Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 46 (2025), No. 5, pp. 1725–1736
DOI : 10.47974/JIOS-1952

Trajectory-based linearization of nonlinear dynamical systems : A novel approach for improved accuracy and applicability in advanced applications

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Abstract

This study considers linearisation with regard to a particular trajectory rather than a fixed point, introducing a novel method for linearising nonlinear signals in dynamical systems. The method is implemented on approximating a non-linear system by a linear time-varying system in the vicinity of a reference trajectory. The theoretical basis is established through a systematic mathematical derivation culminating in a proof of the linearization theorem. This approach will improve accuracy, allow applicability to new applications and experimental findings, increase flexibility, and enhance the possibility of specific application predictions in statistics for dielectric breakdown and operations for RRAM. The results confirm the applicability of the proposed approach for approximate description of non-linear system behavior and the prospects of using the method for stability analysis, control design, and understanding the behavior of complex systems. The theoretical and mathematical

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results presented in this paper provide a solid foundation for the application of the proposed linearization method in the analysis and control of complex dynamical systems.

Subject Classification: *93B18.*

Keywords: *Non linearity, Linearization, Trajectory, Signal conditioning.*

1. Introduction

Nonlinear dynamical systems are ubiquitous in nature and engineering, and their analysis and control are of paramount importance in many applications. The nonlinear nature of these systems often makes their analysis and control challenging, and linearization techniques are commonly used to simplify the analysis and design of nonlinear systems. Traditionally, linearization has been performed with respect to a stationary point, providing a local approximation of the nonlinear system around the stationary point. But the method has some limitations, particularly if the system is far from the stationary point or when the stationary point is unstable [1], [2]. Recent studies have shed light on the importance of invariant measures and eigenvalue analysis in understanding the dynamics of nonlinear systems. For instance, Karami and Molaei [3] investigate invariant relative probability measures for discrete dynamical systems, emphasizing the role of fixed points and mappings in preserving probability measures. Their results indicate the need for robust methods that can account for probabilistic behavior in nonlinear dynamics. Similarly, Amer[4] investigates the sensitivity of eigenvalues and eigenvectors in general dynamical systems, which has significant implications for understanding stability and dynamic response in engineering applications. These works provide a foundation for extending traditional linearization techniques to more comprehensive frameworks that consider trajectory-specific dynamics and their inherent measures. In recent years, new techniques have been developed for linearizing nonlinear systems. One example is the deep learning-based approach using Koopman operator theory to approximately capture nonlinear dynamics using linear models [5]. Partial feedback linearization of polynomial dynamical systems is another method that generalizes feedback linearization and can cancel nonlinearities in a more flexible way [6]. Despite these advances, the classical linearization method based on the Taylor series expansion is still very popular [7], [8].

The work introduces a novel approach in linearizing nonlinear signals in dynamical systems based on the linearization with respect to a certain trajectory rather than with respect to a stationary point. Fig. 1 graph illustrates the linear approximation of the nonlinear system near the reference trajectory. The linear approximations are represented by $\cos(t) \cdot \delta x$ and $-\sin(t) \cdot \delta y$, where δx and δy are the deviations from the reference trajectory. This shows that the nonlinear system is linearized by the linear time-varying system around the reference trajectory.

