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Modified Barzilai-Borwein (MBB) gradient method for solving fuzzy nonlinear equations

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Abstract

Research has demonstrated that conventional analytical techniques such as the SD, BB, CG, and N approaches are insufficient to solve a collection of fuzzy non-linear equations that involve fuzzy number coefficients[1]. Because of this, the two-step extended gradient method is suggested as an alternative way to resolve these kinds of unclear nonlinear equations. This work uses a six-step technique to solve ambiguous nonlinear equations[7]. This method does not require the Jacobi method and only requires finding a line for t = 0. In order to provide clarification and enhance understanding, two well-known numerical examples are provided to demonstrate the suggested concept, along with their practical results implemented in MATLAB in tables and curves.

Subject Classification (2010): *90C26, 46N10, 90Cxx.*

Keywords: *Gradient methods, Barzilai-Borwein (BB) method fuzzy numbers, Fuzzy non-linear equation.*

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1. Introduction and literature review

Throughout history, nonlinear equations have significantly influenced several fields such as engineering, mathematics, medicine, and robotics. Nonlinear problem applications are found in various fields, including weather forecasting, which is characterized by its intrinsic nonlinear feature. An additional intriguing non-linear problem arises in phenomena like chaos, which stems from the challenges associated with solving non-linear equations. Let us consider the set of **n** non-linear equations.

$$f_k(x_1, x_2, \ldots\ldots, x_n) = 0, \qquad k = 1,2,3, \ldots, n \tag{1.1}$$

Considering the following non-linear equation

$$F(x) = 0 \tag{1.1}$$

$\boldsymbol{F}$ is a function that maps from $\boldsymbol{R^n}$ to $\boldsymbol{R^n}$. The aforementioned equation frequently arises in the fields of natural science, engineering, and social science. Various techniques have been suggested and utilized to acquire its resolution. However, in most instances, the parameters of (1.1) are typically represented as Instead of clear numbers, fuzzy numbers. Therefore, the essential reason for the fuzzy (ambiguous) equation must be determined to solve it. The integration underlying fuzzy set theory as initially introduced through a fundamental definition.

Nevertheless, several conventional established analytical methods for solving fuzzy equations were suggested by [13][18][20]. Investigation revealed that these strategies have the capability to only solve imprecise linear and quadratic problems. Therefore, for imprecise non-linear equations such as:

$$\begin{gathered} a\,x^3 + b\,x^2 + c\,x - d = 0 \\ i\,x^2 + j\,cosx = e \end{gathered} \tag{1.2}$$

In the case where (**a, b, c, d, e**) and **x** are all fuzzy values, It's necessary to create numerical techniques for finding the solution to this equation. The authors of references [9] and [10] examined using the steepest descent (SD) approach to solve fuzzy non-linear equations. This strategy is straightforward and involves less computer tasks in acquiring the solution to any given problem. Conversely, this approach travels in perpendicular increments and therefore, exhibits subpar performance [21]. Additionally, it is significantly impacted by poor conditioning and converges in a linear manner towards the solution [14]. The Newton approach, also known as the steepest descent (SD) method under the norm of form $\|\,.\,\|_{G_k}$ [11]: was used by [18] to solve a fuzzy non-linear problem.This approach exhibits rapid convergence when provided with a well-informed beginning estimate. The primary limitation of Newton's approach is the need to compute and store the Jacobian matrix in every iteration. A novel bracketing technique called the Regula Falsi approach was recently developed to solve fuzzy non-linear equations [8]. This strategy is computationally cost-effective and guaranteed to converge as it effectively encloses the root of any given issue properly. Nevertheless, Since gradient information is missing, the linear convergence rate moves slowly toward the solution point. To obtain additional information regarding numerical techniques for solving fuzzy nonlinear equations, please consult references [2],[3][9]. This work proposes the utilization of the improved Barzilai-Borwein gradient method, a type of gradient-based approach, for the resolution of fuzzy nonlinear equations. This approach obtains its step-size by using a two-point approximation to the secant equation that forms the basis for the quasi-Newton method [2]. The modified Barzilai-Borwein approach necessitates fewer computations as it only requires line searches for $k = 0$. Therefore, it accelerates the convergence of the gradient approach. To access the latest discoveries and other sources regarding the Barzilai and Borwein approach, please consult reference.

