

Marshallian and Hicksian demand functions without calculus

Fountas Evangelos *

Department of Informatics

School of Information and Communication Technologies

University of Piraeus

Piraeus 18534

Greece

Milioris Konstantinos [†]

Department of Tourism Management

School of Administrative, Economics, and Social Sciences

University of West Attica

Egaleo 12243

Athens

Greece

Abstract

This study introduces an innovative method for deriving the Marshallian and Hicksian demand functions without the need of calculus, thus enhancing the accessibility of these fundamental economic models for audiences with less mathematical expertise. Conventional derivations of these demand functions employ calculus to optimize utility and expenditure within restrictions, potentially restricting accessibility in educational contexts and practical implementations. We utilize algebraic methods, such as Cauchy's inequality and the Arithmetic Mean-Geometric Mean [1], [3] (AM-GM) inequality, to derive these functions, illustrating that they may be expressed through fundamental algebraic propositions and inequalities.

Our methodology enhances the literature by providing an alternate framework for comprehending demand theory that maintains mathematical precision without necessitating differential calculus. This method is applied to prevalent utility functions, such as the Cobb-Douglas and Stone-Geary models, to demonstrate the usefulness and applicability of algebraic derivations in economic modeling. This study possesses significant importance for economic education, since it corresponds with current initiatives to make quantitative

[†] ORCID-ID: 0000-0001-9283-6582

* E-mail: fountasevan@gmail.com (Corresponding Author)

[†] E-mail: kmil@uniwa.gr

economics more accessible. The proposed algebraic framework may be beneficial in applied economics settings where advanced mathematical tools are scarce. This study seeks to connect intricate calculus-based methodologies with accessible algebraic methods, so expanding the accessibility of microeconomic theory and providing an inclusive option for demand function analysis.

Subject Classification: 62P20, 37N40, 46N10.

Keywords: Demand function, Stone-geary demand functions, Cauchy's inequality.

Introduction

The examination of consumer demand functions is fundamental to microeconomic theory, influencing our comprehension of customer selection, market demand, and pricing tactics. These functions, particularly the Marshallian and Hicksian demand functions, are usually derived using calculus, applying differential methods to optimize utility or spending within restrictions. This calculus-intensive method may present difficulties, particularly for individuals with insufficient mathematical expertise, thereby restricting access to these fundamental concepts in educational and professional settings.

1. Propositions

Generally, in order to find the minimum or maximum value of f function, for one or n variables $x_1, x_2, \dots, x_n$ we use the following propositions:

1.1 Proposition

We have $\max(f) = M$ iff $\max(f + c) = M + c$

$\min(f) = m$ iff $\min(f + c) = m + c,$

where c is a positive constant.

1.2 Proposition

We have $\max(f) = M$ iff $\max(cf) = cM$

$\min(f) = m$ iff $\min(cf) = cm,$

where c is a positive constant.

1.3 Proposition

We have $\max(f) = M > 0$ iff $\max(f^n) = M^n$

$$\min(f) = m > 0 \text{ iff } \min(f^n) = m^n$$

$$\max(f) = M > 0 \text{ iff } \min\left(\frac{1}{f}\right) = \frac{1}{M}$$

$$\min(f) = m > 0 \text{ iff } \max\left(\frac{1}{f}\right) = \frac{1}{m},$$

1.4 Proposition

If n positive variables satisfy $\prod_{i=1}^n x_i = c$, then their sum $\sum_{j=1}^n x_j$ is minimum iff $x_1 = x_2 = \dots = x_n = \sqrt[n]{c}$

1.5 Theorem

If n positive functions have a fixed product, their sum is a minimum if it can be arranged that the functions are equal. On the other hand, if n positive functions have a fixed sum, their product is a maximum if it can be arranged that the functions are equal. [2., p.26]

1.6 Proposition

If the product $\prod_{i=1}^n x_i^{k_i}$ of $x_1^{k_1}, x_2^{k_2}, \dots, x_n^{k_n}$ powers of n positive numbers $x_i, i \in [n]$ with real exponents $k_i, i \in [n]$ is constant, then the sum of $\sum_{i=1}^n x_i, x_i, i \in [n]$ becomes minimum when these numbers are proportional to the exponents, i.e.

