Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 46 (2025), No. 3, pp. 591–599
DOI : 10.47974/JIOS-1322

Edge-odd graceful labeling of Cartesian product of two paths

D. Senthil Kumar *
Department of Mathematics
Takshashila University
Ongur Tindivanam
Tamil Nadu 604305
India

R. Anbarasan †
G. Pushparaj §
Department of Mathematics
Mailam Engineering College
Mailam
Tamil Nadu 604304
India

Abstract

Let G be a finite simple graph with p vertices and q edges is called an edge odd graceful labeling if there exists a bijective map $f : E(G) \to \{1, 3, 5, \ldots, (2k-1)\}$ such that when each vertex is assigned, the sum of all edges incident to it mod $2k$, where $k = max(p, q)$, the resulting vertex labels are distinct. Any (p, q) - graph admitting edge - odd graceful labeling is called an edge-odd graceful graph. In this paper, we investigate the existence of edge odd graceful labeling for the cartesian product of two paths P_m and P_n where m and n are any positive integers.

Subject Classification: *(2010) 05C78.*

Keywords: *Graceful graph, Edge-odd graceful graph, Cartesian product of two graphs.*

1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions.Here we consider only finite simple

* *E-mail:* **zenthil29@gmail.com (Corresponding author)**
† *E-mail:* **abianbu009@gmail.com**
§ *E-mail:* **pushgovindan@gmail.com**

undirected graph.The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$ respectively $p = |V(G)|$ and $q = |E(G)|$.

Lo [5] introduced a labeling of G called edge graceful labeling, which is a bijection f from the set of edges $E(G)$ to the set $\{1, 2, \ldots, q\}$ such that the induced map f^* from the set of vertices $V(G) \to \{0, 1, 2, \ldots, p-1\}$ given by $f^*(u) = \sum uv \in E(G)\ f(uv)\ (mod\ p)$ is a bijection. A graph which admits edge graceful labeling is called an edge graceful graph.

David Kuo and Jing- Ho Yan [4] have proved that the $L(2,1)$ labeling number of Cartesian product of path and cycles. Daoud [3] proved the necessary and sufficient condition for some path and cycle related graph to be edge odd graceful. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

Rosa [6] introduced a labeling of G called β - valuation, later on Solomon W. Golomb called as graceful labeling which is an injection f from the set of vertices $V(G)$ to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge $e = uv$ is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. A graph which admits a graceful labeling is called a graceful graph.

Mohamed Basher [1] motivated to use the cartesian product of two paths.He discussed about the odd- even graceful labeling that means if there exist an injective function $f : V(G) \to \{1, 3, 5,\ \ldots,\ 2q+1\}$ such that induced function $f^* : E(G) \to \{2, 4, 6, \ldots, 2q\}$ defined by $f^*(uv) = |f(u) - f(v)|$ for every $uv \in E(G)$ is a bijection. He proved that the planar grid and prism are odd- even graceful labeling.

Solairaju and Chithra [7] Introduced the concept of edge odd graceful Labeling. If G be graph with p vertices and q edges called an edge odd graceful labeling if there exists a bijective map $f : E(G) \to \{1, 3, 5,\ \ldots,\ (2k-1)\}$ such that its induced map $f^+ : V(G) \to \{0, 1, 2,\ \ldots,\ (2k-1)\}$ defined by $f^+(x) \equiv \sum_{(y \in V, xy \in E)} f(xy)\ (mod\ 2k)$ is injective, where $k = \max\ (p, q)$. [9] Any (p, q) - graph admitting edge odd graceful labeling is called an *edge-odd graceful graph*. Solairaju and Senthil Kumar [8] explained some of the product of path and cycles are edge odd graceful labelling.

