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On companion sequences associated with Leonardo quaternions : Applications over finite fields

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Abstract

It is known that the quaternion algebras are central simple algebras and also clifford algebras. In this paper, we introduce a new class of quaternions called Lucas-Leonardo p-quaternions and derive several fundamental properties of these numbers. Furthermore, we investigate some applications related to companion sequences associated with Leonardo quaternions. In particular, we determine Lucas-Leonardo quaternions and Francois quaternions, which are zero divisors and invertible elements in the quaternion algebra over certain finite fields.

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1. Introduction

Fibonacci sequences and their extensions find numerous applications across various domains, including arts and sciences. Notably, Fibonacci p-numbers, a broader variant of Fibonacci numbers, introduced by Stakhov and Rozin [20] and have found applications in coding theory, particularly in designing error-correcting codes and data compression algorithms. For any given integer $p>0$, the Fibonacci p-sequence is defined by the following recurrence relation

$$F_{p,n}=F_{p,n-1}+F_{p,n-p-1},n>p$$

with initial values $F_{p,0}=0$, $F_{p,k}=1$ for $k=1,2,\ldots,p$. Its companion sequence, the Lucas p-sequence $\{L_{p,n}\}_{n=0}^{\infty}$, [20] also satisfy the same recurrence relation but begins with initial values $L_{p,0}=p+1$, $L_{p,k}=1$ for $k=1,2,\ldots,p$. It is clear to see that when $p=1$, the Fibonacci p-sequence and the Lucas p-sequence reduce to the classical Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and Lucas sequence $\{L_n\}_{n=0}^{\infty}$, respectively. For further information regarding the Fibonacci p-numbers and general second-order linear recurrences, we refer to [1, 12, 20, 24].

There are several non-homogenous extensions of Fibonacci recurrence relation. One among them is the Leonardo p-sequence $\{\mathcal{L}_{p,n}\}_{n=0}^{\infty}$ which was introduced by Tan and Leung [22] and defined by the following non-homogenous relation:

$$\mathcal{L}_{p,n}=\mathcal{L}_{p,n-1}+\mathcal{L}_{p,n-p-1}+p,n>p \tag{1.1}$$

with initial values $\mathcal{L}_{p,0}=\mathcal{L}_{p,1}=\cdots=\mathcal{L}_{p,p}=1$. It is clear to see that when $p=1$, it reduces to the classical Leonardo sequence $\{\mathcal{L}_n\}_{n=0}^{\infty}$. The number of vertices in the n-th Leonardo tree is counted by Leonardo numbers. The Leonardo sequence finds applications in various fields, including mathematics, computer science, and cryptography. For the history of Leonardo sequences, see [A001595] in the On-Line Encyclopedia of Integer Sequences [19], and for the properties of Leonardo numbers, we refer to [2, 3, 5, 13, 18].

A relation between Leonardo p-numbers and Fibonacci p-numbers was given by

$$\mathcal{L}_{p,n}=(p+1)F_{p,n+1}-p \tag{1.2}$$

and a relation between Leonardo p-numbers, Fibonacci p-numbers and Lucas p-numbers was given by

$$\mathcal{L}_{p,n} = L_{p,n+p+1} - F_{p,n+p+1} - p. \tag{1.3}$$

Recently, Zhong et al.[25] have studied a companion sequence of Leonardo p-sequence, called Lucas-Leonardo p-sequence, and defined it by the following non-homogenous relation:

$$\mathcal{R}_{p,n} = \mathcal{R}_{p,n-1} + \mathcal{R}_{p,n-p-1} + p, n > p \tag{1.4}$$

with initial values $\mathcal{R}_{p,0} = p^2 + p + 1, \mathcal{R}_{p,1} = \cdots = \mathcal{R}_{p,p} = 1.$ The non-homogenous relation of Lucas-Leonardo p-sequence can be converted to the following homogenous recurrence relation

$$\mathcal{R}_{p,n} = \mathcal{R}_{p,n-1} + \mathcal{R}_{p,n-p} - \mathcal{R}_{p,n-2p-1,} n > 2p. \tag{1.5}$$

A relation between Lucas-Leonardo p-numbers and Lucas p-numbers is given by

$$\mathcal{R}_{p,n} = (p+1)L_{p,n} - p \tag{1.6}$$

and a relation between Lucas-Leonardo p-numbers and Leonardo p-numbers is given by

$$\mathcal{R}_{p,n} = (p+1)\mathcal{L}_{p,n} - p\mathcal{L}_{p,n-1}. \tag{1.7}$$

In particular, for $p = 1,$ it reduces to the Lucas-Leonardo sequence $\{\mathcal{R}_n\}_{n=0}^{\infty}$ in [A022319]. A relation between Lucas-Leonardo and Lucas numbers is

$$\mathcal{R}_n = 2L_n - 1. \tag{1.8}$$

Another companion sequence of Leonardo sequence, namely the Francois sequence $\{\mathcal{F}_n\}_{n=0}^{\infty},$ see [A022318] in [19], is studied by Diskaya and Menken [6]. The Francois numbers also satisfy the same recurrence relation as Lucas-Leonardo numbers but begins with initial values $\mathcal{F}_0 = 2,$ $\mathcal{F}_1 = 1.$ A relation between Francois numbers, Fibonacci and Lucas numbers is given by

$$\mathcal{F}_n = L_n + F_{n+1} - 1. \tag{1.9}$$

On the other hand, quaternion algebras finds applications in various fields such as mathematics, physics, computer science, engineering, and robotics. It is particularly useful in representing and manipulating spatial rotations and orientations in three-dimensional space due to its compact and efficient representation of such transformations. It is also known that the quaternion algebras are central simple algebras and also clifford algebras. Let F be a field with characteristic not 2. The generalized quaternion algebra over a field F is defined as:

$$\mathbb{Q}_F(a,b) = \{x = x_1 + x_2 i + x_3 j + x_4 k \mid x_1, x_2, x_3, x_4 \in F, \\ i^2 = a, j^2 = b, ij = -ji = k\} \tag{1.40}$$

where a,b are nonzero invertible elements of field F. It is clear to see that when $F = \mathbb{R}$ and $a,b = -1$, we get the real quaternion algebra. For $x \in \mathbb{Q}_F(a,b)$, the norm of x, denoted as $N(x)$ and defined as $N(x) = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$. We recall that a generalized quaternion algebra is a division algebra if and only if a quaternion with a norm of zero is necessarily the zero quaternion. Otherwise, the algebra is called a split algebra. It is known that real quaternion algebra is a division algebra and the quaternion algebra over finite field $\mathbb{Z}_q$, denoted as $Q_{\mathbb{Z}_q}(-1,-1)$ is a split algebra, where q is an odd prime integer, see [9]. Many research studies have concentrated on quaternions whose components stem from distinctive integer sequences such as Fibonacci, Lucas, Leonardo sequences, among various others. In particular, Horadam [10] introduced the Fibonacci quaternions by taking coefficients as Fibonacci numbers. In [16], Savin studied special Fibonacci quaternions in quaternion algebras over finite fields. Recently, in [23], the authors studied Leonardo quaternions over finite fields. For more information related to quaternion algebras and Fibonacci-like quaternion sequences we refer to [4, 7, 8, 11, 14, 15, 17, 21] and the references therein.

