsilon}^*})_t^-, \mathcal{F}_{\hat{\varepsilon}^*}]$ *is an AIvFUPI of* ω $\forall t \in [0,1]$.
(4) $[\mathcal{F}_{\hat{\varepsilon}^*}, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPI of* ω $\forall t \in [0,1]$.
(5) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPI of* ω $\forall s, t \in [0,1]$.
(6) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^-, (\mathcal{F}_{\hat{\varepsilon}^*})_t^-]$ *is an AIvFUPI of* ω $\forall s, t \in [0,1]$ *with* $s \geq t$.
(7) $[(\mathcal{F}_{\hat{\varepsilon}^*})_s^+, (\mathcal{F}_{\hat{\varepsilon}^*})_t^+]$ *is an AIvFUPI of* ω $\forall s, t \in [0,1]$ *with* $s \leq t$.

In Theorems 5.65, 5.66, 5.67, and 5.68, we study inf-types of HFSs of UP-algebras by using the anti-type of CSs. The results of these theorems are a direct result of Theorems 5.12, 5.61, 5.62, 5.63, and 5.64.

Theorem 5.65: *Let* $\hat{\varepsilon}$ *be a HFS on A. Then* $\hat{\varepsilon}$ *is an* inf*-HFUPS of* ω *if and only if* $< \mathcal{H}_{INF}^{\hat{\varepsilon}^*}, \mathcal{F}_{\hat{\varepsilon}} >$ *is an ACUPS of* ω.

Theorem 5.66: *Let* $\hat{\varepsilon}$ *be a HFS on A. Then* $\hat{\varepsilon}$ *is an* inf*-HFUPF of* ω *if and only if* $< \mathcal{H}_{INF}^{\hat{\varepsilon}^*}, \mathcal{F}_{\hat{\varepsilon}} >$ *is an ACUPF of* ω.

Theorem 5.67: *Let* $\hat{\varepsilon}$ *be a HFS on A. Then* $\hat{\varepsilon}$ *is an* inf*-HFUPI of* ω *if and only if* $< \mathcal{H}_{INF}^{\hat{\varepsilon}^*}, \mathcal{F}_{\hat{\varepsilon}} >$ *is an ACUPI of* ω.

Theorem 5.68: *Let* $\hat{\varepsilon}$ *be a HFS on A. Then* $\hat{\varepsilon}$ *is an* inf*-HFSUPI of* ω *if and only if* $< \mathcal{H}_{INF}^{\hat{\varepsilon}^*}, \mathcal{F}_{\hat{\varepsilon}} >$ *is an ACSUPI of* ω.

6. Conclusions and Future Works

In the current study, we introduced the ideas of $\inf_t^\alpha$-HFUPSs (resp., $\inf_t^\alpha$ -HFUPF, $\inf_t^\alpha$ -HFUPI, $\inf_t^\alpha$ -HFSUPI), which are generalizations of IvFUPSs (resp., IvFUPFs, IvFUPIs, IvFSUPIs), of UP-algebras, demonstrated their generalizations, and looked into some of their key characteristics. Then, as illustrated in Figure 1, we obtain the generalization diagram of HFSs in the sense of the infimum of their pictures in UP-algebras. Some inf-HFUPS (resp., inf-HFUPF, inf-HFUPI, inf-HFSUPI) characterizations are examined in terms of sets, FSs, PFSs, HFSs, IvFSs, and CSs.

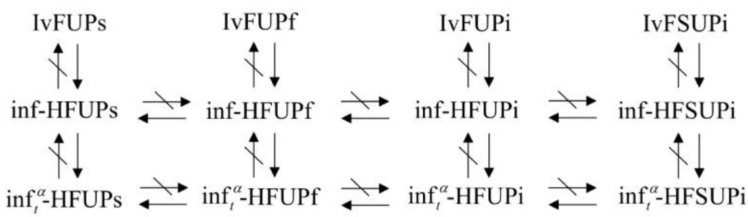

Figure 1
HFSs in UP-algebras

We will be studying UP-algebras in the future with the following goals in mind:

- to increase the number of HFSs outcomes in the sense of the infimum of its images,

- using HFSs to define neutrosophic sets in UP-algebras in terms of the infimum of its images,
- to explain anti-HFSs in terms of the infimum of its images.

