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Radius, diameter, domination number, order and minimum degree

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Abstract

Consider a simple graph G which is connected with given order, diameter, radius and minimum degree. Using set-theory and standard results, tight upper bounds on the domination number $\gamma(G)$ based on the aforementioned graph parameters are proven. The results bring to the literature new bounds on the domination number, apart from strengthening results by Henning and Mukwembi [Domination, radius and minimum degree, *Discrete Applied Mathematics*, **157** (2009), 2964-2968]. We show by construction that the results hold even if the graphs are highly connected. We consider only simple graphs.

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Keywords: *Domination, Radius, Diameter.*

1. Introduction

We examine a graph $G=(V,E)$ with vertex set $V(G)$ and edge set $E(G)$ such that for all $x,y \in V(G)$, there is an $x-y$ path. For most of the definitions, notations and terminologies used in this paper, we refer the reader to [3, 6, 18, 21, 22, 23]. For $v \in V(G)$, the number of edges incident with v in G is its *degree*, $\deg_G(v)$. The *minimum degree* of G, $\delta(G)$, is given by $\delta(G) = \min\{\deg(v) : v \in V(G)\}$ [1]. If $\deg_G(v)=\delta$ for all $v \in V(G)$, then G is called a δ*-regular graph*. Let $S = \{l_i \mid l_i$ is the length of a $u-v$ path in G$\}$. Then, the *distance* $d_G(u,v)$ between u and v is the smallest element of S.

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The *eccentricity* of $v \in V(G)$, $ecc_G(v)$, is defined as $ecc_G(v) = \max\{d_G(u,v) : u \in V(G)\}$ [2]. The radius $\mathrm{rad}(G)$ and diameter $\mathrm{diam}(G)$ of G are given by $\mathrm{rad}(G) = \min\{ecc_G(u) : u \in V(G)\}$ and $\mathrm{diam}(G) = \max\{ecc_G(u) : u \in V(G)\}$, respectively. For a positive integer l, if C_l is a cycle with l edges and G does not contain C_l, then G is a C_l*-free*. A *matching* is a set of pairwise disjoint edges[3]. A set $S \subseteq V(G)$ is *dominating* if $\forall v \in V(G)$, $v \in S$ or there is $u \in S$ such that $uv \in E(G)$. The *domination number*, $\gamma(G)$ of G is given as $\gamma(G) = \min\limits_{S \subseteq V(G)} \{|S| : S \text{ is a dominating set of } G\}$ [4]. If S is connected, then the *connected domination number*, $\gamma_c(G)$ of G is defined as $\gamma_c(G) = \min\limits_{S \subseteq V(G)} \{|S| : S \text{ is a connected dominating set of } G\}$ [5]. The relationship $\gamma(G) \le \gamma_c(G)$ is well known in the literature, (see for instance [3]).

Given $v \in V(G)$, the *open-neighbourhood*, $N_G(v)$, of v in G is $N_G(v) = \{u \in V(G) : d_G(u,v) = 1\}$ [6] and the *closed-neighbourhood*, $N_G[v]$, is given by $N_G[v] = \{v\} \cup N_G(v)$, see also [6]. Let $A \subseteq V(G)$, we write $d_G(u,A) = i$ if there exist $v \in A$ such that $d_G(u,v) = i$ and $d_G(u,y) \ge i$ for all $y \in A$. The set $N(A) = \bigcup\limits_{x \in A} N(x)$. For a positive integer k, Danklemann-Entringer [6] defined a k-packing of G to be a subset $A \subseteq V(G)$ with $d_G(a,b) > k \ \forall a,b \in A$. By $G - \{e\}$ we mean the graph G minus the edge e. If H is a subgraph of G, we write, $H \le G$. $V(G-H)$ denotes the set of vertices in G which are not in H. Where there is no ambiguity, we drop the argument G.

The study on the radius and diameter of a graph in terms of other graph parameters has a long history and the research continues in the literature, see for instance [2, 4, 5, 8, 12, 13, 14, 21, 25, 26, 27, 30] and a survey [7]. Brilliant ideas by Erdös et. al [13] and the Danklemann-Entringer [6] method are some of the powerful tools employed to produce asymptotically tight bounds on the radius and diameter of a graph. Other methods involving the idea of starting with a diametrical path of G also in most cases yielded asymptotically sharp bounds in the aforementioned parameters.