Section 2 provides a concise introduction and precise definitions of the fuzzy nonlinear equations. The **section 3** of this paper introduces the improved Barzilai-Borwein gradient method, This used for the solution of fuzzy non-linear equations. **Section 4** presents numerical results for several examples, while the final section contains the conclusions

2. Basic Fuzzy Definitions

We will provide an overview of fundamental concepts and principles of Fuzzy logic in a concise manner.

Definition 1: [7], [11] "A fuzzy number is a set like $u: R \rightarrow I = [0,1]$ which satisfy the following

I. u is upper semi-continous,

II. $u(x) = 0$ outside some interval [c,d],

III. There are real number a,b such that $c \leq a \leq b \leq d$ and

- $u(x)$ is monotonic increasing on [c,a].
- $u(x)$ is monotonic decreasing on [b,d].
- $u(x) = 1, a \leq x \leq b$.

The set all these fuzzy numbers is denoted by E. An equivalent parametric form is also given in" [10].

Definition 2: [6], [15]. "Fuzzy number u in parametric form is a pair $(\underline{u}, \bar{u})$ of function $\underline{u}(x), \bar{u}(x), \leq \alpha \leq 1$, which satisfies the following requirement:

i. $\underline{u}(x)$ is a bounded monotonic increasing left continuous function,

ii. $\bar{u}(x)$ is a bounded monotonic decreasing left continuous func-tion,

iii. function $\underline{u}(x) \leq \bar{u}(x), \leq \alpha \leq 1$.

A popular fuzzy number is the Trapezoidal fuzzy number $u(x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness σ and right fuzziness β where the membership function is"

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma), & x_0 - \sigma \leq x \leq x_0, x \in [x_0, y_0] \\ \frac{1}{\beta}(y_0 - x + \sigma), & y_0 \leq x \leq y_0 + \beta \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Its parametric form is

$$\underline{u}(\tau)= x_0 - \sigma + \sigma\,\tau, \quad \bar{u}(\tau) = y_0 - \beta + \beta\,\tau$$

TF(R) represents the collection of all fuzzy trapezoidal numbers. The extension principle defines the scalar multiplication and addition operations for fuzzy numbers., and it can be expressed similarly as given in reference [19].

For arbitrary $u =(\underline{u},\bar{u})$, $v = (\underline{v},\bar{v})$, and $k > 0$, the addition $(u + v)$ and multiplication by k scalar are defined as

$$\underline{(u + v)}(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha), \quad \overline{(u + v)}(\alpha) = \bar{u}(\alpha) + \bar{v}(\alpha)$$

$$\underline{ku}(\alpha) = k\underline{u}(\alpha) \qquad \overline{ku}(\alpha) = k\bar{u}(\alpha)$$

3. The Modified BB -GM for solving fuzzy non-linear equation

Applying the objective function to the Taylor formula $f(x)$, we have

$$f(x) \cong f(x_k) + g_k^T(x - x_k) + \frac{1}{2}(x - x_k)^T\nabla^2 f_k(x - x_k), \tag{3.1}$$

Where f_k denotes the function value, g and G are gradient and Hessian respectively at x_k.

Hence substituting $x = x_{k+1}$, We have acquired that

$$f_k \cong f_{k+1} + g_{k+1}^T s_k + \frac{1}{2}{s_k}^T\nabla^2 f_k s_k, \tag{3.2}$$

Therefore

$$\frac{1}{2}{s_k}^T\nabla^2 f_k s_k \cong 2(f_k - f_{k+1}) + 2g_{k+1}^T s_k$$
$$= 2(f_k - f_{k+1}) + (g_k - g_{k+1})^T s_k + y_k^T s_k$$

Considering B_k as a new approximation of $\nabla^2 f_k$ such that

$${s_k}^T B_{k+1}\, s_k = y_k^T s_k + \lambda_k \tag{3.3}$$

Where λ_k is defined by

$$\lambda_k = 2(f_k - f_{k+1}) + (g_k - g_{k+1})^T s_k \tag{3.4}$$

The following new quasi-Newton equation is implied:

$$B_{k+1}\, s_k = z_k, \qquad z_k = y_k + \frac{\lambda_k}{\|s_k\|^2}\, s_k \tag{3.5}$$

Noting that this novel quasi-Newton equation incorporates both the gradient value and function value information from the current and previous steps, it can be argued that the resulting methods will significantly surpass the original approach[5].

