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Continuous time optimal stopping problems with fractional rewards

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Abstract

We consider fractional, or average reward, optimal stopping problems for continuous time stochastic processes. We extend the results in Morimoto [4] to the general case, and then state the characterization of optimal value and optimal stopping time and the efficiency of Denkelbach algorithm for searching an optimal stopping time.

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1. Introduction

Let (Ω, F, P) be a complete probability space and $\{F_t,\ t \in \mathbf{R}_+\}$ be a filtration of sub σ–fields of F which satisfies the usual conditions and $F = \sigma(\cup_{t\in\mathbf{R}_+} F_t)$.

Let $\{X(t), t \in \mathbf{R}_+\}$ be an $\{F_t\}$–adapted right–continuous stochastic process such that

$$E[\sup_{t\in\mathbf{R}_+} X^+(t)] < \infty \ \textit{and}\ E[X^-(t)] < \infty\ (t \in \mathbf{R}_+).$$

And let $\{Y(t), t \in \mathbf{R}_+\}$ be an $\{F_t\}$–adapted right–continuous stochastic process such that

$$E[\sup_{t\in\mathbf{R}_+} Y(t)] < \infty \ \textit{and}\ Y(t) \geq K > 0\ (t \in \mathbf{R}_+).$$

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$$\overline{C_t} = \{\tau \mid \tau : \text{stopping time},\ P(t \le \tau) = 1, E[X^-(\tau)] < \infty\},$$
$$C_t = \{\tau \in \overline{C_t} \mid P(\tau < \infty) = 1\}.$$

We remark that in order that $E[X^-(\tau)] < \infty$, it is necessary and sufficient that for $\theta \in \mathbf{R}$, $E[(X(\tau) - \theta Y(\tau))^-] < \infty$. And we have for $\theta \in \mathbf{R}$,

$$E[\sup_{t \in \mathbf{R}_+}(X(t) - \theta Y(t))^+] < \infty.$$

Then we find a stopping time $\tau^* \in \overline{C_0}(C_0)$, which is called optimal, such that

$$\frac{E[X(\tau^*)]}{E[Y(\tau^*)]} = \sup_{\tau \in \overline{C_0}(C_0)} \frac{E[X(\tau)]}{E[Y(\tau)]}.$$

In order to discuss an optimal stopping point for the above problem, we introduce the following parametric problem: find for $\theta \in \mathbf{R}$, a stopping time $\tau^* \in \overline{C_0}(C_0)$ such that

$$E[X(\tau^*) - \theta Y(\tau^*)] = \sup_{T \in \overline{C_0}(C_0)} E[X(\tau) - \theta Y(\tau)].$$

Now we define the Snell envelopes of the parametric problems by

$$\overline{Z}^{\theta}(t) = \operatorname{ess\,sup}_{\tau \in \overline{C_t}} E[X(\tau) - \theta Y(\tau) | F_t],$$
$$Z^{\theta}(t) = \operatorname{ess\,sup}_{\tau \in C_t} E[X(\tau) - \theta Y(\tau) | F_t].$$

By the way, throughout this paper we make the following assumptions:

$$X(\infty) = \limsup_{t \to \infty} X(t),$$
$$Y(\infty) = \lim_{t \to \infty} Y(t).$$

Then we have

$$\limsup_{t \to \infty}(X(t) - \theta Y(t)) = X(\infty) - \theta Y(\infty).$$

Fractional optimization problems have been studied by several authors, for example, Aggarwal, Chandrasekaran and Nair [1], Iwamoto and Fujita [2], Morimoto [4], Ren and Krogh [5], Sawasaki, Kimura and Tanaka [6], Stancu–Minasian [7], and Tanaka [8]. In particular, Tanaka [8] gave new fractional sequential optimization problems and dicussed their application to fractional multiparameter optimal stopping problems.

We shall consider fractional optimal stopping problems for continuous time stochastic processes with right continuity and quasi–left continuity. Moreover we shall discuss the characterization of optimal value and optimal stopping time and the efficiency of Denkelbach algorithm for searching an optimal stopping time.