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References

[1] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, "Fuzzy translations of a in UP-algebras," *Journal of the Indonesian Mathematical Society*, vol. 23, no. 2, pp. 1–19 (2017).

[2] A. Iampan, "A new branch of the logical algebra: UP-algebras," *Journal of Algebra and Related Topics*, vol. 5, no. 1, pp. 35–54 (2017).

[3] A. Iampan, "Introducing fully UP-semigroups," *Discussiones Mathematicae - General Algebra and Applications*, vol. 38, no. 2, pp. 297–306 (2018).

[4] Y. B. Jun, C. S. Kim, and M. S. Kang, "Cubic subalgebras and ideals of BCK/BCI-algebras," *Far East Journal of Mathematical Sciences*, vol. 44, no. 2, pp. 239–250 (2010).

[5] W. Kaijae, P. Poungsumpao, S. Arayarangsi, and A. Iampan, "UP-algebras characterized by their anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras," *Italian Journal of Pure and Applied Mathematics*, no. 36, pp. 667–692 (2016).

[6] P. Mosrijai, S. Akarachai, and A. Iampan, "New types of hesitant fuzzy sets on UP-algebras," *Mathematica Moravica*, vol. 22, no. 2, pp. 29–39 (2018).

[7] P. Mosrijai and A. Iampan, "Anti-type of hesitant fuzzy sets on UP-algebras," *European Journal of Pure and Applied Mathematics*, vol. 11, no. 4, pp. 976–1002 (2018).

[8] P. Mosrijai, W. Kamti, A. Satirad, and A. Iampan, "Hesitant fuzzy sets on UP-algebras," *Konuralp Journal of Mathematics*, vol. 5, no. 2, pp. 268–280 (2017).

[9] C. Prabpayak and U. Leerawat, "On ideals and congruences in KU-algebras," *Scientia Magna,* vol. 5, no. 1, pp. 54–57 (2009).

[10] A. Satirad, R. Chinram, and A. Iampan, "Pythagorean fuzzy sets in UP-algebras and approximations," *AIMS Mathematics*, vol. 6, no. 6, pp. 6002–6032 (2021).

[11] S. Sripaeng, K. Tanamoon, and A. Iampan, "On anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras," *Journal of Information and Optimization Sciences*, vol. 39, no. 5, pp. 1095–1127 (2018).

[12] T. Senapati, Y. B. Jun, and K. P. Shum, "Cubic set structure applied in UP-algebras," *Discrete Mathematics, Algorithms and Applications,* vol. 10, no. 4, Art. no. 1850049 (2018).

[13] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, "Fuzzy sets in UP-algebras," *Annals of Fuzzy Mathematics and Informatics,* vol. 12, no. 6, pp. 739–756 (2016).

[14] K. Taboon, P. Butsri, and A. Iampan, "A cubic set theory approach to UP-algebras," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 8, pp. 1449–1486 (2020).

[15] V. Torra, "Hesitant fuzzy sets," International Journal of Intelligent Systems, vol. 25, no. 6, pp. 529–539, 2010.

[16] V. Torra and Y. Narukawa, "On hesitant fuzzy sets and decision," in 2009 IEEE International Conference on Fuzzy Systems, Jeju Island, South Korea, pp. 1378–1382 (Aug. 2009).

[17] R. R. Yager, "Pythagorean fuzzy subsets," in 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, pp. 57–61 (Jun. 2013).

[18] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *International Journal of Intelligent Systems,* vol. 28, no. 5, pp. 436–452 (2013).

[19] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353 (1965).

[20] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning I," *Information Sciences*, vol. 8, no. 3, pp. 199–249 (1975).

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