$$\frac{x_1}{k_1} = \frac{x_2}{k_2} = \dots = \frac{x_n}{k_n}$$

1.7 Proposition

If n positive variables satisfy $\sum_{j=1}^n x_j = c$, then their product $\prod_{i=1}^n x_i, i, x_i \in [n]$ is maximum when the numbers are equal, i.e.

$$x_1 = x_2 = \dots = x_n = \frac{c}{n}$$

1.8 *Proposition*

If the sum of n positive numbers $x_j, j \in [n]$ is constant, then the product of their powers $\prod_{i=1}^n x_i^{k_i}, x_i, k_i \in [n]$ with real exponents $k_1, k_2, \dots, k_n$ becomes maximum when these numbers are proportional to their exponents.

$$\frac{x_1}{k_1} = \frac{x_2}{k_2} = \dots = \frac{x_n}{k_n}$$

1.9 *Proposition Cauchy*

Given that $x_1, x_2, \dots, x_n$ are positive real numbers, then (AM-GM)

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n}$$

equality holds iff $x_1 = x_2 = \dots = x_n$.

1.10 *Proposition CBS (Cauchy – Bunyakovsky – Schwarz)*

For real numbers $\alpha_1, \alpha_2, \dots, \alpha_n, b_1, b_2, \dots, b_n$. The following inequality holds:

$$\left(\sum_{j=1}^n \alpha_j^2 \right) \cdot \left(\sum_{j=1}^n \beta_j^2 \right) \geq \left(\sum_{j=1}^n \alpha_j \beta_j \right)^2 \quad (\text{CBS})$$

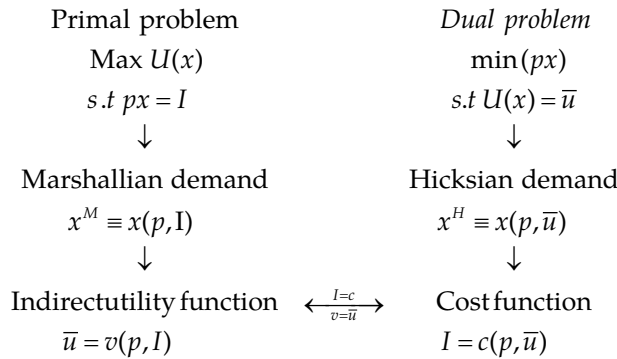
Equality occurs iff the sequences $\alpha_j, j \in [n]$ and $b_j, j \in [n]$ are proportional, i.e.

$$\frac{\alpha_1}{b_1} = \frac{\alpha_2}{b_2} = \dots = \frac{\alpha_n}{b_n}$$

1.11 *Fixed budget & utility maximization Fixed utility & minimization*

PRIMAL PROBLEM	DUAL PROBLEM
CONSUMPTION DUALITY	
Primal problem	Dual problem
Max $U(x)$	min px
subject to $px \leq I$	subject to $U(x) \geq \bar{U}$

Without loss of generality, assume that



2. Applications

2.1 Marshallian Demand Function Utility

Given that x, y, p_x, p_y and I are positive numbers such that $p_x x + p_y y = I$, we will prove that $\max(xy) = \frac{I^2}{4p_x p_y}$.

Proof: If we write $u = p_x x, v = p_y y$, then $p_x x + p_y y = I$ becomes $u + v = I$, (by Proposition 1.7) the product $u \cdot v$ is maximized iff $u = v = \frac{I}{2}$.

$$\text{But we see that } u \cdot v = p_x x \cdot p_y y \therefore xy = \frac{1}{p_x p_y} \cdot uv$$

Hence to $\max(u \cdot v)$ is to $\max(xy)$ also, and this appears in case

$$u = v = \frac{I}{2} = p_x x = p_y y \therefore x^M = \frac{I}{2p_x}, y^M = \frac{I}{2p_y}$$

$$\text{Thus } \max(xy) = \frac{I^2}{4p_x p_y}$$

2.2 Hicksian Demand Function

Will calculate the Hicksian Demand Function, i.e. we will find for which prices of x and y becomes minimum the function $p_x \cdot x + p_y \cdot y = I$.