2. Snell envelopes of parametric problem

In this section we state properties of the Snell envelope and an optimal stopping time for the parametric problem. We refer Morimoto [3].

Theorem 2.1: *For* $\theta \in \mathbf{R}$, *there exists a right–continuous supermartingale* $\{\hat{Z}^\theta(t), t \in \mathbf{R}_+\}$ *majorizing* $\{X(t) - \theta Y(t), t \in \mathbf{R}_+\}$ *and satisfying*

(1) $\hat{Z}^\theta(t) = \overline{Z}^\theta(t) = Z^\theta(t)$ P–a.e. $t \in \mathbf{R}_+$.

(2) $\limsup_{t\to\infty}(X(t) - \theta Y(t)) = \limsup_{t\to\infty}\hat{Z}^\theta(t)$.

(3) $E[\hat{Z}^\theta(t)] = \sup_{\tau\in\overline{C_t}}E[X(\tau) - \theta Y(\tau)] = \sup_{\tau\in C_t}E[X(\tau) - \theta Y(\tau)]$.

(4) *For any* $\xi, \tau \in \overline{C_0}$ *with* $\xi \le \tau$, $E[|\hat{Z}^\theta(\tau)|] < \infty$ *and* $E[\hat{Z}^\theta(\tau)|F_\xi] \le \hat{Z}^\theta(\xi)$.

Theorem 2.2: *Let* $\theta \in \mathbf{R}$ *be fixed. Then* $\xi \in \overline{C_0}$ *is optimal if and only if* $E[X(\xi) - \theta Y(\xi)|F_t] = \hat{Z}^\theta(\xi \wedge t)$ *for* $t \in \mathbf{R}_+$.

Theorem 2.3: *Suppose that for any* $\tau_n \in C_0$ *increasing to* τ,

$$\limsup_{n\to\infty} X(\tau_n) \le X(\tau) \text{ and } \lim_{n\to\infty} Y(\tau_n) = Y(\tau)$$

on $\{\tau < \infty\}$.

Then we have

(1) $\xi = \inf\{t|\overline{Z}^\theta(t) = X(t) - \theta Y(t)\}$ *is optimal in* $\overline{C_0}$.

(2) *There exists an optimal stopping time* $\sigma \in C_0$ *if and only if* $P(\xi < \infty) = 1$.

(3) *If* $\lim_{t\to\infty}X(t) = -\infty$, *then* ξ *is optimal in* C_0.

Remark 2.1: If the condition (3) in the above theorem satisfies, we have $lim_{t\to\infty}(X(t) - \theta Y(t)) = -\infty$ for $\theta \in \mathbf{R}$.

3. Parametric method

For $\theta \in \mathbf{R}$, we set $d(\theta) = \sup_{\tau\in\overline{C_0}}E[X(\tau) - \theta Y(\tau)]$.

Proposition 3.1: The function $\theta \mapsto d(\theta)$ is convex.

Proof: For any $\theta_1 < \theta_2$ and $\lambda \in [0,1]$, we have

$$\begin{aligned} d(\lambda\theta_1 + (1-\lambda)\theta_2) &= \sup_{\tau\in\overline{C_0}} E[X(\tau) - (\lambda\theta_1 + (1-\lambda)\theta_2)Y(\tau)] \\ &= \sup_{\tau\in\overline{C_0}} E[\lambda(X(\tau) - \theta_1 Y(\tau)) + (1-\lambda)(X(\tau) - \theta_2 Y(\tau))] \\ &\le \sup_{\tau\in\overline{C_0}} E[\lambda(X(\tau) - \theta_1 Y(\tau))] + \sup_{\tau\in\overline{C_0}} E[(1-\lambda)(X(\tau) - \theta_2 Y(\tau))] \\ &= \lambda d(\theta_1) + (1-\lambda)d(\theta_2). \end{aligned}$$

Proposition 3.2: The function $\theta \mapsto d(\theta)$ is strictly decreasing.

