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A new parameter to enhance three-term conjugate gradient method with inexact line search

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Abstract

This Work presents a new parameter for the gradient of the three-term conjugate called (TTFGH) method, the 3T-CG algorithms are quite popular for solving large-scale unconstrained problems Due to their appealing practical qualities, such as their computational simplicity, Strong global convergence properties, better descent property, and reduced memory requirements. By employing Wolfe-Powell (sWP) line search condition, these newly developed methods always satisfy the original condition. The proposed approaches' global convergence characteristics are indicated by sWP, assuming Lipschitzian continuity of the objective function. Based on numerical results and preliminary comparisons with the of the modern methods, the proposed methods show that they are effective and promising.

Subject Classification: *(2010) 65K05, 90C06, 90C30, 90C52, 90C56.*

Keywords: *Conjugate gradient, Three term, Adequate descent property, (sWP) line search.*

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1. Introduction

The optimization model is a fundamental mathematical problem in a variety of fields, including economics, engineering, and physics. Conjugate gradient (CG) is the preferred method for solving large-scale unconstrained optimization problems because it is efficient, robust, and avoids the second derivative. This paper discusses nonlinear CG. The method has been tested for the following unconstrained optimizations:

$$\mathrm{Min}_{x\in R^n}\ \hbar\,(x) \tag{1}$$

The smooth function $\hbar: R^n \to \mathrm{R}$, with gradient $q_i = \nabla\hbar(x_i)$, is available [17,19]. CG methods create a sequence $\{x_k\}_{k\geq 0}$ from an initial guess $x_0 \in R$.

$$x_{i+1} = x_i + \partial_i d_i \tag{2}$$

Positive step size ∂_i is step-length, which is typically obtained through an inexact line search.

The CG method is well-known as an efficient iterative method for solving the unconstrained optimization problem [1]. Furthermore, d_i indicates the search direction:

$$d_{i+1} = \begin{cases} -q_{i+1}, \text{if } i = 0 \\ -q_{i+1} + \beta_i d_i, \text{if } i \geq 1 \end{cases} \tag{3}$$

$\beta_i > 0$ is a parameter. CG methods vary depending on the parameter β_i used. Fletcher-Reeves (FR) [12]; Hestenes-Stiefel (HS) [15], Conjugate Descent (CD) by Fletcher [11]; Dai-Yuan (DY) [8]; Polak-Ribiere-Polyak(PRP) [22,23]; and Hager-Zhang(HZ) [13]; are the most important CG methods proposed.

$$\beta_i^{FR} = \frac{q_{i+1}^T q_{i+1}}{q_i^T q_i};\ \beta_i^{HS} = \frac{q_{i+1}^T (q_{i+1}-q_i)}{d_i^T (q_{i+1}-q_i)};\ \beta_i^{CD} = -\frac{\|q_{i+1}\|^2}{d_i^T q_i}$$

$$\beta_i^{PRP} = \frac{q_{i+1}^T (q_{i+1}-q_i)}{\|q_i\|^2};\ \beta_i^{DY} = \frac{q_{i+1}^T q_{i+1}}{y_i^T d_i};\ \beta_i^{HZ} = (y_i - 2d_i \frac{\|y_i\|^2}{-q_i^T d_i}) \frac{q_{i+1}}{d_i^T y_i}.$$

such that $y_i = q_{i+1} - q_i$.These methods are identical when the objective function $\hbar$ is quadratic and exact line search is used [14]. However, their behavior differs for general objective functions.

Line searches can be exact or inexact. Exact line searches are time-consuming, computationally expensive, difficult, and require a significant amount of storage. Inexact line search techniques are popular because of their efficiency and global convergence properties. sWP techniques are among the most well-known inexact line search methods [21], this can be written as:

$$\hbar(x_i + \partial_i d_i) - \hbar(x_i) \leq e_1\, \partial_i q_i^T d_i \tag{4}$$

$$\left|q_{i+1}^T d_i\right| \leq e_2 \left|q_i^T d_i\right| \tag{5}$$

In which $0 < e_1 < 0.5 < e_2 < 1$.

