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Social interactions through signed Petri net

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Abstract

In this paper, dynamical balanceness in Signed Petri net (SiPN) have been introduced. The concept of dynamical balanceness proposed by authors is advantageous as all the signed graphs (directed or undirected) can be simulated by firing of different sequence of transitions in a single SiPN. Also, dynamics associated with a system can be easily represented using SiPN rather than a signed graph.

Subject Classification: *68R10, 05C22.*

Keywords: *Petri net, Signed graph, Balanced signed graph, Firing sequence, SiPN.*

1. Introduction

The concept of signed graph and various developments in the field viz. consistency, balanceness, domination, etc, [10, 4, 3, 1] have grown into a large area of study. Structural balanceness capture the patterns of interactions (positive or negative) and has been widely used to study and analyze the social networks[4, 8, 5, 3, 2]. But the interactions among individuals can change over time. The concept of balanceness in Signed Graph is static in nature, i.e., if the relationships among individuals change, then a new signed graph need to be formulated, which need to be analyzed again. While modeling real social systems which evolve with time and hence are dynamic in nature, a dynamic tool is required so that

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all the possible scenarios can be analyzed with ease without formulating multiple signed graphs. In [7], the authors have introduced an extension to PN called Signed Petri Net (SiPN), to model and analyze dynamical systems [9]. The proposed extension is beneficial since, using the dynamic nature of SiPN, the authors model all the possible interactions between a number of individuals using a single SiPN rather than using multiple structures in the form of signed graphs (sigraphs). For example, if three persons are interacting with one another, directed sigraph (or bidirected sigraphs) has been used to model such interactions. The number of such signed graphs required for analyzing the interactions are eight in number. However, using the concept of SiPN and dynamical balanceness developed by the authors, a single SiPN can be studied to reach the same set of conclusions.

Another benefit of using the SiPN is that even if the interactions between individuals evolve over the period of time (i.e., change from positive to negative or vice-versa), it is easier to represent it using SiPN by mere change of the subset of the transition set of SiPN as shown in section.

2. Basic Definitions

2.1 *Petri Net (PN)*

A *Petri net* [6] is a 5-tuple $N = (P, T, I^-, I^+, \mu_0)$, where

(1) P is the finite, non-empty set of places.

(2) T is the finite, non-empty set of transitions.

(3) $P \cap T = \varnothing$.

(4) $I^-, I^+ : (P \times T) \to \mathbb{N}$ where $\mathbb{N}$ is the set of non-negative integers, are called negative and positive incidence functions respectively.

(5) $\forall p \in P$, $\exists t \in T$ such that $I^-(p,t) \neq 0$ or $I^+(p,t) \neq 0$, and
$\forall t \in T$, $\exists p \in P$ such that $I^-(p,t) \neq 0$ or $I^+(p,t) \neq 0$

(6) $\mu_0 : P \to \mathbb{N}$ is the initial marking which gives the initial distribution of tokens in places.

The arc set of the Petri net N is defined as:

$$E = \{(p,t) : I^-(p,t) > 0\} \cup \{(t,p) : I^+(p,t) > 0\}$$

2.2 Signed Petri Net[7]

Definition 1: Signed Petri Net (SiPN)

A Signed Petri Net is defined as a 3-tuple $N^* = (N', \sigma, \mu_0)$, where

(1) $N' = (P, T, I^-, I^+)$ is a Petri net structure.

(2) $\sigma : E \to \{+, -\}$, where E is the arc set of N'. An arc is called a positive or negative arc respectively according to the sign + or – assigned to it using the function σ.

(3) $\mu_0 = (\mu_0^+, \mu_0^-)$ is the initial marking of SiPN where

(a) $\mu_0^+ : P \to \mathbb{N}$ gives the initial distribution of positive tokens in the places, called positive marking of SiPN.

(b) $\mu_0^- : P \to \mathbb{N}$ gives the initial distribution of negative tokens in the places, called negative marking of SiPN.

Thus, a marking in SiPN can be represented as a vector $\mu = (\mu^+, \mu^-)$ with $\mu^+, \mu^- \in \mathbb{N}^n$, $n = |P|$ such that $\mu(p_i) = (\mu^+(p_i), \mu^-(p_i)) \ \forall \ p_i \in P$.

Graphically, positive and negative arcs in a SiPN are represented by solid and dotted lines respectively. A positive token is represented by a filled circle and a negative token by an open circle.

A SiPN is said to be *negative* if all of its arcs are negative in sign.

Remark 1: $N'' = (N', \sigma)$ is called a SiPN structure where N' is a PN structure and $\sigma : E \to \{+, -\}$.

2.2.1 Execution Rules for Signed Petri Net

Similar to a Petri net, the execution of a SiPN depends on the distribution of tokens in its places. The execution takes place by firing of a transition. A transition may fire if it is enabled.