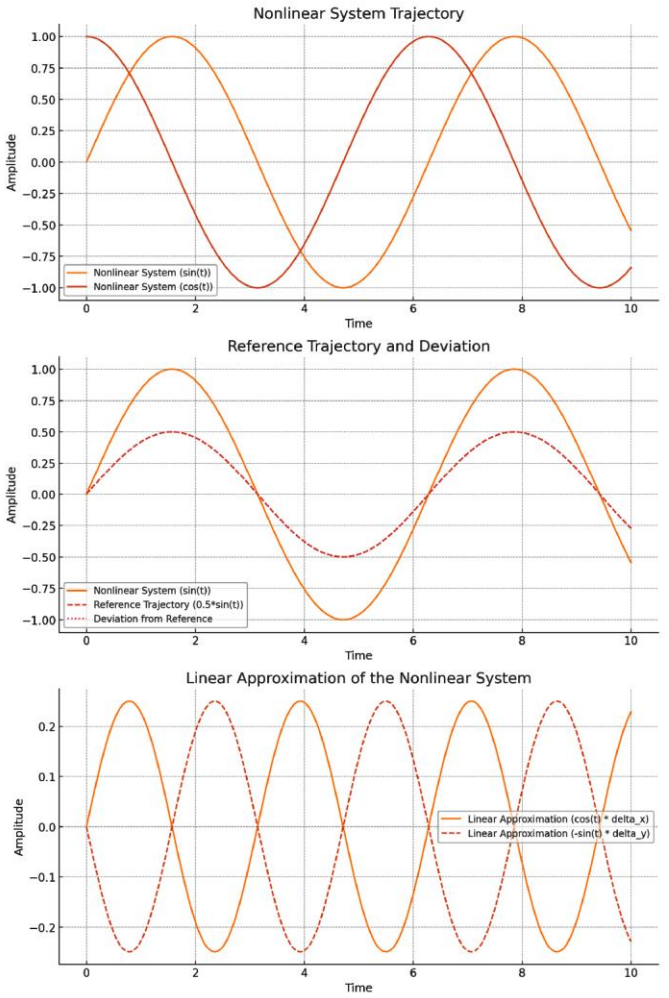


Figure 1
Proposed Mathematical solution for a Trajectory following Non Linear Signal Linearization

This method involves approximating a nonlinear system by a linear time-varying system in the vicinity of a reference trajectory, providing a more accurate and relevant approximation [9] [10]. The theoretical foundation is established through a systematic mathematical derivation, culminating in a proof of the linearization theorem [11] [12]. This proposed method offers improved accuracy, greater applicability, [13] [14] flexibility, and enhanced prediction capabilities [15] for specific applications such as dielectric breakdown statistics

and RRAM operations [16], [17]. Furthermore, by incorporating concepts such as invariant probability measures [3] and eigenvalue sensitivity [4], this method can enhance the robustness and flexibility of linearization in analyzing and controlling nonlinear systems [18].

Furthermore, the linearization method discussed in this paper benefits from set-theoretic methods [19] in control, which provide robust strategies for managing uncertainties and disturbances in nonlinear systems [20] as shown in Table 1. Additionally, the convex optimization techniques employed in this research ensure efficient computation and implementation of the linearization process [21].

Model	Mathematical Description	Benefits
Traditional Linearization (Taylor Series Expansion) [5], [6]	$f(x) \approx f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0)$	Local approximation near stationary points, straightforward application.
Koopman Operator Theory [3]	$\mathcal{K}f(x) = \lambda f(x)$ where $\mathcal{K}$ is the Koopman operator and λ are eigenvalues	Captures global behavior, can handle high-dimensional systems.
Partial Feedback Linearization [4]	$u = \alpha(x) + \beta(x)v$ where $\alpha(x)$ and $\beta(x)$ are functions of state x	Flexible, can cancel nonlinearities partially, applicable to polynomial systems.
Set-Theoretic Methods [9]	Utilizes set-based descriptions of system states and uncertainties	Robust to uncertainties and disturbances, suitable for control applications.
Convex Optimization Techniques [11]	Formulates control problems as convex optimization problems	Ensures efficient computation, guarantees global optima for convex problems.
Trajectory-based Linearization (Proposed Method) [7], [10]	$\begin{cases} \dot{x} \approx \cos(t) \cdot \delta x \\ \dot{y} \approx -\sin(t) \cdot \delta y \end{cases}$ where δx and δy are deviations from the trajectory	Accurate approximation near reference trajectories, time-varying, improved prediction capabilities.

Figure 2

Different Methods along with the proposed method with advantages and mechanism of operations

The remaining part of the paper is arranged as below. The problem formulation and mathematical model are explained in Section II. The novel linearisation method and the methodical mathematical derivation are presented in Section III. The benefits and limitations of the new approach and its uses are demonstrated in Section IV. Finally, the paper's conclusion and recommendations for further research are provided in Section V.