In this paper, we would like to contribute on this topic by introducing a new class of quaternions, called Lucas-Leonardo p-quaternions. We derive several fundamental properties of these numbers including recurrence relations, generating function, and summation formulas. The main motivation of our paper is to consider companion sequences of Leonardo quaternions and derive some special Lucas-Leonardo and Francois quaternions in quaternion algebras over finite fields. In particular, we determine the Lucas-Leonardo quaternions and Francois quaternions which are zero divisors and invertible elements in the quaternion algebra over certain finite fields.

2. Lucas-Leonardo p-quaternions

In this section, we introduce a new class of quaternions, namely Lucas-Leonardo p-quaternions, and explore the fundamental properties of these sequences. Throughout this section, we consider the real quaternion algebra $\mathbb{Q}_{\mathbb{R}}(-1,-1)$. Also, for the simplicity, we denote I as $1 + i + j + k$.

Definition 1: The n^{th} Lucas-Leonardo p-quaternion is defined by

$$\mathbb{Q}\mathcal{R}_{p,n} = \mathcal{R}_{p,n} + \mathcal{R}_{p,n+1} i + \mathcal{R}_{p,n+2} j + \mathcal{R}_{p,n+3} k$$

where $\mathcal{R}_{p,n}$ is the n^{th} Lucas-Leonardo p-number.

From the definition of Lucas-Leonardo p-numbers, we have

$$\begin{aligned}
\mathbb{Q}\mathcal{R}_{p,0} &= \mathcal{R}_{p,0} + \mathcal{R}_{p,1} i + \mathcal{R}_{p,2} j + \mathcal{R}_{p,3} k = I + p(p+1) \\
\mathbb{Q}\mathcal{R}_{p,1} &= \mathcal{R}_{p,1} + \mathcal{R}_{p,2} i + \mathcal{R}_{p,3} j + \mathcal{R}_{p,4} k = I \\
&\vdots \\
\mathbb{Q}\mathcal{R}_{p,p-3} &= \mathcal{R}_{p,p-3} + \mathcal{R}_{p,p-2} i + \mathcal{R}_{p,p-1} j + \mathcal{R}_{p,p} k = I \\
\mathbb{Q}\mathcal{R}_{p,p-2} &= \mathcal{R}_{p,p-2} + \mathcal{R}_{p,p-1} i + \mathcal{R}_{p,p} j + \mathcal{R}_{p,p+1} k = I + (p+1)^2 k \\
\mathbb{Q}\mathcal{R}_{p,p-1} &= \mathcal{R}_{p,p-1} + \mathcal{R}_{p,p} i + \mathcal{R}_{p,p+1} j + \mathcal{R}_{p,p+2} k \\
&= 1 + i + (p^2 + 2p + 2) j + (p^2 + 3p + 3) k \\
&= I + (p+1)((p+1) j + (p+2) k) \\
\mathbb{Q}\mathcal{R}_{p,p} &= \mathcal{R}_{p,p} + \mathcal{R}_{p,p+1} i + \mathcal{R}_{p,p+2} j + \mathcal{R}_{p,p+3} k \\
&= 1 + (p^2 + 2p + 2) i + (p^2 + 3p + 3) j + (p^2 + 5p + 5) k \\
&= I + (p+1)((p+1) i + (p+2) j + (p+4) k).
\end{aligned}$$

It is clear to see that from the relation (1.4) and the definition of Lucas-Leonardo p-quaternions, we have

$$\mathbb{Q}\mathcal{R}_{p,n} = \mathbb{Q}\mathcal{R}_{p,n-1} + \mathbb{Q}\mathcal{R}_{p,n-p-1} + pI, n > p. \tag{2.1}$$

For $n > 2p$, from the relation (1.5) and the definition of Lucas-Leonardo p-numbers, we have

$$\mathbb{Q}\mathcal{R}_{p,n} = \mathbb{Q}\mathcal{R}_{p,n-1} + \mathbb{Q}\mathcal{R}_{p,n-p} - \mathbb{Q}\mathcal{R}_{p,n-2p-1}. \tag{2.2}$$

Now we state the generating function of the Lucas-Leonardo p-quaternions.

Proposition 1: The generating function for Lucas-Leonardo p-quaternions is

$$G(x) = \frac{I + p(p+1) + \sum_{n=1}^{p} (\mathbb{Q}\mathcal{R}_{p,n} - \mathbb{Q}\mathcal{R}_{p,n-1}) x^n + pI \frac{x^{p+1}}{1-x}}{1 - x - x^{p+1}}.$$

Proof: We consider the formal power series representation for the generating function of Lucas-Leonardo p-quaternions, $G(x) = \sum_{n=0}^{\infty}\mathbb{Q}\mathcal{R}_{p,n}x^n$. By using the relation (2.1), we get

$$\begin{aligned}(1-x-x^{p+1})G(x) &= \sum_{n=0}^{\infty}\mathbb{Q}\mathcal{R}_{p,n}x^n - \sum_{n=0}^{\infty}\mathbb{Q}\mathcal{R}_{p,n}x^{n+1} - \sum_{n=0}^{\infty}\mathbb{Q}\mathcal{R}_{p,n}x^{n+p+1} \\ &= \sum_{n=0}^{p}\mathbb{Q}\mathcal{R}_{p,n}x^n - \sum_{n=1}^{p}\mathbb{Q}\mathcal{R}_{p,n-1}x^n + \sum_{n=p+1}^{\infty}(\mathbb{Q}\mathcal{R}_{p,n} - \mathbb{Q}\mathcal{R}_{p,n-1} - \mathbb{Q}\mathcal{R}_{p,n-p-1})x^n \\ &= \sum_{n=0}^{p}\mathbb{Q}\mathcal{R}_{p,n}x^n - \sum_{n=1}^{p}\mathbb{Q}\mathcal{R}_{p,n-1}x^n + pI\sum_{n=p+1}^{\infty}x^n \\ &= \mathbb{Q}\mathcal{R}_{p,0} + \sum_{n=1}^{p}(\mathbb{Q}\mathcal{R}_{p,n} - \mathbb{Q}\mathcal{R}_{p,n-1})x^n + pI\frac{x^{p+1}}{1-x} \\ &= I + p(p+1) + \sum_{n=1}^{p}(\mathbb{Q}\mathcal{R}_{p,n} - \mathbb{Q}\mathcal{R}_{p,n-1})x^n + pI\frac{x^{p+1}}{1-x}\end{aligned}$$

which gives the desired result.