The cartesian product of two paths P_m and P_n whose vertex set is $V(G) \times H(G)$ having mn elements, namely $\{v_1, v_2, \ldots, v_{mn}\}$ and its edge set is $E(P_m \square P_n)$ = $\{\{(v_i v_{i+1} : i \not\equiv 0(mod\ n)\} \cup \{(v_{(j-1)n+i} v_{jn+i} : i = 1$ to $n, j = 1$ to $m)\}\}$ for all positive integers m and n. The cartesian product of two paths P_m and P_n denoted as $[P_m \square P_n]$.

2. Main Result

Here we will discuss about all the positive possible integers for m and n in the following theorem.

Theorem 2.1: *The* $[P_n \square P_n]$ *is edge odd graceful for all integer* $n \geq 5$.

Proof: The graph $[P_n \square P_n]$ is a connected graph whose vertex set is $\{v_1, \ldots, v_{n^2}\}$ and edge set is $\{\{v_i v_{i+1} : i \not\equiv 0 \ (mod\ n)\} \cup \{(v_{(j-1)n+i} v_{jn+i} : i = 1 \text{ to } n,\ j = 1 \text{ to } n)\}\}$ for all integer $n \geq 5$.

Construct a bijective map $f : E(G) \to \{1, 3, 5, \ldots, (2k-1)\}$ in the various steps as follows:

Let i varies from 1 to n^2 and $i \not\equiv 0 (mod\ n)$.

Case (i): n is even.

Claim (1): To get edge-odd labeling of $\{v_i v_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase (1a): Define $f(v_i v_{i+1}) = [(3n^2 - ln - (r-1))]$; where $i = ln + r$; i is odd;

$$l = 0,\ 1, 2, \ldots,\ (n-1);\ r = 1, 3, 5,\ \ldots,\ (n-1);\ \ i \neq 1,\ [(n^2 - n) + 1],\ (n^2 - 1).$$

$$f(v_1 v_2) = 1;\ f(v_{n^2-n+1}\ v_{n^2-n+2}) = (2n^2 - 2n + 1);\ f(v_{n^2-1} v_{n^2}) = (2n^2 - n - 1).$$

Subcase (1b): Define $f(v_i v_{i+1}) = [(3n^2 - 2n + 1) + l(n-2) + (r-2)]$; where $i = ln + r$; i is even; $l = 0, 1, 2, \ldots, (n-1)$; $r = 2, 4, 6, \ldots, (n-2)$;

Claim (2): To get edge-odd labeling to all the edge in $\{v_{((j-1)n+i)}\ v_{jn+i} : i = 1$ to $n, j = 1$ to $n\}$.

Subcase (2a): Define $f(v_{(j-1)n+i} v_{jn+i}) = [i + (j-1)n]$; where $i = 1, 3, 5, \ldots, (n-1)$; $j = 1, 2, 3, \ldots, (n-1)$; when $i \neq 1$ and $j \neq 1$. $f(v_1 v_{n+1}) = (3n^2 - 2n - 1)$.

Subcase (2b): Define $f(v_{(j-1)n+i} v_{jn+i}) = [(n^2 - n + 1) + n(j-1) + (i-2)]$; where $i = 2, 4, \ldots, n; j = 1, 2, \ldots, (n-1)$;

Case (ii): n is odd.

Claim (3): To get edge-odd labeling of all the edges $\{v_i v_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase (3a): Define $f(v_i v_{i+1}) = [(2n-2)l + r]$; where $i = ln + r$; i is odd; $l = 0, 1, 2,\ \ldots,\ (n-1)$; $r = 1, 3, 5,\ \ldots,\ (n-2)$; $i \neq 1$ to $(n-1)$, $(n^2 - n)$ to $(n^2 - 1)$.

$f(v_i v_{i+1}) = (2n^2 - 4n + 3) + 2(i-1)$; where $i = 1$ to $(n-1)$.
$f(v_{((n^2-n)+i)} v_{(n^2-n)+(i+1))} = (2i - 1)$; where $i = 1$ to $(n-1)$.