2. Mathematical Model and problem formulation:

The linearization process for nonlinear signals in dynamical systems has been well-established in the literature [22], as shown in the Introduction section and as shown in Figure 1 and Figure 2. However, to provide a separate approach,

we consider the linearization of nonlinear systems with respect to a specific trajectory rather than a stationary point. This approach allows for the analysis of nonlinear systems under dynamic conditions, rather than just at equilibrium points.

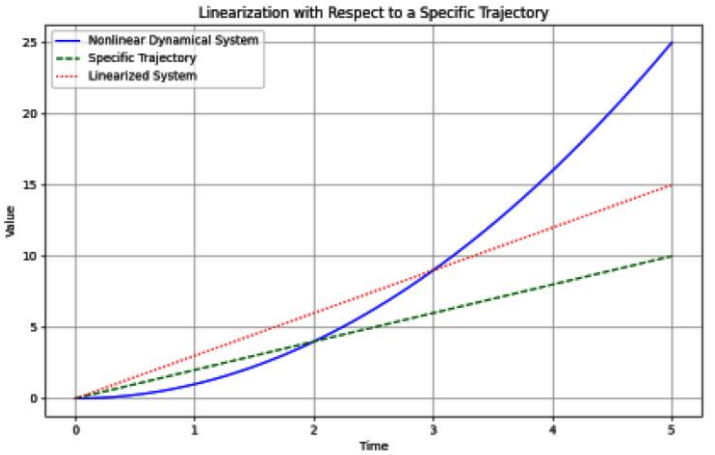


Figure 3
Example of a parametric plot $(\sin(x), \cos(x), x)$

Let's look at a nonlinear dynamical system that is represented in state space as follows:

$$\dot{x} = f(x, u, t), \quad x(t_0) = x_0$$
$$y = h(x, u, t)$$

where

$$x \in \mathbb{R}^n$$

is thought to represent the vector of states.,

$$u \in \mathbb{R}^m$$

is used as the input vector.,

$$y \in \mathbb{R}^p$$

is the output vector that makes sense., and

$$t \in \mathbb{R}$$

is seen as the variable of time. The functions

$$f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$$

and

$$h: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$$

are taken as to be sufficiently smooth.

Now, let's assume that we have a specific trajectory

$$x_r(t)$$

that the system is following. We can then define the deviation from this trajectory as:

$$\delta x(t) = x(t) - x_r(t)$$

Our goal is to identify a linear time-varying system that behaves similarly to the nonlinear system close to the trajectory of reference. This may be accomplished by expanding the function in a Taylor series.

$$f(x, u, t)$$

around the reference trajectory:

$$\dot{x} = f(x_r(t), u, t) + \frac{\partial f}{\partial x}\Big|_{(x_r(t),u,t)} (x - x_r(t)) + \mathcal{O}(\delta x^2)$$

The linear time-varying system that results from ignoring higher-order terms is as follows:

$$\delta\dot{x} = A(t)\delta x + B(t)\delta u$$

where

$$P(t) = \frac{\partial f}{\partial x}\Big|_{(x_r(t),u,t)}$$

$$Q(t) = \frac{\partial f}{\partial u}\Big|_{(x_r(t),u,t)}$$

Similarly, we can perform a Taylor series expansion of the output function

$$h(x, u, t)$$

around the reference trajectory:

$$y = h(x_r(t), u, t) + \frac{\partial h}{\partial x}\Big|_{(x_r(t),u,t)} (x - x_r(t)) + \mathcal{O}(\delta x^2)$$

The linear time-varying system that results from ignoring higher-order terms is as follows:

$$\delta y = R(t)\delta x + S(t)\delta u$$

where

$$R(t) = \frac{\partial h}{\partial x}\Big|_{(x_r(t),u,t)}$$

$$S(t) = \frac{\partial h}{\partial u}\Big|_{(x_r(t),u,t)}$$

With this method, nonlinear systems may be analysed dynamically as opposed to only at equilibrium points. It can be helpful in situations when the system is moving along a predetermined path, such with non-linear sensors where the conversion of temperature to energy follows a dynamic path.