Note that from initial values of Lucas-Leonardo p-quaternions, we have

$$\begin{aligned}&\sum_{n=1}^{p}(\mathbb{Q}\mathcal{R}_{p,n} - \mathbb{Q}\mathcal{R}_{p,n-1})x^n \\ &= (\mathbb{Q}\mathcal{R}_{p,1} - \mathbb{Q}\mathcal{R}_{p,0})x + (\mathbb{Q}\mathcal{R}_{p,2} - \mathbb{Q}\mathcal{R}_{p,1})x^2 + \cdots + (\mathbb{Q}\mathcal{R}_{p,p-3} - \mathbb{Q}\mathcal{R}_{p,p-4})x^{p-3} \\ &\quad + (\mathbb{Q}\mathcal{R}_{p,p-2} - \mathbb{Q}\mathcal{R}_{p,p-3})x^{p-2} + (\mathbb{Q}\mathcal{R}_{p,p-1} - \mathbb{Q}\mathcal{R}_{p,p-2})x^{p-1} + (\mathbb{Q}\mathcal{R}_{p,p} - \mathbb{Q}\mathcal{R}_{p,p-1})x^p \\ &= -p(p+1)x + (p+1)^2kx^{p-2} + (p+1)^2(j+k)x^{p-1} + (p+1)((p+1)i + j + 2k)x^p \\ &= (p+1)(-px + (p+1)kx^{p-2} + (p+1)(j+k)x^{p-1} + ((p+1)i + j + 2k)x^p).\end{aligned}$$

Now we provide some relations between Lucas-Leonardo p-quaternions, Leonardo p-quaternions, Fibonacci p-quaternions, and Lucas p-quaternions. It's worth noting that Fibonacci p-quaternions and Lucas p-quaternions can be defined similarly to Lucas-Leonardo p-quaternions as:

$$\mathbb{Q}F_{p,n} = F_{p,n} + F_{p,n+1}i + F_{p,n+2}j + F_{p,n+3}k$$

and

$$\mathbb{Q}L_{p,n} = L_{p,n} + L_{p,n+1}i + L_{p,n+2}j + L_{p,n+3}k,$$

respectively.

Proposition 2: For $n \geq p$, we have the following relations:

(i) $\mathbb{Q}\mathcal{R}_{p,n} = (p+1)\mathbb{Q}L_{p,n} - pI,$

(ii) $\mathbb{Q}\mathcal{R}_{p,n} = (p+1)\mathbb{Q}\mathcal{L}_{p,n} - p\mathbb{Q}\mathcal{L}_{p,n-1},$

(iii) $\mathbb{Q}\mathcal{R}_{p,n} = \mathbb{Q}\mathcal{L}_{p,n} + p\mathbb{Q}\mathcal{L}_{p,n-p-1} - p^2 I$,

(iv) $\mathbb{Q}\mathcal{R}_{p,n} = p(\mathbb{Q}L_{p,n} - \mathbb{Q}F_{p,n}) + \mathbb{Q}L_{p,n+p+1} - \mathbb{Q}F_{p,n+p+1} - p(p+1)I.$

Proof:

(i) From (1.6), we have

$$\begin{aligned}\mathbb{Q}\mathcal{R}_{p,n} &= \mathcal{R}_{p,n} + \mathcal{R}_{p,n+1}i + \mathcal{R}_{p,n+2}j + \mathcal{R}_{p,n+3}k\\ &= ((p+1)L_{p,n} - p) + ((p+1)L_{p,n+1} - p)i\\ &\quad + ((p+1)L_{p,n+2} - p)j + ((p+1)L_{p,n+3} - p)k\\ &= (p+1)(L_{p,n} + L_{p,n+1}i + L_{p,n+2}j + L_{p,n+3}k) - p(1+i+j+k)\\ &= (p+1)\mathbb{Q}L_{p,n} - pI.\end{aligned}$$

(ii) From (1.7), we have

$$\begin{aligned}\mathbb{Q}\mathcal{R}_{p,n} &= \mathcal{R}_{p,n} + \mathcal{R}_{p,n+1}i + \mathcal{R}_{p,n+2}j + \mathcal{R}_{p,n+3}k\\ &= (p+1)\mathcal{L}_{p,n} - p\mathcal{L}_{p,n-1} + ((p+1)\mathcal{L}_{p,n+1} - p\mathcal{L}_{p,n})i\\ &\quad + ((p+1)\mathcal{L}_{p,n+2} - p\mathcal{L}_{p,n+1})j + ((p+1)\mathcal{L}_{p,n+3} - p\mathcal{L}_{p,n+2})k\\ &= (p+1)(\mathcal{L}_{p,n} + \mathcal{L}_{p,n+1}i + \mathcal{L}_{p,n+2}j + \mathcal{L}_{p,n+3}k)\\ &\quad -p(\mathcal{L}_{p,n-1} + \mathcal{L}_{p,n}i + \mathcal{L}_{p,n+1}j + \mathcal{L}_{p,n+2}k)\\ &= (p+1)\mathbb{Q}\mathcal{L}_{p,n} - p\mathbb{Q}\mathcal{L}_{p,n-1}.\end{aligned}$$

(iii) By using the relation (ii) and (1.1), we can easily obtained the second relation.

