Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 46 (2025), No. 5, pp. 1605–1613
DOI : 10.47974/JIOS-1844

Characteristics of fuzzy membership functions forming Boolean algebra of logic operation

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Abstract

Boolean algebra and fuzzy logic serve as foundational concepts in the fields of computer science, electrical engineering, and mathematics, providing critical frameworks for digital systems and decision-making processes. This paper explores the characteristics of fuzzy membership functions that form Boolean algebra, delving into the nuanced interplay between classical Boolean logic and the flexible, gradational nature of fuzzy sets. By examining the properties and operations of fuzzy membership functions, this research highlights the conditions under which they adhere to the principles of Boolean algebra and discusses their implications for computational applications.

Subject Classification: *03E72.*

Keywords: *Fuzzy logic, Boolean algebra, Extended fuzzy set.*

1. Introduction

Binary logic is encapsulated in the mathematical structure known as Boolean algebra, which was first proposed by George Boole in the middle of the 19th century [2]. It is characterized by operations such as AND, OR, and NOT, which are essential for designing and analyzing digital circuits [13], [14]. Boolean algebras have been extensively studied, with foundational work by Stone on representation theory [15] and Frink on ideals in partially ordered sets [5]. Later, comprehensive texts such as [6] and [9] offered structured insights into the theory and applications of Boolean algebras. Fuzzy logic, on the other

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hand, extends classical Boolean logic by allowing variables to have degrees of truth rather than binary values. This is achieved through fuzzy sets and membership functions, which assign a value between 0 and 1 to each element, representing its degree of membership in the set [17]. The formal basis of fuzzy sets was introduced by Zadeh [17], further expanded by Atanassov with the concept of intuitionistic fuzzy sets [1], and systematized in influential works such as [4], [7], [8], and [12]. In the medical domain, fuzzy modeling has played a crucial role in tumor growth analysis by enhancing prediction accuracy through fuzzy differential equations and multicriteria decision-making models [19]. A fuzzy Boolean algebra can be formed by a family of fuzzy subsets of a finite set, as described in [10]. The complement operation on the fuzzy subset was redefined in this context. Additionally, those works were expanded upon by looking at the fuzzy Boolean algebra's homomorphism, isomorphism, and automorphism features in [10]. In that paper, the qualities of ideals and filters had also been noted [5]. The Boolean algebra's foundation is the set of atoms since any element may be expressed using them [13]. Similar to how fuzzy sets were introduced [17] to extend crisp sets, the idea of fuzzy relations has been developed as an extension of the concept of crisp relations [1], [3]. Among n-dimensional relations, binary relations are particularly important because they can be thought of as generalized mathematical functions [3], [11]. Talee, A. F., Abbasi et al. introduce the fuzzy hyperideals over po-ternary semihypergroups and explore some related properties [18]. They introduce the concept of po-fuzzy point in po-ternary semihypergroups. Moreover, they explain a number of characterizations for regular po-ternary semihypergroups by using the concept of fuzzy hyperideals. There are numerous uses for unique kinds of relations in mathematics, such as equivalency relations and partially ordered relations [5], [11]. In engineering applications, particularly in infrastructure projects like the Lucknow Metro Rail, fuzzy methodologies have also proven effective. Fuzzy inference systems and clustering methods have been used to optimize the performance of Tunnel Boring Machines (TBMs) under varying geological conditions, reducing cost and enhancing precision [20].

This paper aims to explore the characteristics of fuzzy membership functions that enable them to form a structure akin to Boolean algebra. By analyzing these functions, we can understand how fuzzy logic can be employed in systems that traditionally rely on binary logic, thereby enhancing their flexibility and robustness. Numerical examples and extended fuzzy sets are included to demonstrate these characteristics [4], [7], [12], [16].

2. Background and Related Work

2.1 *Boolean Algebra*

Boolean algebra is defined by a set of elements, typically {0, 1}, and operations including conjunction (AND), disjunction (OR), and negation (NOT). These operations satisfy specific axioms such as commutativity, associativity, distributivity, identity elements, and complements.

2.2 *Fuzzy Logic and Membership Functions*

In the sixties, Lotfi Aliasker Zadeh created fuzziness logic, which introduces the idea of part of the truth, where the values of reality fall between 0 and 1. The function known as membership $\mu_A(x)$, which transfers each element x in the domain of discourse to a value between 0 and 1, indicates the degree of membership in the set A. This function is used to characterize fuzzy sets.

3. Characteristics of Fuzzy Membership Functions in Boolean Algebra

To explore how fuzzy membership functions can form a Boolean algebra, we need to examine their properties and the operations defined on them.

3.1 *Properties of Fuzzy Membership Functions*

1 **Normalization**: For a fuzzy set A, there exists at least one element x such that $\mu_A(x) = 1$.

2 **Boundedness**: $0 \leq \mu_A(x) \leq 1$ with each x.

3 **Complement**: The definition of a fuzzy set A's counterpart is $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$.