Proof: For any $\theta_1 < \theta_2$ and $\tau \in \overline{C_0}$, we have

$$\begin{aligned} X(\tau) - \theta_1 Y(\tau) &= X(\tau) - \theta_2 Y(\tau) + (\theta_2 - \theta_1)Y(\tau) \\ &\ge X(\tau) - \theta_2 Y(\tau) + (\theta_2 - \theta_1)K \end{aligned}$$

$$E[X(\tau) - \theta_1 Y(\tau)] \geq E[X(\tau) - \theta_2 Y(\tau)] + (\theta_2 - \theta_1)K.$$

Therefore we obtain $d(\theta_1) \geq d(\theta_2) + (\theta_2 - \theta_1)K > d(\theta_2)$.

Proposition 3.3: *$lim_{\theta\to\infty} d(\theta) = -\infty$ and $lim_{\theta\to-\infty} d(\theta) = \infty$.*

Proof: We may assume that $\theta > 0$.

$$\begin{aligned} d(\theta) &= \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta Y(\tau)] \leq E[\sup_{t\in\mathbf{R}_+} X^+(t) - \theta K] \\ &\to -\infty \ (\theta \to \infty). \end{aligned}$$

Next we may assume that $\theta < 0$.

$$\begin{aligned} d(\theta) &= \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta Y(\tau)] \geq \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta K] \\ &\geq E[X(t)] - \theta K \\ &\geq -E[X^-(t)] - \theta K \\ &\to \infty \ (\theta \to \infty). \end{aligned}$$

Proposition 3.4: There exists only one $\overline{\theta}$ such that $d(\overline{\theta}) = 0$.

Proposition 3.5: The function $\theta \mapsto d(\theta)$ is Lipschitz continuous.

Proof: For any θ_1, θ_2, we have

$$\begin{aligned} |d(\theta_1) - d(\theta_2)| &= |\sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta_1 Y(\tau)] - \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta_2 Y(\tau)]| \\ &\leq \sup_{\tau\in\overline{C_0}} |E[X(\tau) - \theta_1 Y(\tau)] - E[X(\tau) - \theta_2 Y(\tau)]| \\ &= \sup_{\tau\in\overline{C_0}} |\theta_1 - \theta_2| E[Y(\tau)] \\ &\leq |\theta_1 - \theta_2| E[\sup_{t\in\mathbf{R}_+} Y(t)]. \end{aligned}$$

Proposition 3.6: We set $\theta = \frac{E[X(\tau)]}{E[Y(\tau)]}$ for any $\tau \in \overline{C_0}$. Then $d(\theta) \geq 0$.

Proof: For any $\tau \in \overline{C_0}$, we have

$$d(\theta) = \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta Y(\tau)] \geq E[X(\tau) - \theta Y(\tau)] = 0.$$

4. Optimal value

Theorem 4.1: *Suppose that for any $\tau_n \in C_0$ increasing to τ,*

$$\limsup_{n\to\infty} X(\tau_n) \leq X(\tau) \ \textit{and} \ \lim_{n\to\infty} Y(\tau_n) = Y(\tau)$$

on $\{\tau < \infty\}$.

Then we have

$$\overline{\theta} = \frac{E[X(\xi)]}{E[Y(\xi)]} = \sup_{\tau\in\overline{C_0}} \frac{E[X(\tau)]}{E[Y(\tau)]}.$$

Proof: For any $\tau \in \overline{C_0}$, we have

$$0 = d(\overline{\theta}) = E[X(\xi) - \overline{\theta}Y(\xi)] \geq E[X(\tau) - \overline{\theta}Y(\tau)].$$

Then we obtain

$$\overline{\theta} = \frac{E[X(\xi)]}{E[Y(\xi)]} \geq \frac{E[X(\tau)]}{E[Y(\tau)]}.$$

Remark 4.1: If $\lim_{t\to\infty} X(t) = -\infty$, then we have $P(\xi < \infty) = 1$ and

$$\overline{\theta} = \frac{E[X(\xi)]}{E[Y(\xi)]} = \sup_{\tau \in C_0} \frac{E[X(\tau)]}{E[Y(\tau)]}.$$

5. Dinkelbach algorithm

We consider the following algorithm. Let $\delta > 0$ be fixed.