Subcase $(3b)$: Define $f(v_i v_{i+1}) = (2n-2)l + (2n-3) - (r-2)$; where $i = ln + r$; i is even; $l = 1, 2, \ldots, (n-2)$; $r = 2, 4, 6, \ldots, (n-1)$.

Claim (4): To get edge-odd labeling to all the edge in $\{v_{((j-1)n+i)} v_{jn+i} : i = 1$ to $n, j = 1$ to $n\}$.

Define $f(v_{(j-1)n+i} v_{jn+i}) = [(2n^2 - 2n + 1) + (2n-2)(i-1) + 2(j-1)]$; where $i = 1, 2, \ldots, n$
and $j = 1, 2, \ldots, (n-1)$.

In both cases (i), (ii) and all its subcases, the induced map $f^+ : V(G) \to \{1, 2, 3, \ldots, (2k-1)\}$ from f is defined by $f^+(x) \equiv f(xy)(mod\, 2k)$ where this sum runs over all edges through x. Thus the given graph has edge odd graceful labeling from the maps f and f^+. Hence the graph $[P_n \square P_n]$ is edge odd graceful for all integer $n \geq 5$.

Theorem 2.2: *The* $[P_m \square P_n]$ *is edge odd graceful for all positive integers* $m, n \geq 5$, *and* $m \neq n$.

Proof: The graph $[P_m \square P_n]$ is a connected graph whose vertex set is $\{v_1, \ldots, v_{mn}\}$ and edge set is $\{\{v_i v_{i+1} : i \not\equiv 0(mod\, n)\} \cup \{v_{(j-1)n+i} v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}\}$. Here both m and n are the positive integers $m, n \geq 5$ and $m \neq n$.

Construct a bijective map $f : E(G) \to \left\{1, 3, 5, \ldots, (2k-1)\right\}$ in the various steps as follows.

Let i varies from 1 to mn.

Case (1) $m > n$. Here both m and n are even.

Claim (1): To get edge odd labeling to all the edges in $\{v_i v_{i+1} : i \not\equiv 0(mod\, n)\}$.

Subcase $(1a)$: Define $f(v_i v_{i+1}) = [(3mn - 2n - 1) - (r-1) - ln]$; where $i = ln + r; i$ is odd; $l = 0, 1, 2, \ldots, (m-1)$; $r = 1, 3, 5, \ldots, (n-1)$; $i \neq 1$. $f(v_i v_{i+1}) = (2n+1)$.

Subcase $(1b)$: Define $f(v_i v_{i+1}) = [(3mn - 2n + 1) + (r-2) + l(n-2)]$; where $i = ln + r; i$ is even; $l = 0, 1, 2, \ldots, (m-1)$; $r = 2, 4, 6, \ldots, (n-2)$.

Claim (2): To get edge - odd labeling to all the edges in $\{v_{(j-1)n+i} v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}$.

Subcase $(2a)$: Define $f(v_{(j-1)n+i} v_{jn+i}) = [i + n(j-1)]$; where $i = 1, 3, 5, \ldots, (n-1)$ and $j = 1, 2, 3, \ldots, (m-1)$; when $i \neq 1$ and $j \neq 3$. $f(v_{2n+1} v_{3n+1}) = (3mn - 2n - 1)$.

Subcase $(2b)$: Define $f(v_{(j-1)n+i} v_{jn+i}) = [(mn - n + 1) + n(j-1) + (i-2)]$; where $i = 2, 4, 6, \ldots, n$. and $j = 1, 2, 3, \ldots, (m-1)$.

Case (2) $m < n$. Both m and n are even integers.

Claim (3): To get edge odd labeling to all the edges in the set $\{v_i v_{i+1} : i \not\equiv 0 \ (mod\ n)\}$.