4 **Intersection & Union**: For fuzzy sets A and B, the intersection (AND) and union (OR) can be defined using various operators such as:

- **Min Operator**: $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$
- **Max Operator**: $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$
- **Product Operator**: $\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$
- **Probabilistic Sum Operator**: $\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$

3.2 *Fuzzy Logic Operations as Boolean Algebra Operations*

To form a Boolean algebra, the fuzzy logic operations must satisfy the Boolean algebra axioms. We analyze this by considering the following aspects:

1 **Identity Elements**:

- For intersection: $\mu_{A\cap U}(x) = \mu_A(x)$ where U is the universal set with $\mu_U(x) = 1$.
- For union: $\mu_{A\cup\emptyset}(x) = \mu_A(x)$ where $\emptyset$ is the empty set with $\mu_\emptyset(x) = 0$.

2 **Commutativity**:

- Intersection: $\mu_{A\cap B}(x) = \mu_{B\cap A}(x)$
- Union: $\mu_{A\cup B}(x) = \mu_{B\cup A}(x)$

3 **Associativity**:

- Intersection: $\mu_{A\cap(B\cap C)}(x) = \mu_{(A\cap B)\cap C}(x)$
- Union: $\mu_{A\cup(B\cup C)}(x) = \mu_{(A\cup B)\cup C}(x)$

4 **Distributivity**:

- Intersection over union: $\mu_{A\cap(B\cup C)}(x) = \max(\mu_A(x), \min(\mu_B(x), \mu_C(x)))$

- Union over intersection:
  $\mu_{A\cup(B\cap C)}(x) = \min(\mu_A(x), \max(\mu_B(x), \mu_C(x)))$

5 **Complementarity**:

- $\mu_{A\cap\bar{A}}(x) = 0$
- $\mu_{A\cup\bar{A}}(x) = 1$

4. Numerical Examples

To illustrate these concepts, consider the following fuzzy sets and their role of an account:

Let $X = \{1,2,3,4,5\}$.

Define fuzzy sets A and B with the following membership functions:

- $\mu_A(x)$ = {0.1, 0.5, 0.8, 1.0, 0.6}
- $\mu_B(x)$ = {0.7, 0.4, 0.3, 0.9, 0.2}

4.1 *Intersection and Union using Min and Max Operators*

Using the Min and Max operators, we calculate the intersection and union of sets A and B:

- **Intersection (Min Operator)**:
  - $\mu_{A\cap B}(1) = \min(0.1,0.7) = 0.1$
  - $\mu_{A\cap B}(2) = \min(0.5,0.4) = 0.4$
  - $\mu_{A\cap B}(3) = \min(0.8,0.3) = 0.3$
  - $\mu_{A\cap B}(4) = \min(1.0,0.9) = 0.9$
  - $\mu_{A\cap B}(5) = \min(0.6,0.2) = 0.2$
- **Union (Max Operator)**:
  - $\mu_{A\cup B}(1) = \max(0.1,0.7) = 0.7$
  - $\mu_{A\cup B}(2) = \max(0.5,0.4) = 0.5$
  - $\mu_{A\cup B}(3) = \max(0.8,0.3) = 0.8$
  - $\mu_{A\cup B}(4) = \max(1.0,0.9) = 1.0$
  - $\mu_{A\cup B}(5) = \max(0.6,0.2) = 0.6$

4.2 *Complement*

- **Complement of** A:
  - $\mu_{\bar{A}}(1) = 1 - 0.1 = 0.9$
  - $\mu_{\bar{A}}(2) = 1 - 0.5 = 0.5$
  - $\mu_{\bar{A}}(3) = 1 - 0.8 = 0.2$
  - $\mu_{\bar{A}}(4) = 1 - 1.0 = 0.0$
  - $\mu_{\bar{A}}(5) = 1 - 0.6 = 0.4$
- **Complement of** B:
  - $\mu_{\bar{B}}(1) = 1 - 0.7 = 0.3$

- $\mu_{\bar{B}}(2) = 1 - 0.4 = 0.6$
- $\mu_{\bar{B}}(3) = 1 - 0.3 = 0.7$
- $\mu_{\bar{B}}(4) = 1 - 0.9 = 0.1$
- $\mu_{\bar{B}}(5) = 1 - 0.2 = 0.8$

5. Extended Fuzzy Sets

Extended fuzzy sets, such as interval-valued fuzzy sets, type-2 fuzzy sets, and intuitionistic fuzzy sets, provide additional flexibility in representing uncertainty and imprecision.

5.1 *Interval-Valued Fuzzy Sets*

In interval-valued fuzzy sets, the membership degree of each element is an interval rather than a single value. For example, if $X = \{1,2,3\}$, a frequency of the cycle fuzzy set A might be represented as:

- $\mu_A(1) = [0.2,0.4]$
- $\mu_A(2) = [0.5,0.7]$
- $\mu_A(3) = [0.1,0.3]$

5.2 *Fuzzy Sets of Type-2*

The conceptual framework of fuzzy sets is expanded upon by type-2 fuzzy sets, which have fuzzy membership functions. This allows for an additional level of uncertainty. The membership degree itself is a fuzzy set in the interval [0, 1].