Solution: We observe that the product: $(p_x x) \cdot (p_y y) = p_x p_y xy = p_x p_y \bar{u} \equiv \text{constant}$ then the sum $p_x x + p_y y$ becomes minimum when $p_x x = p_y y$ (by proposition 1.7). The function $\bar{u} = xy$ is written:

$$p_x p_y \bar{u} = (p_x x)(p_y y) = (p_x x)^2 \therefore p_x p_y \cdot \bar{u} = p_x^2 \cdot x^2 \therefore x^H = \left(\frac{p_y \bar{u}}{p_x} \right)^{\frac{1}{2}}$$

Similarly it also turns out that Hicksian Demand Function $y^H = \left(\frac{p_x}{p_y} \bar{u} \right)^{\frac{1}{2}}$

2.3 Cobb Douglas Function Utility

Let x, y, p_x, p_y and I be positive numbers such that $p_x x + p_y y = I$. We will prove that

$$\max U(x, y) = \left(\frac{\alpha}{p_x} \right)^{\alpha} \left(\frac{\beta}{p_y} \right)^{\beta} \left(\frac{I}{\alpha + \beta} \right)^{\alpha + \beta}$$

Proof: The function $U(x, y) = x^{\alpha} \cdot y^{\beta}, x, y \in \mathbb{R}^+$ is equivalent to

$$U(x, y) = \frac{1}{p_x^{\alpha} p_y^{\beta}} \cdot (p_x x)^{\alpha} \cdot (p_y y)^{\beta}$$

Because the sum of $p_x x + p_y y$ is constant, the $U(x, y)$ becomes maximum when the factors $p_x x$ and $p_y y$ are analogs of exponents α and β respectively (Proposition 1.8), i.e.

$$\frac{p_x x}{\alpha} = \frac{p_y y}{\beta} = \frac{p_x x + p_y y}{\alpha + \beta} = \frac{I}{\alpha + \beta}$$

hence

$$\left| \begin{aligned} x &= \frac{\alpha I}{p_x(\alpha + \beta)}, y = \frac{\beta I}{p_y(\alpha + \beta)} \\ \max U &= \left(\frac{\alpha}{p_x} \right)^{\alpha} \left(\frac{\beta}{p_y} \right)^{\beta} \left(\frac{I}{\alpha + \beta} \right)^{\alpha + \beta} \end{aligned} \right.$$

Remark: If exponents sum to 1, $\alpha + \beta = 1, 0 < \alpha < 1, 0 < \beta < 1$, we get, we call it a normalized Cobb-Douglas utility function i.e. $\alpha, 1 - \alpha$ we have:

$$\left| \begin{aligned} x^m &= \frac{\alpha I}{p_x}, y^m = \frac{(1 - \alpha) I}{p_y}, \max U = x^{\alpha} \cdot y^{1 - \alpha} = \left(\frac{\alpha}{p_x} \right)^{\alpha} \cdot \left(\frac{1 - \alpha}{p_y} \right)^{1 - \alpha} \cdot I \end{aligned} \right.$$

In a special case $U(K, L) = c \cdot K^{\alpha} \cdot L^{1 - \alpha}$, the typical function form Cobb-Douglas Utility Function, $U(K, L)$: A function of factors, K: Capital and L: Labor c Total Factor Utility .

Alpha, exponent above capital, α output elasticity of capital and $1 - \alpha$ output elasticity of labor.

Without loss of generality, assume that $U(K, L) = K^{\alpha} \cdot L^{1 - \alpha}$

2.4 Hicksian Demand Function

We need to find the minimum of function $p_x \cdot x + p_y \cdot y = I$, when the function $\bar{u} = x^\alpha y^\beta \equiv \text{constant}$, $x, y \in \mathbb{R}^+$.

Solution: Function $p_x x + p_y y$ becomes minimum, iff $*p_x x = p_y y$. Given function $\bar{u} = x^\alpha y^\beta$ is written $p_x^\alpha \cdot p_y^\beta \bar{u} = (p_x x)^\alpha \cdot (p_y y)^\beta$ and so $p_x x = p_y y$ we get

$$\therefore x^{\alpha+\beta} = \frac{p_y^\beta}{p_x^\beta} \cdot \bar{u} \quad \therefore x = \left(\frac{p_y}{p_x} \right)^{\frac{\beta}{\alpha+\beta}} \cdot (\bar{u})^{\frac{1}{\alpha+\beta}}$$

$$\text{Similarly resulting } y = \left(\frac{p_x}{p_y} \right)^{\frac{\alpha}{\alpha+\beta}} \cdot (\bar{u})^{\frac{1}{\alpha+\beta}}$$

$$* \text{It is: hence } \min(p_x x + p_y y) = 2(p_x p_y x y)^{\frac{1}{2}}$$

2.5 Cobb Douglas Generalized Marshallian Function Utility

Deriving a Generalized Marshallian Demand Function for an individual with Cobb Douglas Utility.