A transition t in a SiPN N^* is *enabled* at a marking $\mu = (\mu^+, \mu^-)$ if

$$I^-(p,t) \le \mu^+(p) \ \forall p \in {}^\bullet t \text{ for which } \sigma(p,t) = +$$

$$I^-(p,t) \le \mu^-(p) \ \forall p \in {}^\bullet t \text{ for which } \sigma(p,t) = -$$

An enabled transition t may *fire* at $\mu = (\mu^+, \mu^-)$ provided $\exists\, p_k \in t^\bullet$ such that:

$$\sigma(t, p_k) = \begin{cases} + & \text{if } \sigma(p,t) = + \ \forall p \in {}^\bullet t \\ - & \text{if } \sigma(p,t) = - \ \forall p \in {}^\bullet t \\ + \text{ or } - & \text{if } \sigma(p,t) = + \text{ for some } p \in {}^\bullet t \\ & \text{and } - \text{ for some } p \in {}^\bullet t \end{cases}$$

After firing, it yields a new marking $\mu_1 = (\mu_1^+, \mu_1^-)$ given by the rule:

$\mu_1^+(p) = \mu^+(p) - I^-(p,t) + I^+(p,t)\ \forall p \in P$ where $(p,t)\ \&\ (t,p)$ are positive arcs, if exist

$\mu_1^-(p) = \mu^-(p) - I^-(p,t) + I^+(p,t)\ \forall p \in P$ where $(p,t)\ \&\ (t,p)$ are negative arcs, if exist

We say that μ_1 is reachable from μ and write $\mu \xrightarrow{t} \mu_1$. We restrict the movement of positive (negative) tokens to positive (negative) arcs only.

A marking μ is reachable from μ_0 if there exists a firing sequence η that transforms μ_0 to μ and is written $\mu_0 \xrightarrow{\eta} \mu$. A *firing or occurrence sequence* is a sequence of transitions $\eta = t_1 t_2 \ldots t_k$ such that

$$\mu_0 \xrightarrow{t_1} \mu_1 \xrightarrow{t_2} \mu_2 \xrightarrow{t_3} \mu_3 \ldots \xrightarrow{t_k} \mu$$

Note that a transition $t_j, 1 \le j \le k$ can occur more than once in the firing sequence η.

Let us look at the execution of a SiPN with the help of some examples.

In figure 1, t_1 and t_2 both are enabled at μ_0. Firing of t_1 at μ_0 yields a new marking μ_1 = ((0, 1, 1, 0), (1, 0, 1, 0)) and firing of t_2 at μ_0 yields a new marking μ_1 = ((1, 0, 2, 0), (0, 1, 0, 1)). In figure 2, t_1 is enabled, while t_2 is not. t_1 can fire to give a new marking μ_1 = ((0, 0, 1, 0), (0, 0, 0, 1)).

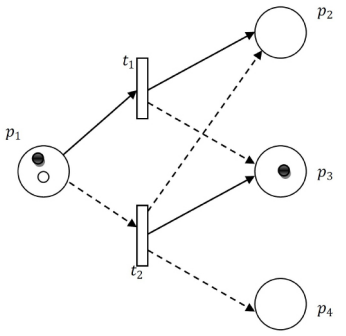


Figure 1

SPN with μ_0 = ((1, 0, 1, 0), (1, 0, 0, 0))

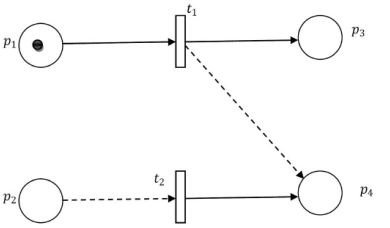


Figure 2

SPN with μ_0 = ((1, 0, 0, 0), (0, 0, 0, 0))

2.3 *Assignment of sign to vertices of a SiPN*

The vertices in a SiPN can also be assigned sign. Transitions are assigned sign by product of sign of arcs (incoming and outgoing) incident on it. In Figures 1 and 2, all transitions are negative in sign.

Places can be assigned sign in one of the two ways:

(1) **With respect to arcs –** Sign is assigned to a place by taking product of incident arcs (incoming and outgoing) on that place. In Figure 1, all the places are negative in sign while in Figure 2, p_1 and p_3 are positive in sign while p_2 and p_4 are negative in sign.

(2) **With respect to marking –** Sign is assigned to a place by taking product of sign of tokens in that place in the given marking. A place without token is considered to be positive. In Figure 1, places p_2, p_3, p_4 are positive in sign while p_1 is negatively signed w.r.t. the marking μ_0. In Figure 2, all the places are positively signed w.r.t. the marking μ_0.

Remark 2: Assigning sign to places with respect to arcs doesn't utilize the most important characteristic (dynamic behaviour) of PN which is a marking. Hence, assigning sign to places with respect to marking has been used throughout the paper.