It is known in the literature that for any spanning tree T of a graph G,

Fact 1: $\mathrm{diam}(T) \ge 2\mathrm{rad}(G) - 1$, *(see for example [9, 18, 20, 22]).*

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Recently, a new technique in the establishment of sharp bounds on the radius and diameter in graphs has been presented in [22] by employing Fact 1 together with the Danklemann-Entringer [6] method. Precisely, the following corollary is derived from [22]:

Corollary 1: *[22] If G is connected with radius* $\operatorname{rad}(G) = r$ *and A is a maximal* 2*-packing of G, then from a forest F with* $V(F) = N[A]$ *and* $E(F)$ *consist of all edges in G incident to a vertex in A, there are* $|A|-1$ *edges found in G, each of them joining two neighbours of distinct vertices of A, such that the attachment of them to F produces a tree* $T' \leq G$ *such that* $\operatorname{diam}(T') \geq 2r-3$.

The domination number of a graph has been studied by several authors, see for instance [1, 16, 18, 29, 31]. For details on application of the domination number, we refer the reader to [16, 17]. The study of domination in graphs is an on going process, see for instance recent research in [10, 11, 19, 22]. Recently, Mahadevan and Vijayalakshmi [24] introduced the clone hop domination number, $CHD(G)$, after the hop domination number, $\gamma_h(G)$ [28]. It is interesting to continue studying on bounds of these domination parameters. Here, we investigate on the bounds for the domination number by applying set theory in conjunction with Fact 1, Corollary 1 and the Danklemann-Entringer [6] method to strengthen results in [18]. Specifically, we improve the following results:

Theorem 1: [18] *For a graph G which is connected with* $\operatorname{rad}(G) = r \geq 1$, *it follows that* $\gamma(G) \geq \frac{1}{3}(\operatorname{diam}(G)+1)$ *and* $\frac{2}{3}r \leq \gamma(G) \leq n - \frac{2}{3}(2r-1)$.

Theorem 2: [18] *Consider a graph G being connected with* $\operatorname{rad}(G) = r \geq 6$ *and* $\delta = \delta(G) \geq 3$. *Then* $\gamma(G) \leq n - \frac{2}{3}(r-6)\delta$ *and apart from additive constant multiples of* δ, *the bound is tight.*

The lower bound on the radius in Theorem 1 was also proved independently in [9]. Since $\gamma(G) \leq \gamma_c(G)$, we observed here that results in [22], for $\gamma(G)$, are an improvement of Theorem 2. This is so because, they imply that for graphs in general, $\gamma(G) \leq n - \frac{2}{3}(r-1)\delta + \frac{4}{3}r - \frac{10}{3}$, $\gamma(G) \leq \leq n - \frac{2}{3}\left(r - \frac{1}{2}\right)\delta + \frac{4}{3}r - \frac{8}{3}$ for $r \equiv 2 \text{ modulo } 3$ and $\gamma(G) \leq n - \frac{2}{3}r\delta + \frac{4}{3}r - 2$ for $r \equiv 0 \text{ modulo } 3$. It can easily be seen that although the results in [22], except one, are sharp for $\gamma_c(G)$, the constructions therein do not yield tight bounds for $\gamma(G)$. The results in this paper fill this gap also by providing sharp bounds for $\gamma(G)$. Similar arguments hold for C_3-free graphs.

For some $\delta \geq 5$, in Section 4 of the paper [15] by Griggs and Wu, highly connected δ-regular graphs were constructed to show that some bounds on the connected domination number are sharp even if we improve the connectivity. That is, such bounds remain valid and sound even if the connectivity is improved. However, the case for $\delta = 3$ was different in such a way that if you improve the connectivity, then the bound for connected domination number improves. In this paper, for $r \equiv 0$ modulo 3 and $\delta \equiv 2$ modulo 3, we give a highly connected family of graphs $H_{\delta,r}$, which is a sub-family of the aforementioned graphs in [15] and is such that it attains the bound in Theorem 2, apart from multiples of the coefficients of δ. In addition, the family $H_{\delta,r}$ attains the bounds in this paper even if we include multiples of the coefficients of δ. Thus for graphs in general, the results of this paper are also sharp even if connectivity is improved. We also strengthen theorems 1 and 2 to triangle-free graphs.