$$\text{Maximize: } u(x_i) \equiv \prod_{i=1}^n x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

$$\text{Subject to: } I = p_1 x_1 + p_2 x_2 + \dots + p_n x_n \equiv \sum_{i=1}^n p_i x_i$$

$$\textbf{Solution:} \text{ We have: } u(x_i) = \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}} (p_1 x_1)^{\alpha_1} \cdot (p_2 x_2)^{\alpha_2} \dots (p_n x_n)^{\alpha_n}$$

$$(\text{by proposition 1.8}), \text{ we have: } \frac{p_1 x_1}{\alpha_1} = \frac{p_2 x_2}{\alpha_2} = \dots = \frac{p_n x_n}{\alpha_n}$$

$$\begin{array}{l} \left| \begin{array}{l} x_1^M = \frac{\alpha_1}{p_1} \cdot \frac{I}{\sum_{i=1}^n \alpha_i} \\ \vdots \\ x_j^M = \frac{\alpha_j}{p_j} \cdot \frac{I}{\sum_{j=1}^n \alpha_j} \end{array} \right. \end{array}$$

hence

2.6 Cobb Douglas Generalized Marshallian Function Utility without Linear Restriction

We will now find the Generalized Marshallian Function form if the restricted function is non Linear, that is, of the form $I \equiv \sum_{i=1}^n p_i x_i^{k_i}$

we have: $u(x_i) \equiv \prod_{i=1}^n x_i^{\alpha_i} \equiv x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$

$$p_1^{\frac{\alpha_1}{k_1}} \cdot p_2^{\frac{\alpha_2}{k_2}} \dots p_n^{\frac{\alpha_n}{k_n}} = \frac{1}{p_1^{\frac{\alpha_1}{k_1}} \cdot p_2^{\frac{\alpha_2}{k_2}} \dots p_n^{\frac{\alpha_n}{k_n}}} \cdot (p_1 x_1^{k_1})^{\frac{\alpha_1}{k_1}} \cdot (p_2 x_2^{k_2})^{\frac{\alpha_2}{k_2}} \dots (p_n x_n^{k_n})^{\frac{\alpha_n}{k_n}}$$

(by proposition 1.8) because the sum $\sum_{i=1}^n p_i x_i^{k_i} = I$ is constant we have:

$$\frac{p_1 x_1^{k_1}}{\frac{\alpha_1}{k_1}} = \frac{p_2 x_2^{k_2}}{\frac{\alpha_2}{k_2}} = \dots = \frac{p_n x_n^{k_n}}{\frac{\alpha_n}{k_n}}$$

$$\therefore \begin{cases} x_1^M = \left(\frac{I}{\frac{\alpha_1}{k_1} + \frac{\alpha_2}{k_2} + \dots + \frac{\alpha_n}{k_n}} \cdot \frac{\alpha_1}{k_1} \cdot \frac{1}{p_1} \right)^{\frac{1}{k_1}} \\ x_2^M = \left(\frac{I}{\frac{\alpha_1}{k_1} + \frac{\alpha_2}{k_2} + \dots + \frac{\alpha_n}{k_n}} \cdot \frac{\alpha_2}{k_2} \cdot \frac{1}{p_2} \right)^{\frac{1}{k_2}} \\ \vdots \end{cases}$$

2.7 Hicksian Demand Function Utility

Solution: Applying Cauchy's inequality:

$$\sum_{i=1}^n p_i x_i \geq n \left(\prod_{i=1}^n p_i x_i \right)^{\frac{1}{n}}$$

and the equality iff $p_1 x_1 = p_2 x_2 = \dots = p_n x_n$

$$i.e. \sum_{i=1}^n p_i x_i = n \left(\prod_{i=1}^n p_i x_i \right)^{\frac{1}{n}}$$