5.3 *Intuitionistic Fuzzy Sets*

Intuitionistic fuzzy sets, introduced by Atanassov, include a level of participation $\mu_A(x)$ and a level of non-membership $\nu_A(x)$ for each element x, with the constraint that $\mu_A(x) + \nu_A(x) \leq 1$.

1. **Intuitionistic Fuzzy Sets (IFS)**:
   - Consider an IFS defined over the universe of discourse
     $$X = \{1, 2, 3, 4, 5\}$$
   - Membership function:
     $$\mu_A(x) = \frac{x}{5}$$
   - Non-membership function:
     $$\nu_A(x) = \frac{5-x}{5}$$
   - Intuitionistic fuzzy set
     $$A = \{(1,0.2,0.8), (2,0.4,0.6), (3,0.6,0.4), (4,0.8,0.2), (5,1,0)\}$$
2. **Picture Fuzzy Sets (PFS)**:
   - Suppose we have a grayscale image pixel with intensity values ranging from 0 to 255.
   - A PFS for a pixel with intensity 150:
   - $Membership function$:

$$\mu_B(x) = \begin{cases} 0, & \text{if } x < 100 \\ \dfrac{x-100}{50}, & \text{if } 100 \le x < 150 \\ 1, & \text{if } x \ge 150 \end{cases}$$

- $Non-membershipfunction$:

$$\nu_B(x) = \begin{cases} 1, & \text{if } x < 100 \\ \frac{255-x}{105}, & \text{if } 100 \le x < 150 \\ 0, & \text{if } x \ge 150 \end{cases}$$

3. **Pythagorean Fuzzy Sets (PyFS)**:
   - Let's define a PyFS for a temperature reading in Celsius:
   - $Membershipfunction$:

$$\mu_C(x) = \frac{1}{1+x^2}$$

   - $Non-membershipfunction$:

$$\nu_C(x) = \frac{x^2}{1+x^2}$$

   - $Pythagoreanfuzzyset$

$$C = \{(20,0.8,0.2), (25,0.6,0.4), (30,0.4,0.6), (35,0.2,0.8)\}$$

4. **Spherical Fuzzy Sets (SFS)**:
   - Imagine a 3D space with coordinates

(x, y, z)

   - An SFS centered at

(2, 3, 4)

   with radius 2:
   - Membership function:

$$\mu_D(x,y,z) = \begin{cases} 1, & \text{if } (x-2)^2 + (y-3)^2 + (z-4)^2 \le 4 \\ 0, & \text{otherwise} \end{cases}$$

5. **Neutrosophic Sets**:
   - Let's consider a neutrosophic set for student performance:

Truth membership:

$$T_E = 0.7$$
$$Indeterminacy\ membership: I_E = 0.2$$
$$Falsity\ membership: F_E = 0.1$$
$$Neutrosophic\ setE = (0.7,0.2,0.1)$$

Spherical fuzzy sets (SFS) are a modified form of traditional fuzzy sets, including intuitionistic fuzzy sets (IFS), Pythagorean fuzzy sets (PFS), and picture fuzzy sets. SFS is a three-dimensional fuzzy set model designed to provide expert viewpoints a clearer, more insightful explanation. Three dimensions are included in SFS, as opposed to regular fuzzy sets, which only employ membership degrees: truth (a), abstention (g), and falseness (b). Each element is given a value between 0 and 1 by the membership function, which indicates the degree of truth (a), and the non-membership function, which

indicates the degree of falseness (b). The level of doubt or hesitation about an element's membership status is captured by the abstention function (g).

Because of its three-dimensional structure, SFS facilitates mathematical operations (addition, subtraction, multiplication, and division), enabling a more accurate depiction of uncertainty and intricate relationships. utilizing language evaluation scales based on SFS, decision-makers can indicate reluctance while utilizing multi-criteria decision-making (MCDM) methodologies. Risk assessment, optimization, and warehouse location selection are examples of practical uses. The benefits of SFS include increased realism, which captures subtleties that typical fuzzy sets could overlook, and a bigger preference domain for decision-makers to communicate preferences and uncertainty.

6. Discussion and Implications

While fuzzy logic provides a more flexible framework compared to classical Boolean logic, the ability of fuzzy membership functions to form a Boolean algebra depends on the specific operators used for intersection and union. The min and max operators, for example, satisfy many of the Boolean algebra properties, making them suitable for constructing fuzzy Boolean algebras. However, other operators like the product and probabilistic sum may not fully adhere to these properties.

Understanding these characteristics is crucial for designing systems that leverage both binary and fuzzy logic, such as fuzzy controllers, decision-making systems, and artificial intelligence applications. By ensuring that fuzzy membership functions adhere to Boolean algebra principles, we can create more robust and versatile computational models.

7. Conclusion

This paper has explored the characteristics of fuzzy membership functions that enable them to form Boolean algebra. Through a detailed analysis of their properties and operations, supplemented by numerical examples and extended fuzzy sets, we have highlighted the conditions under which fuzzy logic can emulate classical Boolean logic. This intersection of fuzzy and Boolean logic opens up new avenues for developing advanced computational systems that benefit from the strengths of both paradigms.

Acknowledgement

This research study is not funded by any organization.

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Received July, 2024