The three-term CG method, introduced by Beale [6], is a significant computational innovation that addresses problem (1). The iterative scheme fulfills

$$d_{i+1} = -q_{i+1} + \beta_i d_i + \gamma_i d_t, \tag{6}$$

Where $\beta_i = \beta_i^{HS}$ $\left(\text{or } \beta_i^{FR}, \beta_i^{LS} \text{ etc}\right)$, and γ_i defined by:

$$\gamma_i = \begin{cases} 0, \text{if } i = t+1 \\ \frac{q_{i+1}^T y_t}{y_t^T d_t}, \text{if } i > t+1 \end{cases},$$

d_t is restart direction. Zhang et al. [28] proposed a modified PRP method with three terms, which are:

$$d_{i+1} = -q_{i+1} + \frac{q_{i+1}^T y_i}{q_i^T q_i} d_i - \frac{q_{i+1}^T d_i}{q_i^T q_i} y_i, i > 0, \quad (7)$$

For the sake of simplicity, we call it the 3TPRP method. One notable feature is that the search direction d_i satisfies $q_{i+1}^T d_{i+1} = -\|q_{i+1}\|^2$ for any k.

Then, Zhang et al. [29] proposed the 3THS method formula, which is written as:

$$d_{i+1} = -q_{i+1} + \frac{q_{i+1}^T y_i}{y_i^T d_i} d_i - \frac{q_{i+1}^T d_i}{y_i^T d_i} d_i, i > 0 \quad (8)$$

When an exact line (ELS) is used, the 3THS method is shortened to the "classic -HS-method." Then Zhang et al. [27] presented a novel Dai-Liao-based 3T-CG method:

$$d_{i+1} = -q_{i+1} + \frac{{q_{i+1}}^T (y_i - \tau s_i)}{s_i^T y_i} s_i - \frac{q_{i+1}^T s_i}{s_i^T y_i} (y_i - \tau s_i), \tau \geq 0 \quad (9)$$

The sufficient descent process repeats, regardless of the line search method used. Baluch et al. [5] extended the approach by proposing a (3T-CG) method. Calculate the search direction (11) using β_i^{BZAU} & ϑ_i^{BZAU}, such that:

$$d_{i+1} = -q_{i+1} + \beta_i^{BZAU} d_i - \vartheta_i^{BZAU} y_i, i \geq 1, \quad (10)$$

Where $\beta_i^{BZAU} = \frac{q_{i+1}^T (q_{i+1} - q_i)}{-\aleph q_i^T d_i + \text{ш} |q_i^T d_{i+1}|}$, $\vartheta_i^{BZAU} = \frac{q_{i+1}^T d_i}{-\aleph q_i^T d_i + \text{ш} |q_i^T d_{i+1}|}$,

for $\aleph \in [1, +\infty), \text{ш} \in [\aleph, +\infty)$. The TTBZAU method uses sWP line search to check the global convergence property of convex and non-convex functions. Furthermore, regardless of the line search used, this method has been shown to be suitable for descent. Fadhilah et al. [10] propose the TTNRMIL method for exact line searches. In addition to traditional RMIL. The new search direction (3TCG) is used; the strategy's search direction is

$$d_{i+1}^{3TNRMIL} = -q_{i+1} + \beta_i^{RMIL} d_i - \beta_i^{RMIL} \vartheta_i d_i, i \geq 1, \quad (11)$$

Recently suggested, Idalisa et al. [16], (TTCG) method of RMIL-CG-method, the suggested technique is called TTRMIL method, and the strategy's search direction is

$$d_{i+1}^{NEWRMIL} = -q_{i+1} + \beta_i^{RMIL} d_i + \frac{q_{i+1}^T d_i}{d_i^T (d_i - q_{i+1})} q_i, i \geq 1. \quad (12)$$

2. A new ttfgh method and its algorithm

This section explains how to derive a new parameter, β_i^{TTFGH}, and the proposed algorithm, which uses inexact line search. Our proposed three-term CG, denoted by TTFGH, defines the search direction as:

$$d_{i+1} = -q_{i+1} + \beta_i^{TTFGH} d_i - \vartheta_i y_i \quad (13)$$

We used ϑ_i, proposed by Sulaiman et al. [25], and defined as follows:

$$\vartheta_i = \frac{q_{i+1}^T q_i}{\|q_i\|^2} \tag{14}$$

Now, multiply both sides of equation (13) by y_i to get:

$$d_{i+1}^T y_i = -q_{i+1}^T y_i + \beta_i^{TTFGH} d_i^T y_i - \vartheta_i y_i^T y_i , \tag{15}$$