Subcase $(3a)$: Define $f(v_i v_{i+1}) = [(3mn-2n-1) - (r-1) - ln]$; where $i = ln + r$; i is odd; $l = 0, 1, 2, \ldots, (m-1)$; $r = 1, 3, 5, \ldots, (n-1)$; $i \neq 2n+1$, $[(m-1)\ n+1]$, $(mn-1)$.

$$f(v_{2n+1} v_{2n+2}) = (3mn-3). \quad f(v_{(m-1)n+1}\ v_{(m-1)n+2)} = (2mn-2n+1).$$

$$f(v_{mn-1} v_{mn}) = (2mn-n-1).$$

Subcase $(3b)$: Define $f(v_i v_{i+1}) = (3mn-2n+1) + (r-2) + l(n-2)$; where $i = ln + r$; i is even; $l = 0, 1, 2, \ldots, (m-1)$; $r = 2, 4, 6, \ldots, (n-2)$; $i \neq 2n+2$. $f(v_{2n+2} v_{2n+3}) = (3mn-4n-1)$.

Claim (4): To get edge-odd labeling to all the edges in $\{v_{(j-1)n+i}\ v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}$.

Subcase $(4a)$: Define $f(v_{(j-1)n+i} v_{jn+i}) = [i + n(j-1)]$;
where $i = 1, 3, 5, \ldots, (n-1)$ and $j = 1, 2, 3, \ldots, (m-1)$.

Subcase $(4b)$: Define $f(v_{(j-1)n+i} v_{jn+i}) = [(mn-n+1) + n(j-1) + (i-2)]$;
where $i = 2, 4, 6, \ldots, n$. and $j = 1, 2, 3, \ldots, (m-1)$.

Case (3) $m > n$. Both m and n are odd integers.

Claim (1): To get edge odd labeling to all the edges in $\{v_i v_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase $(1a)$: Define $f(v_i v_{i+1}) = [(2n-2)l + r]$; where $i = ln + r; i$ is odd; $l = 0, 1, 2, \ldots, (m-1)$; $r = 1, 3, 5, \ldots, (n-2)$; $i \neq 1$ to $(n-1)$, $[(m-1)n+1]$ to $(mn-1)$.

$$f(v_i v_{i+1}) = [(2mn-2m-2n+3) + 2(i-1); \text{ where } i = 1 \text{ to } (n-1).$$

$$f(v_{(m-1)n+i} v_{((m-1)n+i+1)}) = (2i-1); \text{ where } i = 1 \text{ to } (n-1).$$

Subcase $(1b)$: Define $f(v_i v_{i+1}) = [l(2n-2) + (2n-1) - r]$; where $i = ln + r$; i is even;

$$l = 1, 2, \ldots, (m-2); \ r = 2, 4, 6, \ldots, (n-1).$$

Claim (2): To get edge-odd labeling to all the edges in $\{v_{(j-1)n+i}\ v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}$.

Define $f(v_{(j-1)n+i}v_{jn+i}) = [(2n+2)\ (i-1) + (2mn-2m+1) + 2(j-1)]$; where $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, (m-1)$.

Case (4) $m < n$. Both m and n are odd integers.

Claim (3): To get edge - odd labeling to all the edges in $\{v_i v_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase $(3a)$: Define $f(v_i v_{i+1}) = [(2n-2)l + r]$; where $i = ln + r$; i is odd; $l = 0, 1, 2, \ldots, (m-1)$; $r = 1, 3, 5, \ldots, (n-2)$; $i \neq 1$ to $(n-1)$, $[(m-1)n+1]$ to $(mn-1)$.

$f(v_i v_{i+1}) = (2mn-2m-2n+3) + (i-1)$; where $i = 1, 3, 5, \ldots, (n-2)$.

$f(v_i v_{i+1}) = [(2mn-m-n+1) - (i-2)]$; where $i = 2, 4, 6, \ldots, (n-1)$.

$f(v_{(m-1)n+i} v_{(m-1)n+i+1}) = i$; where $i = 1, 3, 5, \ldots, (n-2)$.