$$\begin{aligned}\mathbb{Q}\mathcal{R}_{p,n} &= (p+1)\mathbb{Q}\mathcal{L}_{p,n} - p\mathbb{Q}\mathcal{L}_{p,n-1}\\ &= (p+1)(\mathcal{L}_{p,n} + \mathcal{L}_{p,n+1}i + \mathcal{L}_{p,n+2}j + \mathcal{L}_{p,n+3}k)\\ &\quad -p(\mathcal{L}_{p,n-1} + \mathcal{L}_{p,n}i + \mathcal{L}_{p,n+1}j + \mathcal{L}_{p,n+2}k)\\ &= p(\mathcal{L}_{p,n} - \mathcal{L}_{p,n-1}) + p(\mathcal{L}_{p,n+1} - \mathcal{L}_{p,n})i + p(\mathcal{L}_{p,n+2} - \mathcal{L}_{p,n+1})j + p(\mathcal{L}_{p,n+3} + \mathcal{L}_{p,n+2})k\\ &\quad +(\mathcal{L}_{p,n} + \mathcal{L}_{p,n+1}i + \mathcal{L}_{p,n+2}j + \mathcal{L}_{p,n+3}k)\\ &= p(\mathcal{L}_{p,n-p-1} - p) + p(\mathcal{L}_{p,n-p} - p)i + p(\mathcal{L}_{p,n-p+1} - p)j + p(\mathcal{L}_{p,n-p+2} - p)k\\ &\quad +(\mathcal{L}_{p,n} + \mathcal{L}_{p,n+1}i + \mathcal{L}_{p,n+2}j + \mathcal{L}_{p,n+3}k)\\ &= p(\mathcal{L}_{p,n-p-1} + \mathcal{L}_{p,n-p}i + \mathcal{L}_{p,n-p+1}j + \mathcal{L}_{p,n-p+2}k) - p^2(1+i+j+k) + \mathbb{Q}\mathcal{L}_{p,n}\\ &= p\mathbb{Q}\mathcal{L}_{p,n-p-1} + \mathbb{Q}\mathcal{L}_{p,n} - p^2 I.\end{aligned}$$

(iv) By using the relation (iii) and (1.3), we obtain the desired result.

$$\begin{aligned}
\mathbb{Q}\mathcal{R}_{p,n} &= p\mathbb{Q}\mathcal{L}_{p,n-p-1} + \mathbb{Q}\mathcal{L}_{p,n} - p^2 I. \\
&= p(\mathcal{L}_{p,n-p-1} + \mathcal{L}_{p,n-p} i + \mathcal{L}_{p,n-p+1} j + \mathcal{L}_{p,n-p+2} k) \\
&\quad +(\mathcal{L}_{p,n} + \mathcal{L}_{p,n+1} i + \mathcal{L}_{p,n+2} j + \mathcal{L}_{p,n+3} k) - p^2 I \\
&= p(L_{p,n} - F_{p,n} - p) + p(L_{p,n+1} - F_{p,n+1} - p)i \\
&\quad +p(L_{p,n+2} - F_{p,n+2} - p)j + p((L_{p,n+3} - F_{p,n+3} - p))k \\
&\quad +(L_{p,n+p+1} + L_{p,n+p+2} i + L_{p,n+p+3} j + L_{p,n+p+4} k) \\
&\quad -(F_{p,n+p+1} + F_{p,n+p+2} i + F_{p,n+p+3} j + F_{p,n+p+4} k) - pI - p^2 I \\
&= p(L_{p,n} + L_{p,n+1} i + L_{p,n+2} j + L_{p,n+3} k) - p(F_{p,n} + F_{p,n+1} i + F_{p,n+2} j + F_{p,n+3} k) \\
&\quad +\mathbb{Q}L_{p,n+p+1} - \mathbb{Q}F_{p,n+p+1} - pI - p^2 I \\
&= p(\mathbb{Q}L_{p,n} - \mathbb{Q}F_{p,n}) + \mathbb{Q}L_{p,n+p+1} - \mathbb{Q}F_{p,n+p+1} - p(p+1)I.
\end{aligned}$$

Proposition 3: For $n \geq p,$ we have

$$\sum_{k=0}^{n} \mathbb{Q}\mathcal{R}_{p,k} = (p+1)(\mathbb{Q}L_{p,n+p+1} - \mathbb{Q}L_{p,p}) - p(n+1)I.$$

Proof: By using the relation (i) in Proposition 2 and the summation formula for Lucas p-quaternions $\sum_{r=0}^{n} \mathbb{Q}L_{p,r} = \mathbb{Q}L_{p,n+p+1} - \mathbb{Q}L_{p,p},$ we get

$$\begin{aligned}
\sum_{r=0}^{n} \mathbb{Q}\mathcal{R}_{p,r} &= (p+1)\sum_{r=0}^{n} \mathbb{Q}L_{p,r} - \sum_{r=0}^{n} pI \\
&= (p+1)\sum_{r=0}^{n} \mathbb{Q}L_{p,r} - p(n+1)I \\
&= (p+1)(\mathbb{Q}L_{p,n+p+1} - \mathbb{Q}L_{p,p}) - p(n+1)I.
\end{aligned}$$

Next we give a convoluted relation for Lucas-Leonardo p-quaternions. To do this, we need the following relations in [1, Theorem 3] and [24, Property 6], respectively:

$$\sum_{t=1}^{p} L_{p,n-t} F_{p,t} = L_{p,n+p} - F_{p,p+1} L_{p,n}, \tag{2.3}$$

$$\sum_{t=0}^{n} F_{p,t} = F_{p,n+p+1} - F_{p,p}. \tag{2.4}$$

Proposition 4: For $n \geq p,$ we have

$$\sum_{t=1}^{p}\mathbb{Q}\mathcal{R}_{p,n-t}F_{p,t}=(p+1)(\mathbb{Q}L_{p,n+p}-\mathbb{Q}L_{p,n})-pI(F_{p,2p+1}-1).$$

Proof: By using the relation (i) in Proposition 2 and (2.4), we have

$$\begin{aligned}\sum_{t=1}^{p}\mathbb{Q}\mathcal{R}_{p,n-t}F_{p,t}&=\sum_{t=1}^{p}((p+1)\mathbb{Q}L_{p,n-t}-pI)F_{p,t}\\&=(p+1)\sum_{t=1}^{p}\mathbb{Q}L_{p,n-t}F_{p,t}-pI\sum_{t=1}^{p}F_{p,t}\\&=(p+1)\sum_{t=1}^{p}(L_{p,n-t}+L_{p,n-t+1}i+L_{p,n-t+2}j+L_{p,n-t+3}k)F_{p,t}-pI(F_{p,2p+1}-F_{p,p})\\&=(p+1)\left(\sum_{t=1}^{p}L_{p,n-t}F_{p,t}+\sum_{t=1}^{p}L_{p,n-t+1}F_{p,t}i\right.\\&\left.+\sum_{t=1}^{p}L_{p,n-t+2}F_{p,t}j+\sum_{t=1}^{p}L_{p,n-t+3}F_{p,t}k\right)-pI(F_{p,2p+1}-1).\end{aligned}$$