3. Dynamical Balanceness in SiPN

The *underlying graph* of a SiPN N^*, denoted by N^v is the graph where directions of the arcs of N^* have been removed.

A *path(cycle)* in a SiPN N^* is the path(cycle) in its underlying graph N^v.

3.1 *Sign of a cycle in a SiPN*

The *sign of a cycle* in a SiPN N^* can be defined in three ways. In method (1) and (2), cycles are considered in an underlying graph while in method (3), directed cycles are considered.

(1) **With respect to arc sign**

In this case, sign of the cycle is given by product of sign of the arcs on the cycle. e.g. the sign of cycle $p_1t_1p_3t_2p_1$ in the figure 1 is positive.

(2) **With respect to vertex sign**

In this case, the sign of the cycle is given by the product of sign of the vertices on the cycle with the sign of end vertex considered twice. e.g. the sign of cycle $p_1t_1p_3t_2p_1$ in the figure 1 is positive.

(3) **With respect to flow of tokens**

The sign of a cycle with respect to flow of tokens is defined as the product of sign of tokens that flow on the output arcs of the transitions lying on the cycle, when all the transitions on the cycle fire.

Note that we use this method of assigning sign to a cycle for social interactions when the input and output arcs for any transition are of same sign because the concept of balanceness is used to study stability of social interactions.

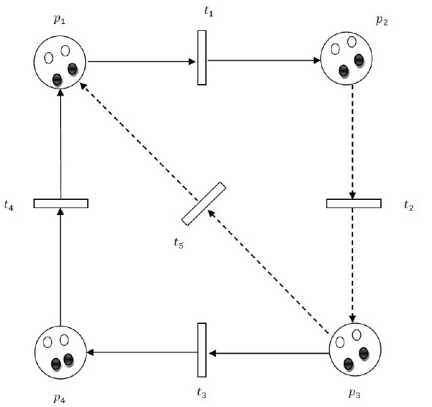


Figure 3

A SiPN

In the figure 3, the sign of cycle $p_1t_1p_2t_2p_3t_5p_1$ is positive since when t_1, t_2 and t_5 fire, one positive and two negative tokens flow on the output arcs of these transitions. The product of the sign of these tokens is positive. Similarly, the sign of cycle $p_1t_1p_2t_2p_3t_3p_4t_4p_1$ is negative since when t_1, t_2, t_3 and t_4 fire, three positive and one negative token flow on the output arcs of these transitions and the product of the sign of these tokens is negative, hence the sign of cycle $p_1t_1p_2t_2p_3t_3p_4t_4p_1$ w.r.t. flow of tokens is negative.

Remark 3: If there exist at least one transition on a directed cycle which does not fire, then sign of such a cycle is not defined with respect to flow of tokens.

Remark 4: Throughout the paper, the sign of a cycle is defined with respect to flow of tokens.

3.2 *Dynamically Balanced SiPN*

A SiPN N^* is said to be dynamically balanced with respect to $T' \subseteq T$, if the sign of all the directed cycles consisting of transitions from T' only is positive with respect to flow of tokens else, the SiPN is not dynamically balanced.

Let us consider the SiPN given in figure 3 with $T' = \{t_1, t_2, t_5\}$. The SiPN is dynamically balanced w.r.t. T' since the cycle $p_1 t_1 p_2 t_2 p_3 t_5 p_1$ is positive. If we consider the SiPN with $T' = \{t_1, t_2, t_5, t_4, t_3\}$, it is not dynamically balanced w.r.t. T' since out of the two cycles $p_1 t_1 p_2 t_2 p_3 t_3 p_4 t_4 p_1$ and $p_1 t_1 p_2 t_2 p_3 t_5 p_1$, the former is negative.

4. Equivalence between Balanced Signed Graphs and Dynamically Balanced SiPN

Consider a SiPN N^* given in figure 4 which represents all possible interactions between three persons represented by $p_1, p_2, \& p_3$ places. An interaction between p_i and p_j can be positive or negative $\forall i, j \in \{1,2,3\}$, $i \neq j$. All the transitions in N^* represent the initiation of an interaction between any two individuals. The positive and negative arcs in N^* indicate positive (friendly) and negative (antagonistic) interactions respectively.