Now, for fuzzy nonlinear equation $F(x) = 0$, the parametric form is defined as

$$\begin{cases} \underline{F}(\underline{x},\bar{x},\tau) = \underline{c}(\tau), \\ \bar{F}(\underline{x},\bar{x},\tau) = \bar{c}(\tau), \end{cases} \forall\ \tau \in [0,1] \tag{3.6}$$

The objective of this study is to transform the parametric representation of a fuzzy nonlinear equation into an unconstrained optimization problem, followed by the use of the novel conjugate gradient approach to obtain the optimal solution[18]. By using the approved version of the Barzilai-Borwein gradient method, the goal is to obtain the solution to the parameterized problem discussed above. Let us now establish the definition of a function $G_\tau: R^2 \to R$ in the following:

$$G_\tau(\underline{x}, \bar{x}, \tau) = \left[\underline{F}(\underline{x}, \bar{x}, \tau) + \bar{F}(\underline{x}, \bar{x}, \tau)\right]^2, \quad \forall\ \tau \in [0,1] \tag{3.7}$$

whose gradient $\nabla G_\tau(x)$ at point $x(\tau) = (\underline{x}(\tau), \bar{x}(\tau))$ is given as

$$\nabla G_\tau(x) = \left(\frac{\partial G_\tau}{\partial \underline{x}}, \frac{\partial G_\tau}{\partial \bar{x}}\right) \tag{3.8}$$

We establish a suitable approach for determining the step size between two points:

$$x_{k+1}(\tau) = (\underline{x}_{k+1}(\tau), \bar{x}_{k+1}(\tau)) \text{ as}$$

$$x_{k+1}(\tau) = x_k(\tau) - \alpha_k \nabla G_\tau(x_k) \tag{3.9}$$

Suppose $D_k = \alpha_k I$, where I is an identity matrix, the in (2) is rewritten as

$$x_{k+1}(\tau) = x_k(\tau) - D_k \nabla G_\tau(x_k)$$

Considering that the two-point approximation determines the step-size, we calculate α_k such that:

$$min\|s_k - D_{k+1} z_k\| \tag{3.10}$$

Based on (3.10), it is evident that D_k will exhibits the quasi-Newton characteristic. Therefore,

$$\alpha_k = \frac{s_k^T z_k}{\|z_k\|^2} \tag{3.11}$$

Where $s_k(\tau) = x_{k+1}(\tau) - x_k(\tau)$ and $s_k(\tau) = g_{k+1}(\tau) - g_k(\tau)$ for $g_k(\tau) = \nabla G_\tau(x_k)$, $k = 0,1,2,3, \ldots$. However, by symmetry in (3) can be written as

$$min\|D_{k+1}^{-1} s_k - z_k\|$$

Hence, we get

$$\alpha_k = \frac{\|s_k\|^2}{s_k^T z_k} \tag{3.12}$$

Upon solving the fuzzy nonlinear equations, it is found that the obtained results satisfy the inequality $\underline{x}(0) \le \bar{x}(1) \le \bar{x}(0)$, one can choose the following fuzzy number

$$x_0 = \left(\underline{x}(0), \bar{x}(1), \bar{x}(0)\right) \tag{3.13}$$

As the first estimate [2], [16], [17], and its parametric form is provided as

$$\underline{x}(\tau) = \underline{x}(0) + \left(\underline{x}(1) - \underline{x}(0)\right)\tau \tag{3.14}$$

And

$$\bar{x}(\tau) = \bar{x}(0) + (\bar{x}(1) - \bar{x}(0))\tau,$$

The explanation above leads to the expanded BB algorithm.

Algorithm 1. Modified BB algorithm for fuzzy nonlinear equation:

Step 1. Transform the given fuzzy nonlinear equation to its parametric form and solve for $\tau = 0$ and $\tau = 1$ to obtain the Initial guess x_0.

Step 2. Evaluate G_r at $\left(\underline{x}(0), \overline{x}(0)\right)$ and compute the gradient $\nabla G_\tau(x_0)$.

Step 3. If $\|\nabla G_\tau(x_0)\| \leq 0$**, stop. Else, let** $d_0 = -\nabla G_\tau(x_0)$.