From the relation (2.3) and since $F_{p,p+1}=1,$ we get

$$\begin{aligned}\sum_{t=1}^{p}\mathbb{Q}\mathcal{R}_{p,n-t}F_{p,t}&=(p+1)(L_{p,n+p}-F_{p,p+1}L_{p,n}+(L_{p,n+p+1}-F_{p,p+1}L_{p,n+1})i\\&\quad+(L_{p,n+p+2}-F_{p,p+1}L_{p,n+2})j+(L_{p,n+p+3}-F_{p,p+1}L_{p,n+3})k)\\&\quad-pI(F_{p,2p+1}-1)\\&=(p+1)(L_{p,n+p}+L_{p,n+p+1}i+L_{p,n+p+2}j+L_{p,n+p+3}k)\\&\quad-(L_{p,n}+L_{p,n+1}i+L_{p,n+2}j+L_{p,n+3}k)-pI(F_{p,2p+1}-1)\\&=(p+1)\mathbb{Q}L_{p,n+p}-(p+1)\mathbb{Q}L_{p,n}-pI(F_{p,2p+1}-1).\end{aligned}$$

3. Lucas-Leonardo quaternions and Francois quaternions over finite fields

In this section, we consider the quaternion algebra $\mathbb{Q}_{\mathbb{Z}q}(-1,-1),$ for simplicity $\mathbb{Q}_{\mathbb{Z}q}$ where q is an odd prime integer. First, we determine the Lucas-Leonardo quaternions which are zero divisors in the quaternion algebra $\mathbb{Q}_{\mathbb{Z}q}$ for $q=3,5,$ and 7. It's important to note that finding zero divisors and invertible elements in the Lucas-Leonardo p-quaternions is more challenging task than in the Fibonacci quaternions studied by Savin [16]. This is mainly because the norm of Lucas-Leonardo p-quaternions is more complex. So, we focus on the standard Lucas-Leonardo quaternion case instead. Similar approach will be used for the Francois quaternions.

Let $\mathbb{Q}\mathcal{R}_n$ be the nth Lucas-Leonardo quaternion defined as

$$\mathbb{Q}\mathcal{R}_n = \mathcal{R}_n + \mathcal{R}_{n+1}i + \mathcal{R}_{n+2}j + \mathcal{R}_{n+3}k$$

where the basis $\{1, i, j, k\}$ satisfies the multiplication rules: $i^2 = j^2 = k^2 = ijk = -1$. By using the relation (1.8) for the Lucas-Leonardo quaternion and using the relations $L_n^2 + L_{n+1}^2 = 5F_{2n+1}$, $F_n^2 + F_{n+1}^2 = F_{2n+1}$, $F_n + F_{n+4} = 3F_{n+2}$, $L_n F_{n+1} = F_{2n+1} + (-1)^n F_1$, $L_n + L_{n+2} = 5F_{n+1}$, $F_n + F_{n+2} = L_{n+1}$ the norm of Lucas-Leonardo quaternion can be obtained as follows:

$$\begin{aligned} N(\mathbb{Q}\mathcal{R}_n) &= \mathcal{R}_n^2 + \mathcal{R}_{n+1}^2 + \mathcal{R}_{n+2}^2 + \mathcal{R}_{n+3}^2 \\ &= (2L_n - 1)^2 + (2L_{n+1} - 1)^2 + (2L_{n+2} - 1)^2 + (2L_{n+3} - 1)^2 \\ &= 4(L_n^2 + L_{n+1}^2 + L_{n+2}^2 + L_{n+3}^2) - 4(L_n + L_{n+1} + L_{n+2} + L_{n+3}) + 4 \qquad (3.1) \\ &= 20(F_{2n+1} + F_{2n+5}) - 4(L_{n+2} + L_{n+4}) + 4 \\ &= 60F_{2n+3} - 20F_{n+3} + 4 = 4(15F_{2n+3} - 5F_{n+3} + 1). \end{aligned}$$

Proposition 5: A Lucas-Leonardo quaternion $\mathbb{Q}\mathcal{R}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_3}$ if and only if $n \equiv 0, 2, 3 \pmod 8$.

Proof: A Lucas-Leonardo quaternion $\mathbb{Q}\mathcal{R}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_3}$ if and only if $N(\mathbb{Q}\mathcal{R}_n) = \overline{0}$ in $\mathbb{Z}_3$. By using the relation (3.1), we have

$$F_{n+3} + 1 \equiv 0 \pmod 3 \Leftrightarrow F_{n+3} \equiv 2 \pmod 3$$

Since the cycle of the Fibonacci numbers is modulo 3 is

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, \ldots, \qquad (3.2)$$

the cycle length of Fibonacci numbers modulo 3 is 8. So

$$F_{n+3} \equiv 2 \pmod 3 \Leftrightarrow n + 3 \equiv 3, 5, 6 \pmod 8 \Leftrightarrow n \equiv 0, 2, 3 \pmod 8.$$

Proposition 6: All Lucas-Leonardo quaternions $\mathbb{Q}\mathcal{R}_n$ are invertible in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_5}$.

Proof: Since $N(\mathbb{Q}\mathcal{R}_n) = \overline{4} \neq \overline{0}$ in $\mathbb{Z}_5$, all Lucas-Leonardo quaternions $\mathbb{Q}\mathcal{R}_n$ are invertible in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_5}$.

From previous Proposition, there are no Lucas-Leonardo quaternions which are zero divisors in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_5}$.

Proposition 7: A Lucas-Leonardo quaternion $\mathbb{Q}\mathcal{R}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_7}$ if and only if $n \equiv 0, 6, 7, 9 \pmod{16}$.

Proof: A Lucas-Leonardo quaternion $\mathbb{Q}\mathcal{R}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_7}$ if and only if $N(\mathbb{Q}\mathcal{R}_n)=\overline{0}$ in $\mathbb{Z}_7$. By using the relation (3.1), we have

$$\begin{aligned}N(\mathbb{Q}\mathcal{R}_n)=\overline{0} &\Leftrightarrow 4(15F_{2n+3}-5F_{n+3}+1)\equiv 0 \pmod 7)\\ &\Leftrightarrow 4(F_{2n+3}+2F_{n+3}+1)\equiv 0 \pmod 7)\\ &\Leftrightarrow F_{2n+3}+2F_{n+3}+1\equiv 0 \pmod 7)\\ &\Leftrightarrow F_{n+1}^2+F_{n+2}^2+2(F_{n+1}+F_{n+2})+1\equiv 0 \pmod 7)\\ &\Leftrightarrow (F_{n+1}+1)^2+(F_{n+2}+1)^2\equiv 1 \pmod 7).\end{aligned}$$