(1) For $\tau_1 \in \overline{C_0}$ such that $\theta_1 = \frac{E[X(\tau_1)]}{E[Y(\tau_1)]}$, we have $d(\theta_1) \geq 0$.
   (i) If $d(\theta_1) < \delta$, then stop. In particular, τ_1 is exactly optimal if $d(\theta_1) = 0$.
   (ii) If $d(\theta_1) \geq \delta$, then set $\theta_2 = \theta_1 = \frac{E[X(\tau_1)]}{E[Y(\tau_1)]}$ and go to (2).

(2) Find $\tau_2 \in \overline{C_0}$ such that $d(\theta_2) = \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta_2 Y(\tau)] = E[X(\tau_2) - \theta_2 Y(\tau_2)]$.
   (i) If $d(\theta_2) < \delta$, then stop. In particular, τ_2 is exactly optimal if $d(\theta_2) = 0$.
   (ii) If $d(\theta_2) \geq \delta$, then set $\theta_3 = \frac{E[X(\tau_2)]}{E[Y(\tau_2)]}$ and go to (3).

(3) Find $\tau_3 \in \overline{C_0}$ such that $d(\theta_3) = \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta_3 Y(\tau)] = E[X(\tau_3) - \theta_3 Y(\tau_3)]$.
   (i) If $d(\theta_3) < \delta$, then stop. In particular, τ_3 is exactly optimal if $d(\theta_3) = 0$.
   (ii) If $d(\theta_3) \geq \delta$, then set $\theta_4 = \frac{E[X(\tau_3)]}{E[Y(\tau_3)]}$ and go to (k).

(k) Find $\tau_k \in \overline{C_0}$ such that $d(\theta_k) = \sup_{\tau\in\overline{C_0}} E[X(\tau) - \theta_k Y(\tau)] = E[X(\tau_k) - \theta_k Y(\tau_k)]$.
   (i) If $d(\theta_k) < \delta$, then stop. In particular, τ_k is exactly optimal if $d(\theta_k) = 0$.
   (ii) If $d(\theta_k) \geq \delta$, then set $\theta_{k+1} = \frac{E[X(\tau_k)]}{E[Y(\tau_k)]}$, $k := k + 1$ and go to (k).

Proposition 5.1: For any k such that $d(\theta_k) \geq \delta$, we have $\theta_{k+1} > \theta_k$.

Proof:

$$0 < d(\theta_k) = E[X(\tau_k) - \theta_k Y(\tau_k)] = \theta_{k+1} E[Y(\tau_k)] - E[\theta_k Y(\tau_k)]$$
$$= (\theta_{k+1} - \theta_k) E[Y(\tau_k)],$$

from which we obtain $\theta_{k+1} > \theta_k$.

Proposition 5.2: $lim_{k\to\infty}\theta_k = \overline{\theta}$.

Proof: By Proposition 5.1, there exists $\theta := \lim_{k\to\infty}\theta_k$. Suppose that $\theta < \overline{\theta}$. From the proof of Proposition 5.1, we have

$$0 < d(\theta_k) = E[X(\tau_k) - \theta_k Y(\tau_k)] = \theta_{k+1}E[Y(\tau_k)] - E[\theta_k Y(\tau_k)] \leq (\theta_{k+1} - \theta_k)E[\sup_{t\in\mathbf{R}_+} Y(t)],$$

and then $\lim_{k\to\infty} d(\theta_k) = 0$. Therefore we obtain $d(\theta) = 0$. On the other hand, from $\theta < \overline{\theta}$, we have $d(\theta) < d(\overline{\theta})$. This is a contradiction. By the same arguments, we can conclude $\theta = \overline{\theta}$.

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