Step 4. If k equals zero, determine the value of α_k using the line search process. Otherwise, if k is greater than zero, calculate α_k using equation (3.11) or (3.12).

Step 5. Revise the current value $x_{k+1}(\tau) = x_k(\tau) + \alpha_k d_k$

Step 6. Repeat steps 1 through 5 using$k = k + 1$ until tolerance is satisfied

Lemma 3.1: *Assuming level set* Ł $= \{x: f(x) < f(x_0)\}$ *is bounded with f and g are Lipschitz then*

$$\|z_k\| \leq M\|s_k\| \tag{3.15}$$

The newly chosen step size possesses a desirable attribute whereby, for every value α_k, as indicated by equations (3.12) and (3.14) that α_k is greater than zero.

Lemma 3.2: *The computing step-size α_k is bounded and positive at all k.*

We have that

$$s_k^T z_k \leq L s_k^T s_k \tag{3.16}$$

This implies $\alpha_k \geq L^{-1}$, for some positive L.

Theorem 3.3: *Suppose that x_0 is a starting point for which the level set* Ł $= \{x: f(x) < f(x_0)\}$ *is bounded and its gradient ∇f_k is Lipschitz continuous in some neighborhood* Ɲ *of* Ł. *Consider it iterates $\{x_k\}$ generated by the algorithm of Modified BB the either $\nabla f_k = 0$ or for some finite k $lim_{k\to\infty} \nabla f_k = 0$ [4].*

4. Numerical experiment

This section demonstrates the numerical resolution of many instances to exemplify the enhanced BB gradient method. The calculations are performed by a 'MATLAB' programmer using a computer with double precision.

First Counter Example:[21] Considering fuzzy nonlinear equation

$$(3,4,5)x^2 + (1,2,3)x - (1,2,3) = 0$$

Without loss of generality, let's purpose x is positive, we give the parametric form of the above equation as

$$(3 + \tau)\underline{x}^2(\tau) + (1 + \tau)\,\underline{x}(\tau) - (1 + \tau) = 0,$$

$$(\tau - 5)\overline{x}^2(\tau) + (3 - \tau)\,\overline{x}(\tau) - (3 - \tau) = 0.$$

Subsequently, we get the initial approximation by solving the aforementioned parametric equation for $\tau = 0$ and $\tau = 1$.

At $\tau = 0$, we have

$$(3)\underline{x}^2(0) + \underline{x}(0) - 1 = 0,$$
$$(5)\overline{x}^2(0) + (3)\,\overline{x}(0) - 3 = 0.$$

For $\tau = 1$, we have

$$(4)\underline{x}^2(1) + (2)\,\underline{x}(1) - 2 = 0,$$
$$(4)\overline{x}^2(\tau) + (2)\,\overline{x}(1) - 2 = 0.$$

After computation, we have $\underline{x}(0) = 0.434, \overline{x}(0) = 0{,}681$ and $\underline{x}(1) = \overline{x}(1) = 0{,}5$. From (11) we have $x_0 = (0{,}434, 0.5, 0.681)$. The answer was achieved after 6 iterations, with a tolerance of 10^{-5}. For detailed information on the solution, please consult Figure 1.

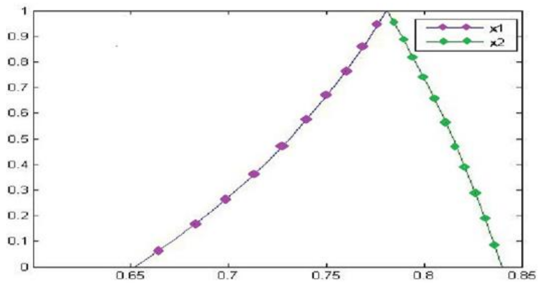

Figure 1

The solution of problem 1 using modified BB.