Since the cycle of Fibonacci numbers modulo 7 is

$$0,1,1,2,3,5,1,6,0,6,6,5,4,2,6,1,\ldots,$$

the cycle length of Fibonacci numbers modulo 7 is 16. See [19, A105870]. So, we get

$$(F_{n+1}+1)^2+(F_{n+2}+1)^2\equiv 1 \pmod 7) \Leftrightarrow n\equiv 0,6,7,9 \pmod{16}).$$

Next, we consider the Francois quaternions over certain finite fields. Let $\mathbb{Q}\mathcal{F}_n$ be the nth Francois quaternion defined as

$$\mathbb{Q}\mathcal{F}_n=\mathcal{F}_n+\mathcal{F}_{n+1}i+\mathcal{F}_{n+2}j+\mathcal{F}_{n+3}k$$

where the basis $\{1,i,j,k\}$ satisfies the real quaternion multiplication rules. By using the relation (1.9) for the Francois quaternion, the norm of Francois quaternion can be obtained as follows:

$$\begin{aligned}N(\mathbb{Q}\mathcal{F}_n)&=\mathcal{F}_n^2+\mathcal{F}_{n+1}^2+\mathcal{F}_{n+2}^2+\mathcal{F}_{n+3}^2\\ &=(L_n+F_{n+1}-1)^2+(L_{n+1}+F_{n+2}-1)^2+(L_{n+2}+F_{n+3}-1)^2+(L_{n+3}+F_{n+4}-1)^2\\ &=(L_n^2+L_{n+1}^2+L_{n+2}^2+L_{n+3}^2)+(F_{n+1}^2+F_{n+2}^2+F_{n+3}^2+F_{n+4}^2)+4\\ &\quad+2(L_nF_{n+1}+L_{n+1}F_{n+2}+L_{n+2}F_{n+3}+L_{n+3}F_{n+4})-2(L_n+L_{n+1}+L_{n+2}+L_{n+3})\\ &\quad-2(F_{n+1}+F_{n+2}+F_{n+3}+F_{n+4})\\ &=5F_{2n+1}+5F_{2n+5}+3F_{2n+5}+4\\ &\quad+2(F_{2n+1}+(-1)^n+F_{2n+3}+(-1)^{n+1}+F_{2n+5}+(-1)^{n+2}+F_{2n+7}+(-1)^{n+3})\\ &\quad-2(L_{n+2}+L_{n+4})-2(F_{n+3}+F_{n+5})\\ &=5F_{2n+1}+5F_{2n+5}+3F_{2n+5}+4+2(F_{2n+1}+F_{2n+3}+F_{2n+5}+F_{2n+7})\\ &\quad-2(5F_{n+3})-2(F_{n+3}+F_{n+5})\\ &=7F_{2n+1}+10F_{2n+5}+4+2(F_{2n+1}+F_{2n+2})+2(F_{2n+5}+F_{2n+6})-12F_{n+3}-2F_{n+5}\end{aligned}$$

$$= 9F_{2n+1} + 12F_{2n+5} + 4 + 2F_{2n+2} + 2(F_{2n+4} + F_{2n+5}) - 34F_{n+1} - 18F_n$$
$$= 39F_{2n+1} + 48F_{2n+2} - 34F_{n+1} - 18F_n + 4.$$

Proposition 8: A Francois quaternion $Q\mathcal{F}_n$ is a zero divisor in quaternion algebra $Q_{\mathbb{Z}_3}$ if and only if $n \equiv 0,1,6 \pmod 8$.

Proof: A Francois quaternion $Q\mathcal{F}_n$ is a zero divisor in quaternion algebra $Q_{\mathbb{Z}_3}$ if and only if $N(Q\mathcal{F}_n) = \overline{0}$ in $\mathbb{Z}_3$. From (3.2), the cycle length of Fibonacci numbers modulo 3 is 8. Thus, we have

$$N(\mathbb{Q}\mathcal{F}_n) = \overline{0} \pmod 3 \Leftrightarrow -F_{n+1} + 1 \equiv 0 \pmod 3 \Leftrightarrow F_{n+1} \equiv 1 \pmod 3)$$
$$\Leftrightarrow n+1 \equiv 1,2,7 \pmod 8 \Leftrightarrow n \equiv 0,1,6 \pmod 8.$$

Proposition 9: A Francois quaternion $\mathbb{Q}\mathcal{F}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_5}$ if and only if $n \equiv 5,8,10,19 \pmod{20}$.

Proof: A Francois quaternion $\mathbb{Q}\mathcal{F}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_5}$ if and only if $N(\mathbb{Q}\mathcal{F}_n) = \overline{0}$ in $\mathbb{Z}_5$. By using the norm relation, we have

$$\begin{aligned} N(\mathbb{Q}\mathcal{F}_n) = \overline{0} &\Leftrightarrow 4F_{2n+1} + 3F_{2n+2} + F_{n+1} + 2F_n + 4 \equiv 0 \pmod 5 \\ &\Leftrightarrow -F_{2n+1} - 2F_{2n+2} + F_{n+2} + F_n + 4 \equiv 0 \pmod 5 \\ &\Leftrightarrow -F_{2n+3} - F_{2n+2} + F_{n+2} + F_n + 4 \equiv 0 \pmod 5 \qquad (3.3) \\ &\Leftrightarrow F_{2n+3} + F_{2n+2} - F_{n+2} - F_n + 1 \equiv 0 \pmod 5 \\ &\Leftrightarrow F_{2n+4} - F_{n+2} - F_n + 1 \equiv 0 \pmod 5. \end{aligned}$$

We recall that the cycle of Fibonacci numbers modulo 5 is

$$0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1.$$

So the cycle length of Fibonacci numbers modulo 5 is 20. See [19, A082116]. To find n such that the congruence (3.3) is satisfied, we need to consider the following cases:

