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Geodetic sets in zero-divisor graphs

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Abstract

In this study we analyze the characteristics of finite rings by examining the structural properties of graphs derived from algebraic frameworks and specifically aims to explore the unique attributes of geodetic sets within the zero-divisor graph associated with certain classes of finite rings.

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Keywords: *Geodetic number, Geodetic sets, Zero-divisor graph, Commutative ring.*

1. Introduction

Graphs play a crucial role in representing various algebraic systems. The zero-divisor graph concept was first proposed by Beck in 1998 [7], and Anderson and Naseer [6] later developed it. Further refinements to its definition, particularly concerning zero divisors, were made by Anderson and Livingston. In a given commutative ring $\mathcal{R}$ with unity, let $\mathcal{Z}(\mathcal{R})$ represent the set of zero divisors and $\mathcal{Z}^*(\mathcal{R}) = \mathcal{Z}(\mathcal{R}) - \{0\}$ represent the set of non-zero zero divisors. The zero-divisor graph, $\Gamma(\mathcal{R})$ is a simple graph where the vertex set consist of elements in $\mathcal{Z}^*(\mathcal{R})$, and two vertices x and y are connected by an edge if and only if their product is zero (x.y = 0). Notably, when $\mathcal{R}$ is an integral domain,

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$\Gamma(\mathcal{R})$ is an empty graph, as it contains no zero divisors. Various authors have examined the characteristics of $\Gamma(\mathcal{R})$ in relation to finite rings ([1], [2], [3], [4], [8], [9], [11], [12], [13], [14], [15]). Anderson and Livingston defined $\Gamma(\mathcal{R})$ in 1999 [5]. The geodetic number is an important idea in graph theory, first introduced by Harary [10]. A geodesic of two vertices u and v is the shortest path between them which includes not just these two vertices but also any intermediate vertices along the way. The collection of all such shortest paths for a given set of vertices S forms $\mathcal{J}[\mathcal{S}] = \bigcup_{u,v\in\mathcal{S}} \mathcal{J}[u,v]$. If this set covers every vertex in G, then $\mathcal{S}$ is called a geodetic set and the cardinality of minimum geodetic set gives geodetic number of the graph G and is denoted by g(G).

2. Geodetic sets in certain classes of graphs $\Gamma(Z_n)$

In this paper, we endeavor to explore the geodetic sets of specific classes of $\Gamma(\mathcal{R})$ that stem from the commutative ring.

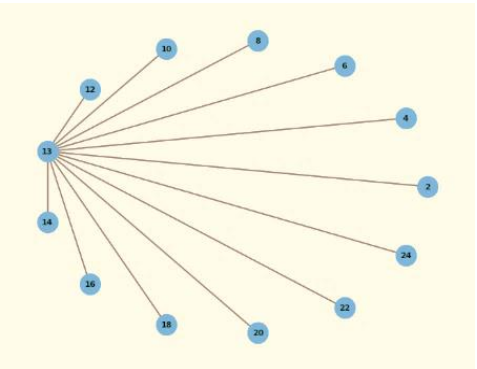
Theorem 2.1: *For the graph $\Gamma[Z_{pq}]$, where p<q and the geodetic number can be determined by g ($\Gamma[Z_{pq}]$) =* $\begin{cases} q-1, if\ p=2\ , q\ is\ an\ odd\ prime \\ p-1,\ if\ p\ and\ q\ are\ odd\ primes \end{cases}$

Proof: The graph $\Gamma[Z_{pq}]$) consists of two complete sub graphs: one containing multiples of p and another containing multiple of q. Every vertex in one sub graph connects to every vertex in the other sub graph creating a bipartite structure. The shortest path between elements in different sub graphs must pass through at least one vertex from the opposite set. Hence, the graph $\Gamma[Z_{pq}]$) = $K_{p-1,q-1}$.

Case 1: If p = 2, q > 2 , then V_1 is a geodetic set, because V_2 contains only one element, making it unable to form a geodetic set. Hence, the geodetic number being q-1.