The function $\bar{u} \equiv x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ is equivalent to

$$\bar{u} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} = (p_1 x_1)^{\alpha_1} \cdot (p_2 x_2)^{\alpha_2} \dots (p_n x_n)^{\alpha_n}$$

$$\bar{u} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} = (p_1 x_1)^{\alpha_1 + \alpha_2 + \dots + \alpha_n}$$

hence
$$x_1^H = \left(\frac{1}{p_1} \right) \cdot (p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \cdot \bar{u})^{\frac{1}{\alpha_1 + \alpha_2 + \dots + \alpha_n}}$$

Finally we have:

$$\left| \begin{array}{l} x_1^H = \frac{1}{p_1} (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \cdot \bar{u})^{\frac{1}{\alpha_1 + \alpha_2 + \dots + \alpha_n}} \\ x_2^H = \frac{1}{p_2} (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \cdot \bar{u})^{\frac{1}{\alpha_1 + \alpha_2 + \dots + \alpha_n}} \\ \vdots \\ x_n^H = \frac{1}{p_n} (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \cdot \bar{u})^{\frac{1}{\alpha_1 + \alpha_2 + \dots + \alpha_n}} \end{array} \right.$$

2.8 Cobb Douglas Generalized

For $U = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ (utility function) in special case where $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ with subject to $p_1 x_1 + p_2 x_2 + \dots + p_n x_n = I \equiv \text{constant}$, we will prove that

$$x_1^M = \frac{I\alpha_1}{p_1}, x_2^M = \frac{I\alpha_2}{p_2}, \dots, x_n^M = \frac{I\alpha_n}{p_n}$$

Proof: Marshallian Function is

$$\max U = \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}} (p_1 x_1)^{\alpha_1} (p_2 x_2)^{\alpha_2} \cdots (p_n x_n)^{\alpha_n}$$

(by proposition 1.8), because the sum $\sum_{i=1}^n p_i x_i = I \equiv \text{constant}$,

hence
$$x_1^M = \frac{I\alpha_1}{p_1}, x_2^M = \frac{I\alpha_2}{p_2}, \dots, x_n^M = \frac{I\alpha_n}{p_n}$$

2.9 Hicksian Demand Function Utility

We will find the minimum of the function $\sum_{i=1}^n p_i x_i$ when $\sum_{i=1}^n \alpha_i = 1$ and $\bar{u} = \prod_{i=1}^n x_i \equiv \text{constant}$

Solution: Because $\sum_{i=1}^n \alpha_i = 1$, previous types of 2.7 become:

$$x_1^H = \frac{1}{p_1} (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}) \cdot \bar{u} \quad \therefore \left| \begin{array}{l} x_1^H = (p_1^{\alpha_1-1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}) \cdot \bar{u} \\ x_2^H = (p_1^{\alpha_1} p_2^{\alpha_2-1} \cdots p_n^{\alpha_n}) \cdot \bar{u} \\ \vdots \\ x_n^H = (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n-1}) \cdot \bar{u} \end{array} \right.$$

2.10 General model of Stone-Geary

Consider the n goods Stone-Geary utility function

$$\prod_{i=1}^n (x_i - \beta_i)^{\alpha_i}$$

Derive the Marshallian demand functions for goods $x_i, i \in [n]$.

Solution: Derive the Marshallian demand functions for goods $x_i \in [n]$, $i \in p[n]$

We have:

$$\begin{aligned} & \prod_{i=1}^n (x_i - \beta_i)^{\alpha_i} \\ & \equiv \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}} (p_1 x_1 - p_1 \beta_1)^{\alpha_1} \cdot (p_2 x_2 - p_2 \beta_2)^{\alpha_2} \dots (p_n x_n - p_n \beta_n)^{\alpha_n} \end{aligned}$$

by 1.8 we have

$$\begin{aligned} \frac{p_1 x_1 - p_1 \beta_1}{\alpha_1} &= \frac{p_2 x_2 - p_2 \beta_2}{\alpha_2} = \dots = \frac{p_n x_n - p_n \beta_n}{\alpha_n} \\ &= \frac{(p_1 x_1 - p_1 \beta_1) + (p_2 x_2 - p_2 \beta_2) + \dots + (p_n x_n - p_n \beta_n)}{\alpha_1 + \alpha_2 + \dots + \alpha_n} = \frac{I - c}{\sum_{i=1}^n \alpha_i} \end{aligned}$$

hence

$$\left| \begin{array}{l} x_1^M = \beta_1 + \frac{\alpha_1(I-c)}{p_1 \sum_{i=1}^n \alpha_i} \\ \vdots \\ x_i^M = \beta_i + \frac{\alpha_i(I-c)}{p_i \sum_{i=1}^n \alpha_i} \end{array} \right.$$