Case 1: If $n \equiv 0 \pmod{20}$, then $n+2 \equiv 2 \pmod{20}$, $2n+4 \equiv 4 \pmod{20}$, $F_{2n+4} \equiv 3 \pmod 5$, $F_{n+2} \equiv 1 \pmod 5$, $F_n \equiv 0 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 3 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 2: If $n \equiv 1 \pmod{20}$, then $n+2 \equiv 3 \pmod{20}$, $2n+4 \equiv 6 \pmod{20}$, $F_{2n+4} \equiv 3 \pmod 5$, $F_{n+2} \equiv 2 \pmod 5$, $F_n \equiv 1 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 1 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 3: If $n \equiv 2 \pmod{20}$, then $n+2 \equiv 4 \pmod{20}$, $2n+4 \equiv 8 \pmod{20}$, $F_{2n+4} \equiv 1 \pmod 5$, $F_{n+2} \equiv 3 \pmod 5$, $F_n \equiv 1 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 3 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 4: If $n \equiv 3 \pmod{20}$, then $n+2 \equiv 5 \pmod{20}$, $2n+4 \equiv 10 \pmod{20}$, $F_{2n+4} \equiv 0 \pmod 5$, $F_{n+2} \equiv 0 \pmod 5$, $F_n \equiv 2 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 4 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 5: If $n \equiv 4 \pmod{20}$, then $n+2 \equiv 6 \pmod{20}$, $2n+4 \equiv 12 \pmod{20}$, $F_{2n+4} \equiv 4 \pmod 5$, $F_{n+2} \equiv 3 \pmod 5$, $F_n \equiv 3 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 4 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 6: If $n \equiv 5 \pmod{20}$, then $n+2 \equiv 7 \pmod{20}$, $2n+4 \equiv 14 \pmod{20}$, $F_{2n+4} \equiv 2 \pmod 5$, $F_{n+2} \equiv 3 \pmod 5$, $F_n \equiv 0 \pmod 5$. So, the congruence (3.3) is satisfied. Therefore, we have $N(\mathbb{Q}\mathcal{F}_n) = \overline{0} \Leftrightarrow n \equiv 5 \pmod{20}$.

Case 7: If $n \equiv 6 \pmod{20}$, then $n+2 \equiv 8 \pmod{20}$, $2n+4 \equiv 16 \pmod{20}$, $F_{2n+4} \equiv 2 \pmod 5$, $F_{n+2} \equiv 1 \pmod 5$, $F_n \equiv 3 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 4 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 8: If $n \equiv 7 \pmod{20}$, then $n+2 \equiv 9 \pmod{20}$, $2n+4 \equiv 18 \pmod{20}$, $F_{2n+4} \equiv 4 \pmod 5$, $F_{n+2} \equiv 4 \pmod 5$, $F_n \equiv 3 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 3 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 9: If $n \equiv 8 \pmod{20}$, then $n+2 \equiv 10 \pmod{20}$, $2n+4 \equiv 0 \pmod{20}$, $F_{2n+4} \equiv 0 \pmod 5$, $F_{n+2} \equiv 0 \pmod 5$, $F_n \equiv 1 \pmod 5$. So, the congruence (3.3) is satisfied. Therefore, we have $N(\mathbb{Q}\mathcal{F}_n) = \overline{0} \Leftrightarrow n \equiv 8 \pmod{20}$.

Case 10: If $n \equiv 9 \pmod{20}$, then $n+2 \equiv 11 \pmod{20}$, $2n+4 \equiv 2 \pmod{20}$, $F_{2n+4} \equiv 1 \pmod 5$, $F_{n+2} \equiv 4 \pmod 5$, $F_n \equiv 4 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 4 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 11: If $n \equiv 10 \pmod{20}$, then $n+2 \equiv 12 \pmod{20}$, $2n+4 \equiv 4 \pmod{20}$, $F_{2n+4} \equiv 3 \pmod 5$, $F_{n+2} \equiv 4 \pmod 5$, $F_n \equiv 0 \pmod 5$. So, the congruence (3.3) is satisfied. Therefore, we have $N(Q\mathcal{F}_n) = \overline{0} \Leftrightarrow n \equiv 10 \pmod{20}$.

Case 12: If $n \equiv 11 \pmod{20}$, then $n+2 \equiv 13 \pmod{20}$, $2n+4 \equiv 6 \pmod{20}$, $F_{2n+4} \equiv 3 \pmod 5$, $F_{n+2} \equiv 3 \pmod 5$, $F_n \equiv 4 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 2 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 13: If $n \equiv 12 \pmod{20}$, then $n+2 \equiv 14 \pmod{20}$, $2n+4 \equiv 8 \pmod{20}$, $F_{2n+4} \equiv 1 \pmod 5$, $F_{n+2} \equiv 2 \pmod 5$, $F_n \equiv 4 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 1 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 14: If $n \equiv 13 \pmod{20}$, then $n+2 \equiv 15 \pmod{20}$, $2n+4 \equiv 10 \pmod{20}$, $F_{2n+4} \equiv 0 \pmod 5$, $F_{n+2} \equiv 0 \pmod 5$, $F_n \equiv 3 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 3 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 15: If $n \equiv 14 \pmod{20}$, then $n+2 \equiv 16 \pmod{20}$, $2n+4 \equiv 12 \pmod{20}$, $F_{2n+4} \equiv 4 \pmod 5$, $F_{n+2} \equiv 2 \pmod 5$, $F_n \equiv 2 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 1 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 16: If $n \equiv 15 \pmod{20}$, then $n+2 \equiv 17 \pmod{20}$, $2n+4 \equiv 14 \pmod{20}$, $F_{2n+4} \equiv 2 \pmod 5$, $F_{n+2} \equiv 2 \pmod 5$, $F_n \equiv 0 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 1 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 17: If $n \equiv 16 \pmod{20}$, then $n+2 \equiv 18 \pmod{20}$, $2n+4 \equiv 16 \pmod{20}$, $F_{2n+4} \equiv 2 \pmod 5$, $F_{n+2} \equiv 4 \pmod 5$, $F_n \equiv 2 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 2 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 18: If $n \equiv 17 \pmod{20}$, then $n+2 \equiv 19 \pmod{20}$, $2n+4 \equiv 18 \pmod{20}$, $F_{2n+4} \equiv 4 \pmod 5$, $F_{n+2} \equiv 1 \pmod 5$, $F_n \equiv 2 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 2 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 19: If $n \equiv 18 \pmod{20}$, then $n+2 \equiv 0 \pmod{20}$, $2n+4 \equiv 0 \pmod{20}$, $F_{2n+4} \equiv 0 \pmod 5$, $F_{n+2} \equiv 0 \pmod 5$, $F_n \equiv 4 \pmod 5$. It results $F_{2n+4} - F_{n+2} - F_n + 1 \equiv 2 \pmod 5$, therefore the congruence (3.3) is not satisfied.

Case 20: If $n \equiv 19 \pmod{20}$, then $n+2 \equiv 1 \pmod{20}$, $2n+4 \equiv 2 \pmod{20}$, $F_{2n+4} \equiv 1 \pmod 5$, $F_{n+2} \equiv 1 \pmod 5$, $F_n \equiv 1 \pmod 5$. So, the congruence (3.3) is satisfied. Therefore, we have $N(\mathbb{Q}\mathcal{F}_n) = \overline{0} \Leftrightarrow n \equiv 19 \pmod{20}$.