Dai and Liao [26] proposed this conjugacy condition for an inexact line search.

$$d_{i+1}^T y_i = -\varsigma\, \partial_i q_{i+1}^T d_i, \text{ where } \varsigma > 0 \tag{16}$$

Use (16) in (15), We get:

$$-\varsigma\, \partial_i q_{i+1}^T d_i = -q_{i+1}^T y_i + \beta_i^{TTFGH} d_i^T y_i - \vartheta_i \|y_i\|^2$$

Then after doing some arithmetic operations:

$$\beta_i^{TTFGH} = \frac{q_{i+1}^T y_i - \varsigma\, \partial_i q_{i+1}^T d_i + \vartheta_i \|y_i\|^2}{d_i^T y_i},\ \vartheta_i = \frac{q_{i+1}^T q_i}{\|q_i\|^2}, \varsigma = 0.1 \tag{17}$$

2.1 *Algorithem (TTFGH method with SWP)*

- **Step1**: Data $x_0 \in R\ n\ , e \geq 0$, compute $\hbar(x_0)$ when $i = 0$, and,

$$q_0 = -\nabla\hbar(x_0), d_0 = -q_0$$

- **Step 2:** If $\|q_i\| \leq e$, then stop; If not, move on to the next step.
- **Step 3:** Calculate a step length ∂_i using sWP described in (4) and (5).
- **Step 4:** Calculate $x_{i+1} = x_i + \partial_i d_i$, $q_{i+1} = q(x_{i+1})$, if $\|q_{i+1}\| \leq e$, stop.
- **Step 5:** Calculate d_{i+1} based on formula (15), β_i based on formula (17), and ϑ_i by (14).
- **Step 6:** if $|q_{i+1}^T q_i| \geq 0.2\|q_{i+1}\|^2$, is satisfied, set $d_{i+1} = -q_{i+1}$, otherwise go to next step.
- **Step 7:** set $i = i + 1$ proceed to 'step 2'.

3. The sufficiengt descent property and the global convergence

In this section, we will use the sWP line search to demonstrate the algorithm's global convergence and sufficient descent properties, Najm, H. Y et al. [20]. The following assumptions are needed.

3.1 *Assumption*

The level was set $\mathfrak{d} = \{x \in R^n; \hbar(x) \leq \hbar(x_0)\}$, has a bound, that $z > 0$, exists, such that $\|x\| \leq z, \forall x \in \mathfrak{d}$.

3.2 Assumption

In certain neighbourhoods, Since $\hbar$ is a continuously differentiable objective function & has a Lipschitz continuous gradient, there is a value Ł>0, which means that:

$$\|\nabla\hbar(x) - \nabla\hbar(y)\| \leq Ł\|x - y\|, \forall\, x, y \in M \tag{18}$$

According to these presumptions, a positive constant $(\aleph, \bar{\aleph}, v \,\&\, \bar{v})$ exists, such that

$$\aleph \le q_{i+1} \le \bar{\aleph}, v \le q_i \le \bar{v} \text{ and } \|d_i\| \le \tau, . \forall x \in ƀ , \tag{19}$$

Al-Naemi [2].

Sufficient descent is a crucial condition for global convergence of CG methods, facilitating a more structured convergence.

Sufficient descent is a key condition for global convergence of CG methods, facilitating the convergence structure. The algorithm generates a search direction that meets the sufficient descent condition (c > 0).

$$q_{i+1}^T d_{i+1} \le -\mathcal{P}\|q_{i+1}\|^2, \forall i \ge 0. \tag{20}$$

Theorem 3.1: *Suppose that the line search satisfies sWP conditions (4) and (5). Then d_{i+1} given by (13) and (17) satisfies a sufficient condition (20).*

Proof: Induction is used to prove the proposition. When $i = 0$ is met, $d_0^T q_0 = \|q_0\|^2 \le -\mathcal{P}\|q_0\|^2$. As a result, condition (20) is valid. The next step is to demonstrate that condition (20) also true for $k \ge 1$.