$f(v_{(m-1)n+i} v_{(m-1)n+i+1}) = [(2n-3) - (i-2)]$; where $i = 2, 4, 6, \ldots, (n-1)$.

Subcase $(3b)$: Define $f(v_i v_{i+1}) = [l(2n-2) + (2n-3) - (r-2)]$; where $i = ln + r$; i is even; $l = 1, 2, \ldots, (m-2)$; $r = 2, 4, 6, \ldots, (n-1)$.

Claim (4): To get edge-odd labeling to all the edges in $\{v_{(j-1)n+i}\ v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}$.

Subcase $(4a)$: Define $f(v_{(j-1)n+i} v_{jn+i}) = [(2mn-2m+1) + (i-1) + 2n(j-1)]$; where $i = 1, 3, 5, \ldots, n$ and $j = 1, 2, 3, \ldots, (m-1)$.

Subcase $(4b)$: Define $f(v_{(j-1)n+i} v_{jn+i}) = [(2mn-2m+2n-1) - (i-2) + 2n(j-1)]$ where $i = 2, 4, \ldots, (n-1)$ and $j = 1, 2, 3, \ldots, (m-1)$.

Case (5) $m > n$. Here m is even and n is odd positive integer.

Claim (1): To get edge odd labeling to all the edges in $\{v_i v_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase $(1a)$: Define $f(v_i v_{i+1}) = [(mn+1) + l(n-1) + (r-1)]$; where $i = ln + r$; i is odd; $l = 0, 1, 2, \ldots, (m-1)$; $r = 1, 3, 5, \ldots, (n-2)$; $i \neq [(m-1)n-2]$.

$$f(v_{((m-1)n-2)} v_{((m-1)n-1)}) = (3mn-3m+1).$$

Subcase $(1b)$: Define $f(v_i v_{i+1}) = [(2mn-m+1) + l(n-1) + (r-2)]$; where $i = ln + r$; i is even; $l = 0, 1, 2, \ldots, (m-1)$; $r = 2, 4, 6, \ldots, (n-1)$; $i \neq [(m-1)n-1]$.

$$f(v_{((m-1)n-1)} v_{(m-1)n}) = (2mn-2m+1).$$

Claim (2): To get edge-odd labeling to all the edges in $\{v_{(j-1)n+i}\ v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}$.

Subcase (2a): Define $f(v_{(j-1)n+i}v_{jn+i}) = [j+(n+1)(i-1)]$; where $i = 1, 2, \ldots, (n-2)$ and $j = 1, 3, 5, \ldots, (m-1)$; when $i \neq n-1$ and $j \neq 1$; when $i \neq n-1$ and $j \neq 3, 5, \ldots, (m-1)$; when $i \neq n$ and $j \neq 1, 2, 3, \ldots, (m-1)$.

$$f(v_{n-1}v_{2n-1}) = (3mn-2m-2n+n^2+2). \quad f(v_{(j-1)n+i}v_{jn+i}) = [j+(n+1)i];$$

where $i = (n-1)$ and $j = 3, 5, 7, \ldots, (m-1)$.

$f(v_{(j-1)n+i}v_{jn+i}) = [j+(n+1)(i-2)]$; where $i = n$ and $j = 1, 3, 5, \ldots, (m-1)$.

Subcase (2b): Define $f(v_{(j-1)n+i}v_{jn+i}) = [(3mn-2m+1)+(j-2)+(i-1)(n-1)]$; where $i = 1, 2, \ldots, (n-2)$; and $j = 2, 4, 6, \ldots, (m-2)$; when $i \neq n-1$ and $j \neq 2$; when $i \neq n-1$ and $j \neq 4, 6, \ldots, (m-1)$; when $i \neq n$ and $j \neq 2, 4, 6, \ldots, (m-1)$.