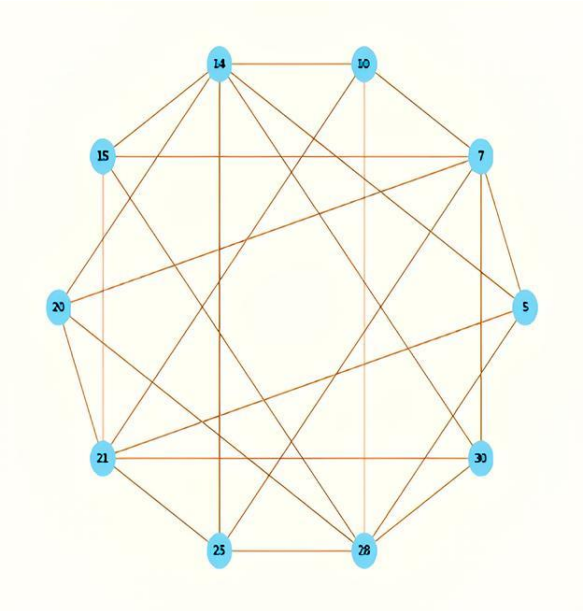
Case 2: If p = 3, $q > 3$, the vertex set V consists of {q, 2q, 3, 6, 9… 3(q-1)}. Two disjoint subsets can be separated from it as V_1 = {q, 2q} and V_2 = {3, 6, 9… 3(q-1)}. When n=3q and q≠3 is an odd prime number, $\Gamma[Z_n]$ is a bipartite graph. The geodetic sets are {q, 2q} and {3, 6, 9… 3(q-1)}; then g($\Gamma[Z_n]$) = 2, and the minimal geodetic set is {p, 2p}. Generally, the geodetic sets of $\Gamma[Z_{pq}]$ are vertex sets V_1 and V_2, and the geodetic number is p-1.

Example 2.2. Consider the graph $\Gamma[Z_{26}]$) where $Z_{26} = Z_{2\times13}$, and the graph is given below,



The only geodetic set is S = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24} and the geodetic number is g ($\Gamma[Z_{26}]$) =12.

Example 2.3: Consider the graph $\Gamma[Z_{5q}]$, where q is an odd prime and q =7. The edge set is {(5, 7), (5, 14), (5, 21), (5, 28), (10, 7), (10, 14), (10, 21), (10, 28), (15, 7), (15, 14), (15, 21), (15, 28), (20, 7), (20, 14), (20, 21), (20, 28), (25, 7), (25, 14), (25, 21), (30, 21), (30, 28)} and the graph is as follows. The geodetic sets are {7, 14, 21, 28} and {5, 10, 15, 20, 25, 30}. The geodetic number of $\Gamma[Z_{35}]$ is g($\Gamma[Z_{35}]$) = 4.



Theorem 2.4: *Given p as a prime number, the graph $\Gamma[Z_{p^2}]$ has a geodetic number of p - 1*

Proof: Consider the graph, $\Gamma[Z_{p^2}]$ and the elements in the vertex set V are of the form kp, where k ranges from 1 to p-1. If kp and mp are two zero divisors then their product is zero in Z_{p^2}.Since this holds for any two distinct zero divisors, every vertices are adjacent, forming a graph K_{p-1} which is a complete graph. In K_{p-1} every vertex must be included in the geodetic set for shortest path coverage. Hence $g(\Gamma[Z_{p^2}]) = p - 1$.

Theorem 2.5: *For any prime number p, the graph $\Gamma[Z_{p^m}]$ where m>2, the geodetic number is $g(\Gamma[Z_{p^m}]) = p^{m-1} - p$.*

Proof: **Case: m=3**.

For m=3, the graph $\Gamma[Z_{p^3}]$ consists vertex set partition sets as follows:

Set V_1: $V_1=\{k_1p: k_1=1, 2, \text{------}, p\text{-}1, \text{-----}, p^2\text{-}1;$where $p\nmid k_1\}$.

Set V_2: $V_2= \{k_2 p^2: k_2=1, 2, 3, \text{-----}p\text{-}1\}$.

Now, $|V_1| = p^2\text{-}p$, $|V_2| = p\text{-}1$

Then $\Gamma[Z_{p^3}]$ is a complete graph. Since all shortest paths pass through vertices in V_1, the minimum geodetic set must include all elements of V_1 resulting in geodetic number of $g(\Gamma[Z_{p^3}]) = p^2\text{-}p$.