Multiply (13) by q_{i+1}^T , yielding

$$q_{i+1}^T d_{i+1} = -\|q_{i+1}\|^2 + \beta_i^{TTFG} q_{i+1}^T d_i - \vartheta_i q_{i+1}^T y_i, \tag{21}$$

Now substituting (17) into (21), we obtain

$$q_{i+1}^T d_{i+1} = -\|q_{i+1}\|^2 + \frac{q_{i+1}^T y_i - \varsigma\, \partial_i q_{i+1}^T d_i + \vartheta_i \|y_i\|^2}{d_i^T y_i} q_{i+1}^T d_i - \frac{q_{i+1}^T q_i}{\|q_i\|^2} q_{i+1}^T y_i$$

By Powell restart criteria, we have

$$q_{i+1}^T y_i = \|q_{i+1}\|^2 - q_{i+1}^T q_i \le 1.2\|q_{i+1}\|^2 \tag{22}$$

By using one side of (5) with $q_i^T d_i \le -\mathcal{P}\|q_i\|^2$:

$$q_{i+1}^T d_i \le -e_2 \mathcal{P}\|q_i\|^2 \tag{23}$$

$$q_{i+1}^T d_{i+1} \le -\|q_{i+1}\|^2 + \frac{1.2\|q_{i+1}\|^2 - \varsigma\, \partial_i e_2 \mathcal{P}\|q_i\|^2 - \frac{0.2\|q_{i+1}\|^2}{\|q_i\|^2}\|y_i\|^2}{d_i^T y_i} d_i^T y_i - \frac{0.2\|q_{i+1}\|^2}{\|q_i\|^2} q_{i+1}^T y_i$$

Using Lipschitz condition in (18) , $s_i = \partial_i d_i$ and (19), we get

$$\|y_i\| \le Ł\|s_i\| = \partial_i Ł\|d_i\| \tag{24}$$

$$q_{i+1}^T d_{i+1} \le 0.2\|q_{i+1}\|^2 - \varsigma\, \partial_i e_2 \mathcal{P}\, \frac{\|q_i\|^2\|q_{i+1}\|^2}{\|q_{i+1}\|^2} - \frac{0.2\|q_{i+1}\|^2}{\|q_i\|^2}\partial_i Ł\|d_i\| + \frac{0.24\|q_{i+1}\|^2}{\|q_i\|^2}\|q_{i+1}\|^2$$

$$= -\left[0.2 + \varsigma\, \partial_i e_2 \mathcal{P}\, \frac{\|q_i\|^2}{\|q_{i+1}\|^2} + \frac{0.2\, \partial_i Ł\|d_i\|}{\|q_i\|^2} - \frac{0.24\|q_{i+1}\|^2}{\|q_i\|^2}\right]\|q_{i+1}\|^2 ,$$

Using (19), we get:

$$\frac{\|q_i\|^2}{\|q_{i+1}\|^2} \le \frac{\bar{v}^2}{\aleph^2} = \omega, \varsigma = 0.1, 0 < e_2 < 1, \& \ 0 \le \partial_i \le 1. \tag{25}$$

$$q_{i+1}^T d_{i+1} \le -\left[0.2 + 0.1\, \partial_i e_2 \mathcal{P}\omega + \frac{0.2\, \partial_i Ł\tau}{v^2} - \frac{0.24}{\omega}\right]$$

$\mathrm{q}_{\mathrm{i}+1}^{\mathrm{T}}\mathrm{d}_{\mathrm{i}+1} \leq -\mathcal{P}\|\mathrm{q}_{\mathrm{i}+1}\|^2$ with $\mathcal{P} > 0$.

This indicates that $\mathrm{d}_{\mathrm{i}+1}$ is an appropriate descent direction. As a result $\mathrm{q}_{\mathrm{i}+1}^{\mathrm{T}}\mathrm{d}_{\mathrm{i}+1} \leq -\mathcal{P}\|\mathrm{q}_{\mathrm{i}+1}\|^2$ is correct. The proof is finished.

The Zoutendijk condition [30] is critical for analyzing the CG method's global convergence properties in the next Lemma

Lemma 3.1: *Assume that Assumption 3.1 and 3.2 are true. The direction* d_{i+1} *generated by (13), and (17),* ∂_i *is obtained from the (sWP) (4) and (5). If*

$$\sum_{\mathrm{i}\geq 1}\frac{1}{\|\mathrm{d}_{\mathrm{i}+1}\|^2} = \infty. \tag{26}$$