$f(v_{2n-1}v_{3n-1}) = [(n+1)(n-1)+1]$; $f(v_{(j-1)n+i}v_{jn+i}) = [(3mn-2m+1) + (j-2) + i(n-1)]$; where $i = n-1$ and $j = 4, 6, \ldots, (m-2)$; $f(v_{(j-1)n+i}v_{jn+i}) = [(3mn-2m+1) + (j-2) + (i-2)(n-1)]$; where $i = n$ and $j = 2, 4, \ldots, (m-2)$.

Case (6) $m < n$. Here m is even and n is odd positive integers.

Claim (3): To get edge-odd labeling to all the edges $\{v_iv_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase (3a): Define $f(v_iv_{i+1}) = [(mn+1)+l(n-1)+(r-1)]$; where $i = ln+r; i$ is odd; $l = 0, 1, 2, \ldots, (m-1)$; $r = 1, 3, 5, \ldots, (n-2)$;

Subcase (3b): Define $f(v_iv_{i+1}) = [(2mn-n+2) + l(n-1) + (r-2)]$; where $i = ln+r; i$ is even; $l = 0, 1, 2, \ldots, (m-1)$; $r = 2, 4, 6, \ldots, (n-1)$;

Claim (4): To get edge-odd labeling to all the edges in $\{v_{(j-1)n+i}\ v_{jn+i} :\ i = 1$ to $n, j = 1$ to $m\}$.

Subcase (4a): Define $f(v_{(j-1)n+i}v_{jn+i}) = [j+(n-1)(i-1)]$; where $i = 1, 2, 3, \ldots, n$. and $j = 1, 3, 5, \ldots, (m-1)$; when $i \neq n$ and $j \neq 1$; when $i \neq n$ and $j \neq (m-1)$.

$$f(v_nv_{2n}) = (mn-1); \quad f(v_{(m-1)n}v_{mn}) = (mn-m+1)$$

Subcase (4b): Define $f(v_{(j-1)n+i}v_{jn+i}) = [(3mn-2n+3)+(j-2)+(i-1)(n-3)]$

Case (7) $m < n$. Here m is odd and n is even integers.

Claim (1): To get edge odd labeling to all the edges in $\{v_iv_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase (1a): Define $f(v_iv_{i+1}) = [(3mn-2n-1)-ln-(r-1)]$; where $i = ln+r; i$ is odd; $l = 0, 1, 2, \ldots, (m-1)$; $r = 1, 3, 5, \ldots, (n-1)$; $i \neq 1$. $f(v_1v_2) = (n+1)$.

Subcase (1b): Define $f(v_i v_{i+1}) = [(3mn-2n+1)+l(n-2)+(r-2)]$; where $i = ln + r; i$ is even; $l = 0,1,2,\ldots,(m-1)$; $r = 2,4,6,\ldots,(n-2)$;

Claim (2): To get edge-odd labeling to all the edges in $\{v_{(j-1)n+i}\ v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}$.

Subcase (2a): Define $f(v_{(j-1)n+i} v_{jn+i}) = [i + n(j-1)]$; where $i = 1,3,5,\ldots,(n-1)$ and $j = 1,2,\ldots,(m-1)$;

Subcase (2b): Define $f(v_{(j-1)n+i} v_{jn+i}) = [(mn-n+1)+(i-2)+n(j-1)]$ where $i = 2,4,6,\ldots,n$; and $j = 1,2,3,\ldots,(m-1)$; when $i \neq 1$ and $j \neq 2$.

$$f(v_{n+1} v_{2n+1}) = (3mn-2n-1).$$

Case (8) $m > n$. Here m is odd and n is even integers.

Claim (3): To get edge odd labeling to all the edges in $\{v_i v_{i+1} : i \not\equiv 0 (mod\ n)\}$.