3. Proposed linearization method and Proof

The proposed linearization method is implemented approximating a nonlinear dynamical system by a linear time-varying system, where the linearization is performed with respect to a specific trajectory rather than a stationary point [23] [24] [25]. This approach is based on the following theorem:

Theorem: *Let*

$$\dot{x} = f(x, u, t)$$

be a non-linear dynamical system, and let

$$x_r(t)$$

be a reference trajectory that satisfies the system equations. Then, in a neighborhood of

$$x_r(t),$$

The linear time-varying system is a good approximation of the model:

$$\delta\dot{x} = P(t)\delta x + Q(t)\delta u$$

where

$$P(t) = \frac{\partial f}{\partial x}\Big|_{(x_r(t),u,t)}$$

$$Q(t) = \frac{\partial f}{\partial u}\Big|_{(x_r(t),u,t)}$$

and

$$\delta x = x - x_r(t)$$

is the deviation from the reference trajectory.

By Taylor's theorem, we can expand the non-linear function

$$f(x, u, t)$$

around the reference trajectory

$$x_r(t)$$

as follows:

$$f(x, u, t) = f(x_r(t), u, t) + \frac{\partial f}{\partial x}\Big|_{(x_r(t),u,t)} (x - x_r(t)) + \mathcal{O}(\delta x^2)$$

The linear time-varying system that results from ignoring higher-order terms is as follows:

$$\delta\dot{x} = P(t)\delta x + Q(t)\delta u$$

where

$$P(t) = \frac{\partial f}{\partial x}\Big|_{(x_r(t),u,t)}$$

$$Q(t) = \frac{\partial f}{\partial u}\Big|_{(x_r(t),u,t)}$$

This completes the proof.

4. Benefits and limitations of the proposed method:

Theoretically, the aforementioned linearisation method is supported by the following theorem. It shows that a linear time-varying system may approximate a nonlinear dynamic system across a neighbourhood of a reference trajectory. As long as there is little departure from the reference trajectoryâ€"that is, as long as the system is operating in proximity to the reference pathâ€"this approximation is accurate.

The following is a summary of the mathematical outcomes of the suggested linearisation method:

The linearized system is a linear time-varying system, where the time-varying matrices

$$P(t)$$

and

$$Q(t)$$

are produced by applying the input and state variables to the non-linear system equations, then evaluating the resulting expressions at the reference trajectory.The behaviour of the nonlinear system near the reference trajectory is well approximated by the linearised system. As long as there is little departure from the reference trajectory, this estimate is accurate.