Then

$$\lim_{\mathrm{i}\to\infty}\inf\|\mathrm{q}_{\mathrm{i}+1}\| = 0. \tag{27}$$

Theorem 3.2: *Assume that assumptions 3.1 & 3.2 are valid. The TTFG method can generate the sequences* $\{\mathrm{q}_{i+1}\}$ *and* $\{d_{i+1}\}$*, where* ∂_i *satisfies the sWP. Then (26) is valid.*

Proof: We have $\mathrm{d}_{i+1} \neq 0$ if we use lemma (3.1) and the descent condition holds, It must be demonstrated that $\|\mathrm{d}_{i+1}\|$ bounded above. From (13) and (17)

$$\|\mathrm{d}_{i+1}\| = \left\|-\mathrm{q}_{i+1} + \frac{\mathrm{q}_{i+1}^T y_i - \varsigma\partial_i \mathrm{q}_{i+1}^T d_i + \vartheta_i\|y_i\|^2}{y_i^T d_i}\mathrm{d}_i - \frac{\mathrm{q}_{i+1}^T \mathrm{q}_i}{\|\mathrm{q}_i\|^2}y_i\right\|$$

$$\leq \|\mathrm{q}_{i+1}\| + \frac{\|\mathrm{q}_{i+1}\|.\|y_i\| + |\varsigma|.|\partial_i|.\|\mathrm{q}_{i+1}\|.\|d_i\| + \frac{\|\mathrm{q}_{i+1}\|.\|\mathrm{q}_i\|}{\|\mathrm{q}_{i+1}\|^2}\|y_i\|}{\|y_i\|\|d_i\|}\|d_i\| + \frac{\|\mathrm{q}_{i+1}\|.\|\mathrm{q}_i\|}{\|\mathrm{q}_i\|^2}\|y_i\|$$

$$= 2\|\mathrm{q}_{i+1}\| + |\varsigma||\partial_i|\frac{\|\mathrm{q}_{i+1}\|}{\|y_i\|}\|d_i\| + \frac{\|\mathrm{q}_i\|}{\|\mathrm{q}_{i+1}\|} + \frac{\|\mathrm{q}_{i+1}\|}{\|\mathrm{q}_i\|}\|y_i\|$$

Using (25), we get

$$\|\mathrm{d}_{i+1}\| \leq (2 + 0.1\partial_i\frac{\overline{\aleph}^2}{\sigma}\tau + 2\frac{\overline{\vee}}{\aleph})\|\mathrm{q}_{i+1}\| \leq C\overline{\aleph} = R$$

$$\sum_{i\geq 1}\frac{1}{\|\mathrm{d}_{i+1}\|^2} \geq \frac{1}{R^2}\sum_{i\geq 1}1 = +\infty.$$

This is a contradiction. As a result, equation (27) holds true.

4. Result and discussion

We present the results numerical of a group of (30) unrestricted test problems for nonlinear optimization in this part. from the CUTE library [3], with extensive optimization problems introduced by Andrei [4] and Bongartz [7].

For dimensions N = 100, 1000 &10000, each test function is executed three times, all employed algorithms sWP line search terms (4) - (5) with $e_1 = 0.001$ and $e_2 = 0.9$.

In the experiments, we employ gradient errors to determine the level of the responses, so that we can easily compare them to other codes; in other words, the iteration is forced to end when the gradient errors reach a certain threshold.

$$\|\mathrm{q}_{\mathrm{i}}\| \leq 10^{-6} \tag{28}$$

Where q_i is the objective function's gradient. TNOF is the total sum of the number of instances in which the function has been evaluated. The total sum of number of iteration evaluations is represented by the TNOI. To solve these test functions, we use the following algorithms:

(1) RMIL: where RMIL denotes Rivaie, Mustafa, Ismail and Leong [24].

(2) Liu & Wu: where Liu & Wu method denotes, Jinkui Liua and Xuesha Wu [18].

(3) 3TNRMIL: the three-term version of RMIL termed as 3TNRMILproposed in Fadhilah N et al. [10].

(4) ttRMIL: the three-term version of RMIL methods termed as,ttRMIL proposed in Idalisa et al. [16].

(5) TTFGH: Our proposed algorithm.