Subcase(3a):Define $f(v_i v_{i+1}) = [(3mn-2n-1)-ln-(r-1)]$; where $i = ln + r; i$ is odd; $l = 0,1,2,\ldots,(m-1)$; $r = 1,3,5,\ldots,(n-1)$; $i \neq 1, [(m-2)n+1]$.

$$f(v_1 v_2) = (2n+1). \quad f(v_{(m-2)n+1} v_{(m-2)n+2}) = (mn-2n+1).$$

Subcase (3b): Define $f(v_i v_{i+1}) = [(3mn-2n+1)+l(n-2)+(r-2)]$; where $i = ln + r; i$ is even; $l = 0,1,2,\ldots,(m-1); r = 2,4,6,\ldots,(n-2)$;

Claim (2): To get edge-odd labeling to all the edges in $\{v_{(j-1)n+i}\ v_{jn+i} : i = 1$ to $n, j = 1$ to $m\}$.

Subcase (4a): Define $f(v_{(j-1)n+i} v_{jn+i}) = [i + n(j-1)]$; where $i = 1,3,5,\ldots,(n-1)$ and $j = 1,2,3,\ldots,(m-1)$; when $i \neq 1$ and $j \neq 3$; when $i \neq 1$ and $j \neq (m-2)$.

$$f(v_{2n+1} v_{3n+1}) = (3nm-2n-1). \quad f(v_{(m-2)n+1} v_{(m-1)n+1}) = (2mn-1).$$

Subcase (4b): Define $f(v_{(j-1)n+i} v_{jn+i}) = [(mn-n+1)+(i-2)+n(j-1)]$ where $i = 2,4,6,\ldots,n$; and $j = 1,2,3,\ldots,(m-1)$.

All the above cases and all its subcases,are induced map $f^+ : V(G) \to \{1,2,3,\ldots,(2k-1)\}$ from f is defined by $f^+(x) \equiv f(xy)(mod\ 2k)$ where this sum runs over all edges through x. Thus the given graph has edge odd graceful labeling from the maps f and f^+. Hence for all positive odd integer $m \geq 5$ and positive even integer $n \geq 6$.

Acknowledgements

We would like to thank the anonymous referees for carefully reading an earlier version of this paper, and providing valuable remarks and suggestions that helped to improve its exposition.

References

[1] Mohammad Basher, "Odd-even graceful labeling of planar grid and prism graphs," *Journal of Information Optimization Science*, pp. 1–6 (2020).

[2] Gary Chartrand and Linda Lesniak, *Graphs and Digraphs*, 4th ed. Boca Raton, FL: Chapman and Hall/CRC (2005).

[3] Salem N. Daoud, "Edge odd graceful labeling of some path and cycle related graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 14, pp. 178–203 (2017).

[4] David Kuo and Jing-Ho Yan, "On L(2,1)-labelings of Cartesian products of paths and cycles," *Discrete Mathematics*, vol. 283, pp. 137–144 (2004).

[5] Shih-Pin Lo, "On edge-graceful labeling of graphs," *Congressus Numerantium*, vol. 50, pp. 231–241 (1985).

[6] A. Rosa, "On certain valuations of the vertices of a graph," in *Theory of Graphs*, Internat. Symp., Rome, Italy, July 1966, Gordan and Breach, New York and Paris, pp. 349–355 (1967).

[7] A. Solairaju and K. Chithra, "Edge-odd graceful labeling of some graphs," *Proceedings of the ICMCS Bulletin of the ICA*, vol. 1, pp. 101–107 (2008).

[8] A. Solairaju and D. Senthil Kumar, "Edge-Odd Gracefulness of Graphs [P3 ◊ Pn], [P3 ◊ 2Pn], [C3 ◊ Pn], [C3 ◊ 2Pn], [P3 ◊ Pn], [P3 ◊ 2Pn], [C3 ◊ Pn], and [C3 ◊ 2Pn]," *Global Journal of Pure and Applied Mathematics*, vol. 11, pp. 129–137 (2015).

[9] Joseph A. Gallian, "A Dynamic Survey of Graph Labeling," *The Electronic Journal of Combinatorics*, vol. 17 (2010).

Received February, 2022

Revised November, 2022