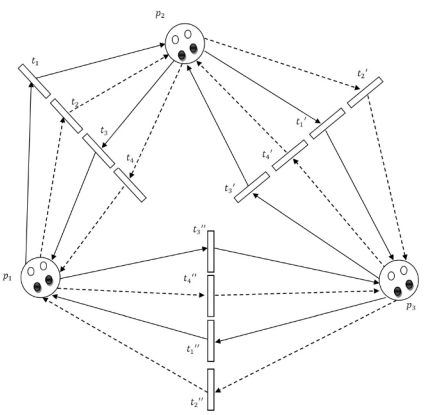


Figure 4

A SiPN representing three persons interactions

The positive and negative tokens in the places are the positive and negative thoughts of the person who leads to positive and negative interactions respectively. A subset of transition set of SiPN N^*, say T' can be used to obtain a sub-SiPN from a given SiPN which contains all the transitions from T' only along with the places connected to these transitions via incident arcs of the transitions. The given sub-SiPN obtained using T' can describe a situation in sigraph where between three individuals some type of interactions are taking place. On changing T', other situations can be represented. Hence, using a single SiPN all the possible interactions can be represented by change of this subset of transitions.

Example: If $T' = \{t_1, t_3, t_2', t_4', t_1'', t_3''\}$, then corresponding sub-SiPN for the SiPN in figure 4 is given in figure 5. Clearly, the sub-SiPN in the figure is not dynamically balanced w.r.t. T'. Since, cycle $p_1 t_1 p_2 t_2' p_3 t_1'' p_1$ is negative in sign w.r.t. flow of tokens. The signed graph equivalent to this sub-SiPN is represented in figure 6 which by theory of structural balance for sigraphs is unbalanced which agrees with the result for SiPN .

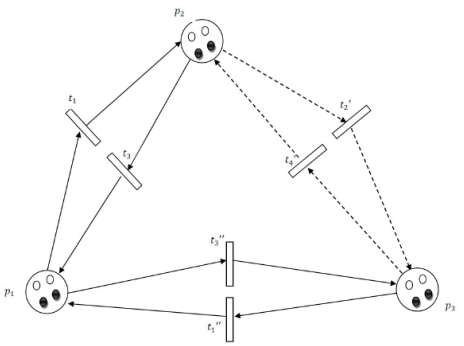


Figure 5

Sub-SiPN corresponding to $T' = \{t_1, t_3, t_1'', t_3'', t_2', t_4'\}$

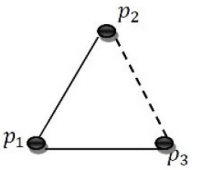


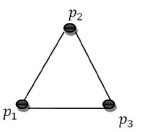
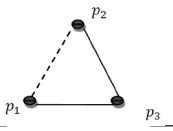
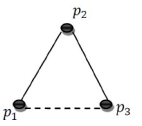
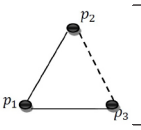
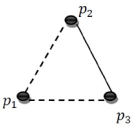
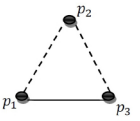
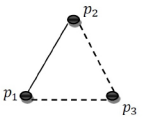
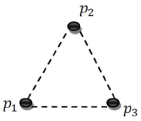
Figure 6

Equivalent Signed Graph

Similarly, all the possible signed graphs can be obtained if different subsets of transition set of SiPN are considered. This is shown in the Table 1. The results in the table are obtained for bi-directed signed graphs, however, for the same SiPN the results for uni-directed signed graphs can also be obtained.

Table 1

Table for the corresponding SiPN for a subset of transitions

Subset of transitions T'	Corresponding Signed graph	Nature
$\{t_1, t_3, t_1', t_3', t_1'', t_3''\}$		Balanced
$\{t_2, t_4, t_1', t_3', t_1'', t_3''\}$		Not Balanced
$\{t_1, t_3, t_1', t_3', t_2'', t_4''\}$		Not Balanced
$\{t_1, t_3, t_2', t_4', t_1'', t_3''\}$		Not Balanced
$\{t_2, t_4, t_1', t_3', t_2'', t_4''\}$		Balanced
$\{t_2, t_4, t_2', t_4', t_1'', t_3''\}$		Balanced
$\{t_1, t_3, t_2', t_4', t_2'', t_4''\}$		Balanced
$\{t_2, t_4, t_2', t_4', t_2'', t_4''\}$		Not Balanced

This concept of dynamical balanceness can be extended if the number of individuals increase. As seen above, a single SiPN can simulate all sigraphs by change in the marking of SiPN due to firing of a set of transitions. It can be seen that exactly similar results are obtained with less

efforts. Also, the dynamics of SiPN help in studying all interactions in a single SiPN structure rather than multiple signed graphs.

5. Conclusion

The authors utilize the concept of a Signed Petri Net(SiPN) to introduce the notion of dynamical Balanceness because, the concept of balanceness which is prevalent in signed graphs is static in nature. Hence, to analyze social interactions between individuals, multiple signed graphs need to be formulated. This limitation can be overcome by SiPN, since using a single SiPN, exactly same results can be obtained by varying the marking of a SiPN, which make the proposed notion advantageous.

Data Availability

No data were used to support this study.

Conflict of Interest

The authors declare that there are no conflicts of interest

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