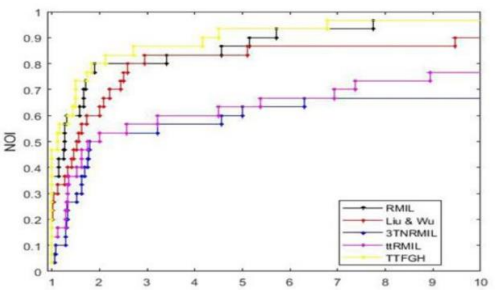
All algorithms are implemented in FORTRAN. to assess the TTFGH algorithm's numerical performance, the 1, 2, 3, and 4 algorithms are also tested as a comparison group. Table 1 displays the results and We can see that the performance improvement of the new algorithm is as shown in Figure 1, Dolan-More [9].

Table 1
Comparison of test results for five algorithms

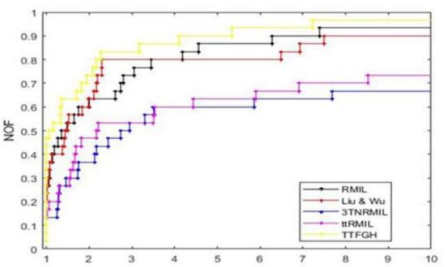
Test Function	Two-terms RMIL		Three-terms Liu & Wu		Three-terms 3TNRMIL		Three-terms ttRMIL		Three-terms TTFGH	
	TNOI	TNOF	TNOI	TNOF	TNOI	TNOF	TNOI	TNOF	TNOI	TNOF
Beal	96	187	76	177	1454	2911	1500	3048	85	206
Cubic	48	132	106	262	1500	1015	1307	2770	102	255
Diagonal4	6	18	12	27	30	63	27	63	27	57
Dixmaani	1121	3563	1523	3052	1500	3750	1500	3759	982	2986
EX Freudenstein & Roth	24	63	60	138	1500	3006	1500	4038	163	336
Ex Quadratic PenaltyQP1	36	9409	547	1471	38	132	19	40	33	289
Ex Three Exponential	68	978	568	1169	108	381	537	1330	60	156
Fred	30	73	153	506	1500	3879	1500	3546	657	1593
Powell3	651	1119	1500	3221	719	1442	840	1701	114	246
Quartc	3	12	4	26	4	26	6	26	4	26
Rosen	187	328	141	329	1500	3083	602	1345	112	303
Scaled Quadratic SQ1	1403	2809	1138	2297	1500	1003	1500	3552	1131	2281
Wolfe	1229	1461	239	581	767	1537	611	1236	276	561
Wood	1619	3635	1030	2089	1800	3614	1800	3783	1484	2835
Dixmaana	28	56	518	1541	22	52	22	53	22	52
SUMM	77	408	84	426	99	646	80	506	59	303
Recip	18	54	44	124	245	749	297	744	75	221
QuadraticFunctio	1939	3887	1193	2374	1500	3006	1500	3649	1169	2350

(QF1)										
QuadraticFunctio((QF2)	1458	9929	1164	2532	1500	3012	1500	3042	1160	2376
Perturbed Quadratic(PQ)	1945	2921	1172	2297	1500	3003	1500	3721	1142	2287
OSP	1330	8149	1169	4062	1255	4673	1250	4471	773	2665
Hager	1070	2175	608	1906	1069	3806	754	3142	235	294
Ex.Rosen	87	228	141	329	1500	3084	602	1345	130	303
Ex Wood	925	3072	1027	2081	1500	3015	1500	3182	1385	2779
Diagonal2	121	522	12	27	1245	4985	1245	4986	22	46
Ex EP1	12	30	12	30	13	30	12	30	12	30
Ex Beale (Matrix Rom)	133	287	115	222	1022	2063	39	105	106	219
Ex Block Diagonal	72	180	105	216	76	158	80	166	61	168
Ex Himmelbau	40	1031	69	946	71	1522	53	518	60	939
Ex Penalty	186	362	50	118	534	1082	523	1051	24	405

We can see from Table 1 are listed results numerical compared with several methods are listed.



(a) TNOI 'Dolan-More' performance profile



(b) TNOF 'Dolan-More' performance profile

Figures 1
Display the performance of all algorithms in solving unconstrained optimization problems

Conclusion

In this research, we suggest a new way to solve a math problem called the TTFGH method. It helps us find the right direction to solve the problem using three different steps. With sWP line search, it is demonstrated that the proposed directions always satisfy the adequate descent property. We demonstrate convergence properties of new schemes under standard assumptions. The preliminary numerical experiment on a subset of the test function collection shows that the new algorithms work.

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