Second counter Example: considering

$$(4{,}6{,}8)x^2 + (2{,}3{,}4)x - (8{,}12{,}16) = (5{,}6{,}7)$$

Let us assume that x is positive, the parametric form of this equation is as follows, maintaining generality. [3], [12]:

$$(4 + 2\,\tau)\underline{x}^2(\,\tau) + (2 +\,\tau)\,\underline{x}(\,\tau) - (8 + 4\,\tau) = (5 +\,\tau),$$
$$(8 - 2\,\tau)\overline{x}^2(\,\tau) + (4 -\,\tau)\,\overline{x}(\,\tau) - (16 - 4\,\tau) = (7 -\,\tau).$$

Or equality of

$$(4 + 2\,\tau)\underline{x}^2(\,\tau) + (2 +\,\tau)\,\underline{x}(\,\tau) - (3 + 3\,\tau) = 0,$$
$$(8 - 2\,\tau)\overline{x}^2(\,\tau) + (4 -\,\tau)\,\overline{x}(\,\tau) - (9 - 3\,\tau) = 0.$$

To obtain an initial guess, we employ the mentioned system. For $\tau = 0$ and $\tau = 1$ to get the first guess. For $\tau = 0$, we have

$$(4)\underline{x}^2(0) + 2\underline{x}(0) - 3 = 0,$$
$$(8)\overline{x}^2(0) + (4)\,\overline{x}(0) - 9 = 0.$$

For $\tau = 1$, we have

$$(6)\underline{x}^2(1) + (3)\,\underline{x}(1) - 6 = 0,$$

$$(6)\overline{x}^2(1) + (3)\ \overline{x}(1) - 6 = 0.$$

Hence, $\underline{x}(0)$=0.65139, $\overline{x}(0) = 0.83972$, and $\underline{x}(1) = \overline{x}(1) = 0.78078$. The Fuzzy number can be used as an initial guess.

$$x_0 = \left(\underline{x}(0), \overline{x}(1), \overline{x}(0)\ \right)$$
$$= (0.65139, 0.78078, 0.83972).$$

The aforementioned equations are solved by following steps 2 to 5 iteratively until the solution is achieved with a maximum error ε < 10^{-5}. Figure 2 displays the specifics of the solution for values of τ ranging from 0 to 1.

This diagram demonstrates the rapid convergence of Modified BB approach towards a solution, The assumption is that a starting value close enough to the real answer is found. Nevertheless, It's challenging to ascertain the beginning values that will result in a solution. Inadequate starting values might not provide the intended effect . The answer is obtained using Modified BB approach after three iterations, with a maximum error of less than 10^{-5} (Table 1).

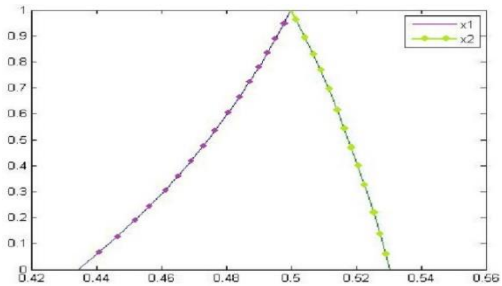

Figure 2

The solution of problem 1 using modified BB.

Table 1

Give the details for the performance of the suggested methods and other related methods.

Problem	Tolerance	BB(MBB)	BB(MBB)	BB(MBB)	BB(MBB)
		Initial Point	**Iteration**	**Optimal Point**	**f −minimum**
1	$1x10^{-3}$	0.434, 0.5, 0.5, 0.681	13(11)	(0.1996 0.9203 0.9203 0.6810)	6.5618 e-21 (9.6966e-12)
1	$1x10^{-5}$	0.434, 0.5, 0.5, 0.681	19(15)	(0.0039, 1.0000, 1.0000, 0.6810)	3.7626 e-18 (1.4907e-12)
2	$1x10^{-3}$	0.6514, 0.7808, 0.8397	13(9)	(0.0112 1.0000 1.0000 0.8397)	5.7593 e-16 (-0.3157 1.8007 0.6810)
\`2	$1x10^{-5}$	0.6514, 0.7808, 0.8397	20(14)	(0.0112 9823, 9823, 0.8397)	5.7593 e-16 (1.9511e-08)

Conclusion

This work presents a gradient-based approach modified (BB) method for solving fuzzy non-linear equations. This approach does not necessitate any matrix whatever matrix calculation, as well as the only line search required when k equals zero. Therefore, the suggested technique decreases the computing expense in every iteration. The indeterminate quantities are transformed into parametric form and subsequently resolved using the BB gradient approach. The efficacy of the proposed strategy is demonstrated through numerical results provided for several examples. Research groups in cognate disciplines can utilize the findings to enhance the application of the methodology.

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