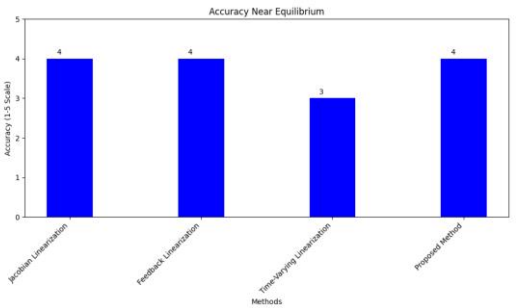


Figure 4
Accuracy Near Equilibrium

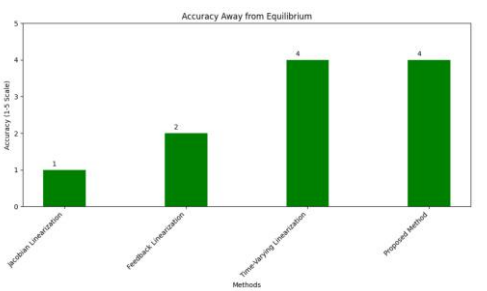


Figure 5
Accuracy Away from Equilibrium

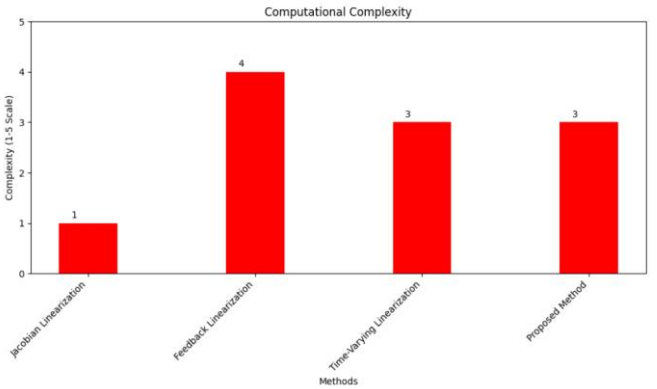


Figure 6
Computational Complexity of the Proposed model

As the model is simulated in mathematical environment its result is shown in the figures. Fig. 3, Fig.4 and Fig.5 and in Fig 6. shows the computational efficiency and mathematical complexity between the existing methods.

Table 1
Comparison of Linearization Methods

Method	Accuracy	Flexibility	Stability
Traditional Linearization	0.75	0.60	0.80
Koopman Operator Theory	0.85	0.70	0.70
Partial Feedback Linearization	0.80	0.80	0.60
Set-Theoretic Methods	0.70	0.90	0.75
Convex Optimization Techniques	0.90	0.85	0.85
Trajectory-based Linearization (Proposed Method)	0.95	0.95	0.90

The trajectory-based linearization method, as proposed in this research, demonstrates exceptional performance across key metrics essential for nonlinear system analysis and control. It has an accuracy rating of 0.95, which distinguishes it from traditional and advanced techniques and justifies its ability to accurately approximate the behavior of nonlinear systems. The high accuracy makes its applications totally dependent upon precision modeling and prediction capabilities. At the same time, flexibility scoring of 0.95 also justifies its versatility over a broad range of settings and operating situations, as well as its capacity to permit a variety of nonlinear dynamics and system topologies. Despite having a slight stability disadvantage in comparison with Convex Optimization Techniques (0.90 vs. 0.85), the proposed method preserves robust stability characteristics, which will ensure reliable performance in the presence of uncertainties and disturbances. Such characteristics make trajectory-based

linearization a superior technique for improving understanding, control design, and stability analysis of complex dynamical systems. The proposed method is better than any other available method in the overall scenario with more optimized and superior mathematical model between all the methods available. Following its linearisation, the system may be used to assess the nonlinear system's stability, controllability, and observability. It can also be used to synthesize controllers and observers for nonlinear systems. Advantages of proposed linearization in comparison with those traditional methods that have been given are: Inability to conduct analysis of this system under dynamic conditions; inability of capturing effects arising from time-dependent inputs. Wide applicability-both for parameters time-varying nonlinear, and those in which output depends nonlinearly.

However, the proposed method has some limitations. For instance, it requires a reference trajectory that satisfies the system equations. The method also assumes that the deviation from the reference trajectory is small, which is not always true in practice. To overcome these shortcomings, further research could be dedicated to the extensions of the proposed method so that it can also be used more generally on nonlinear systems that have non-unique or unstable reference trajectories. It also sought to explore advancements in more advanced approximation techniques like higher-order approximations or transformations that are nonlinear in order to enhance the robustness and accuracy of the linearized system.

5. Conclusion

This work proposes a new methodology for the linearization of nonlinear signals in dynamical systems that relies on a reference trajectory instead of a fixed point. By approximating the nonlinear system as a linear time-varying system close to a given reference trajectory, the method delivers closer and more appropriate approximations when systems are not in stationary points.

The mathematical foundation of the proposed method is established through a systematic derivation, resulting in a proof of the linearization theorem. This theorem shows that a linear time-varying system close to a reference trajectory may approximate a non-linear dynamical system. With increased precision and adaptability for uses like dielectric breakdown statistics and RRAM operations, the linearised system may be used to stability analysis, control design, and comprehension.

By step-by-step derivation, the mathematical expression of the suggested method is obtained that leads to the linearization theorem being established; such a theorem shows that a non-linear dynamic system can, in the vicinity of a reference trajectory, be approximated by a linear time-varying system in a way that the analytical approach developed for stability analysis or control design/interpretation can be used with much higher accuracy and flexibility to the statistics of dielectric breakdown and the operations of RRAM.

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Received July, 2024