Case: m=4

For m=4, the vertex set V of $\Gamma[Z_{p^4}]$ is divided into three disjoint subsets:

Set V_1: $V_1= \{k_1p: k_1=1, 2,\ldots., p\text{-}1, p+1,\ldots, p^3\text{-}1;$ where $p\nmid k_1\}$

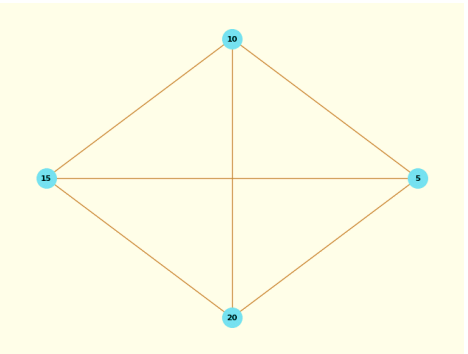
Set V_2:$V_2 =\{k_2p^2: k_2=1, 2,..., p\text{-}1, p+1,...,.., p^2\text{-}1;$ where $p\nmid k_2\}$

Set V_3: $V_3 = \{k_3p^3: k_3=1, 2, 3,\ldots\ldots, p\text{-}1\}$. Then $|V_1|= p^3\text{-}p^2$, $|V_2| = p^2\text{-}p$ and $|V_3| = p\text{-}1$. The adjacency relations are: (i) no two vertices in V_1 or V_2 are adjacent, (ii) all vertices in V_3 form a complete graph, (iii) Since all of the elements in V_1 and V_2 must be included in the smallest geodetic set, every vertex in V_1 and V_2 is adjacent to every vertex in V_3. Now, $g(\Gamma[Z_{p^4}])=| V_1 |+ |V_2 | = p^3\text{-}p^2+p^2\text{-}p = p^3\text{-}p$.

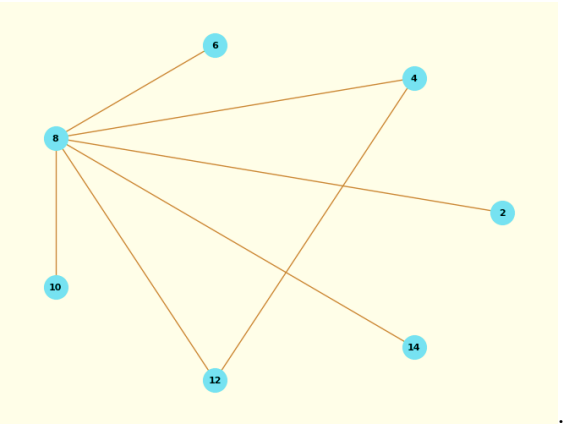
General case: m>2

For a general power m, graph $\Gamma[Z_{p^m}]$ consist of vertex set V which is divided into m-1 subsets: For $1\leq i\leq$m-1, $V_i= \{k_ip^i: k_i= 1, 2,\ldots\ldots, p^{n-i}\text{-}1$and $p\nmid k_i\}$.Also, the vertices in V_i (for $1\leq i\leq$m-2) are not adjacent to one another, while all vertices in V_{m-1} are mutually incident. Each vertex in V_i (for i<m-1) is adjacent to all vertices in V_{m-1} . Since the geodetic set must contain all elements from $V_1, V_2, \ldots, V_{m-2}$ the geodetic number follows as : $g(\Gamma[Z_{p^m}])= p^{m-1}\text{-}p$

Example 2.6: Consider the graph $\Gamma[Z_{p^2}]$, where p=5 is given with vertex set {5, 10, 15, 20} and edges occur between every pair of vertices. This is the complete graph K_4 and the geodetic set is {5, 10, 15, 20}



Example 2.7: Consider the graph $\Gamma[Z_{p^4}]$, where p = 2 is given with vertex set $V(\Gamma[Z_{2^4}])$ = {2, 4, 6, 8, 10, 12, 14},the minimum geodetic set is {2, 4, 6, 10, 12, 14} and geodetic number is 6.



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3. Conclusions

In this study, we identified the geodetic sets of certain classes of zero-divisor graphs and by analyzing the structural properties of these graphs, we determined conditions under which minimal geodetic sets can be established. Our results contribute to the broader understanding of graph-theoretic properties of algebraic structures and provide a foundation for further exploration in geodetic graph theory within ring theory. Future work may extend these findings to more general ring classes or explore applications in computational algebra and network theory.

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