Thus we obtain the desired result.

4. Conclusion

Quaternions find broad applications across numerous fields, notably in physics and mathematics. Therefore, it is valuable to explore and analyze the properties of specific types of quaternions. Numerous studies have focused on quaternions whose components originate from special integer sequences, including Fibonacci, Lucas, Leonardo sequences, among others. Here, we introduce a new class of quaternions which can be seen as a companion quaternion sequence of Leonardo p-quaternions. We investigate basic properties of these quaternions and give some applications over finite fields. The results of this paper can be summarized as follows:

- We derive a new class of quaternions which takes Lucas-Leonardo p-numbers as a components. When $p = 1$, the Lucas-Leonardo p-quaternions reduce to the Lucas-Leonardo quaternions. We obtain generating function, recurrence relations and sum identities for these new quaternion class.
- We obtain that
  - a Lucas-Leonardo quaternion $\mathbb{Q}\mathcal{R}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_3}$ if and only if $n \equiv 0, 2, 3 \pmod 8$,
  - all Lucas-Leonardo quaternions are invertible in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_5}$,
  - a Lucas-Leonardo quaternion $\mathbb{Q}\mathcal{R}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_7}$ if and only if $n \equiv 0, 6, 7, 9 \pmod{16}$.
- We obtain that
  - A Francois quaternion $Q\mathcal{F}_n$ is a zero divisor in quaternion algebra $Q_{\mathbb{Z}_3}$ if and only if $n \equiv 0, 1, 6 \pmod 8$,
  - A Francois quaternion $\mathbb{Q}\mathcal{F}_n$ is a zero divisor in quaternion algebra $\mathbb{Q}_{\mathbb{Z}_5}$ if and only if $n \equiv 5, 8, 10, 19 \pmod{20}$.

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References

[1] S. Abbad, H. Belbachir, and B. Benzaghou, "Companion sequences associated to the r-Fibonacci sequence: algebraic and combinatorial properties," *Turkish J. Math.*, vol. 43, no. 3, pp. 1095–1114 (2019).

[2] Y. Alp and E. G. Kocer, "Hybrid Leonardo numbers," Chaos Solitons Fractals, vol. 150, p. 111128 (2021).

[3] P. Catarino and A. Borges, "On Leonardo numbers," Acta Math. Univ. Comenianae, vol. 89, no. 1, pp. 75–86 (2019).

[4] A. Dagdeviren, F. Kürüz, and P. Catarino, "On Leonardo Pisano dual quaternions," *J. Inf. Optim. Sci.*, vol. 45, no. 1, pp. 131–143 (2024).

[5] E. W. Dijkstra, "Archive: Fibonacci numbers and Leonardo numbers," EWD 797, [Online]. Available: www.cs.utexas.edu. Jul. (1981).

[6] O. Diskaya, H. Menken, and P. Catarino, "On the hyperbolic Leonardo and hyperbolic François quaternions," *J. New Theory*, vol. 42, pp. 74–85 (2023).

[7] C. Flaut and D. Savin, "Some special number sequences obtained from a difference equation of degree three," Chaos Solitons Fractals, vol. 106, pp. 67–71 (2018).

[8] C. Flaut and D. Savin, "Some examples of division symbol algebras of degree 3 and 5," *Carpathian J. Math.*, vol. 31, no. 2, pp. 197–204 (2015).

[9] J. M. Grau, C. Miguel, and A. M. Oller-Marcen, "On the structure of quaternion rings over Z/nZ," *Adv. Appl. Clifford Algebras*, vol. 25, no. 4, pp. 875–887 (2015).

[10] A. F. Horadam, "Complex Fibonacci numbers and Fibonacci quaternions," *Am. Math. Mon.*, vol. 70, pp. 289–291 (1963).

[11] S. Halici and A. Karatas, "On a generalization for quaternion sequences," *Chaos Solitons Fractals*, vol. 98, pp. 178–182 (2017).

[12] E. G. Kocer, N. Tuglu, and A. Stakhov, "On the m-extension of the Fibonacci and Lucas p-numbers," *Chaos Solitons Fractals*, vol. 40, pp. 1890–1906 (2009).

[13] K. Kuhapatanakul and J. Chobsorn, "On the generalized Leonardo numbers," Integers, vol. 22, #A48 (2022).

[14] M. C. S. Mangueira, F. R. V. Alves, and P. M. M. C. Catarino, "Hybrid quaternions of Leonardo," *Trends Comput. Appl. Math.*, vol. 23, no. 1, pp. 51–62 (2022).

[15] C. J. Miguel and R. Serodio, "On the structure of quaternion rings over Zp," *Int. J. Algebra*, vol. 5, no. 27, pp. 1313–1325 (2011).

[16] D. Savin, "About special elements in quaternion algebras over finite fields," *Adv. Appl. Clifford Algebras*, vol. 27, no. 2, pp. 1801–1813 (2017).

[17] D. Savin, "About division quaternion algebras and division symbol algebras," *Carpathian J. Math.*, vol. 32, no. 2, pp. 233–240 (2016).

[18] A. G. Shannon, "A note on generalized Leonardo numbers," *Notes Number Theory Discrete Math.*, vol. 25, no. 3, pp. 97–101 (2019).

[19] N. J. A. Sloane, The On-line Encyclopedia of Integer Sequences, The OEIS Foundation Inc., [Online]. Available: http://oeis.org.

[20] A. Stakhov and B. Rozin, "Theory of Binet formulas for Fibonacci and Lucas p-numbers," *Chaos Solitons Fractals*, vol. 27, no. 5, pp. 1162–1177 (2006).

[21] E. Tan and H. H. Leung, "Some results on Horadam quaternions," Chaos Solitons Fractals, vol. 138 (2020).

[22] E. Tan and H. H. Leung, "On Leonardo p-numbers," Integers, vol. 23, #A7 (2023).

[23] E. Tan, D. Savin, and S. Yılmaz, "A new class of Leonardo hybrid numbers and some remarks on Leonardo quaternions over finite fields," *Mathematics*, vol. 11, p. 4701 (2023).

[24] N. Tuglu, E. Gokcen Kocer, and A. Stakhov, "Bivariate Fibonacci-like p-polynomials," *Appl. Math. Comput.*, vol. 217, pp. 10239–10246 (2011).

[25] J. Zhong, J. Yao, and C.-L. Chung, "A note on incomplete Fibonacci-Lucas relations," *Symmetry*, vol. 15, p. 2113 (2023).

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