2.11 Hicksian General model of Stone-Geary

We will find the minimum of the function $I \equiv \sum_{i=1}^n p_i x_i$, $x_i, i \in [n]$

subject to $\bar{u} \equiv \prod_{i=1}^n (x_i - \beta_i)^{\alpha_i}$

Solution: The function $\bar{u}$ is equivalent to

$$\bar{u} = \frac{1}{p_1^{\alpha_1} p_1^{\alpha_2} \dots p_n^{\alpha_n}} (p_1 x_1 - p_1 \beta_1)^{\alpha_1} \cdot (p_2 x_2 - p_2 \beta_2)^{\alpha_2} \dots (p_n x_n - p_n \beta_n)^{\alpha_n}$$

Function I is written:

$$(p_1x_1 - p_1\beta_1) + (p_2x_2 - p_2\beta_2) + \cdots + (p_nx_n - p_n\beta_n) = I - c = C^*$$

and becomes minimum iff

$$p_1x_1 - p_1\beta_1 = p_2x_2 - p_2\beta_2 = \cdots = p_nx_n - p_n\beta_n$$

and $\bar{u}$ is written

$$\bar{u} = \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}} \cdot (p_1x_1 - p_1\beta_1)^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}$$

$$\text{hence} \quad \left| \begin{array}{l} x_1^H = \beta_1^H + \frac{1}{p_1} (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \cdot \bar{u})^{\frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}} \\ x_2^H = \beta_2^H + \frac{1}{p_2} (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \cdot \bar{u})^{\frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}} \\ \vdots \\ x_n^H = \beta_n^H + \frac{1}{p_n} (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \cdot \bar{u})^{\frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}} \end{array} \right.$$

Remark: For $x_i = 0, i \in [n]$ the Stone-Gear function reduces to the generalized Cobb -Douglas function. The Stone-Gear utility function was first derived by Roy C. Geary, in a comment on earlier work by Laurence Klein and Herman Rubin.

2.12 Partial case of Stone – Geary

Consider the two-good Stone-Geary utility function

$$U(A, B) \equiv (A - \alpha)^\gamma \cdot (B - \beta)^{1-\gamma}$$

Solution: The u function:

$$U \equiv \frac{1}{p_A^\gamma \cdot p_B^{1-\gamma}} \cdot (p_A \cdot A - p_A \cdot \alpha)^\gamma \cdot (p_B \cdot B - p_B \cdot \beta)^{1-\gamma}$$

$$\text{hence} \quad A = \alpha + \frac{[(I-c)\gamma]}{p_A}, \quad B = \beta + \frac{[(I-c)(1-\gamma)]}{p_B}$$

2.13 Proposition of Marshallian's general model

We will prove the inequality $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$

with $\sum_{i=1}^n \alpha_i = 1$ and $x_j, j \in [n]$, non-negative numbers.

Proof: If set $\alpha_i = \frac{m_i}{n}$, then is $m_1 + m_2 + \dots + m_n = n$.

Applying Cauchy's inequality [3]

we get

$$\sqrt[n]{x_1^{m_1} \cdot x_2^{m_2} \dots x_n^{m_n}} \leq \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{n}$$

whereat, eventually, resulting that $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \leq \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$

hence $\max(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$

Acknowledgement

I owe many thanks to Prof. Antonios Panayiotopoulos for his valuable thoughts.

References

- [1] P. Fuleky, Anatomy of C-D Production/Utility Functions in Three Dimensions (2006).
- [2] I. Niven, Maxima and Minima Without Calculus, MAA Am. Math. Soc. (1981).
- [3] V. M. Tikhomirov, Stories about Maxima and Minima, Am. Math. Soc. (1990).

Received August, 2022