Further, we provide several other families that attain the bounds of this paper. The results of this paper also imply new sharp lower bounds for the radius and diameter based on the domination number.

2. Main Results

We start by strengthening Theorem 2 for graphs in general. For all $\delta > 2$ the result strengthen Theorem 1 also.

Theorem 3: *If G is connected with* $\mathrm{rad}(G) = r$ *and* $\delta \geq 2$, *then* $\gamma(G) \leq n - \frac{1}{3}(\mathrm{diam}(G)+1)\delta$ *and* $\gamma(G) \leq n - \frac{2}{3}\left(r - \frac{1}{2}\right)\delta$. *Moreover,* $\gamma(G) \leq n - \frac{2}{3} r\delta$ *for* $r \equiv 0$ modulo 3. *All bounds are sharp.*

Proof: To establish the first bound, let $P(u_0, u_d) = u_0, u_1, u_2, \cdots, u_d$ be the diametrical path of G and $A_0 = \{u_{3i} : 0 \leq i \leq \frac{d}{3}\}$. Then $|A_0| \geq \frac{1}{3}(\mathrm{diam}(G)+1)$. Thus, $V(G) \setminus N(A_0)$ dominates G. So, $\gamma(G) \leq n - |N(A_0)|$ and the result follows, since $u_i u_j \notin E(G)$ and $N(u_i) \cap N(u_j) = \varnothing$ for all $u_i, u_j \in A_0$ with $i \neq j$.

To settle the second bound, we note first that for $\delta = 2$, Theorem 1 produces the result. Consider $\delta \geq 3$ and apply the Danklemann-Entringer technique. Choose a center vertex $v \in V(G)$ and define $A = \{v\}$. If there exist $u \in V(G)$ such that $d_G(u, A) = 3$, add u to A. Repeat the procedure of adding vertices u to A such that $d_G(u, A) = 3$, till every vertex in $V(G) \setminus A$ is a distance at most 2 from A. Let F be the forest with $V(F) = N[A]$ and

$E(F)$ being the set of all edges $e \in E(G)$ such that e is incident to a vertex in A. By the construction of A and the Danklemann-Entringer technique, we add to F, $|A|-1$ suitable edges of $E(G)$, to form a tree, see also Corollary 1. Let $T' \leq G$, be the tree formed such that T' has the smallest diameter amongst all subtree of G that can be constructed by this operation.

In addition, let T' be such that each vertex in A is distance preserving from the center vertex. Now by the construction of T', each vertex in $V(G-T')$ has a neighbour in T'. Define $T \leq G$ to be a tree formed by attaching each vertex in $V(G-T')$ to one of its neighbours in T'. It follows that

$$3|A| \geq \mathrm{diam}(T')+1 \text{ and } \mathrm{diam}(T') \leq \mathrm{diam}(T) \leq \mathrm{diam}(T')+2.$$

Let $P' = v_1, v_2, v_3, \cdots, v_{d-1}$ be a path such that $\mathrm{diam}(T') = |E(P')|$. Then

$$\gamma(G) \leq n - |N(A)| \leq n - \frac{1}{3}(diam(T')+1)\delta.$$

So, the theorem holds when $\mathrm{diam}(T') \geq 2r-2$. Note also that if there exist a vertex of A which is a distance at least 3 from the diametrical path of T', then

$$|A| \geq \frac{1}{3}(\mathrm{diam}(T')+1)+1.$$

Therefore, the result follows, since $\mathrm{diam}(T') \geq 2r-3$, see Corollary 1.

Consider each element of $V(G)$ being a distance at most 2 from P' and that $|E(P')| = 2r-3$. Let v_0 and v_d be vertices in G such that $d_G(v_{r-1}, v_d) = r$ and $d_G(v_0, v_r) = r$. Evidently, $v_0, v_d \in V(G-T')$ and there exist $x \in N(v_2) \setminus \{v_3\}$ and $y \in N(v_{d-2}) \setminus \{v_{d-3}\}$ such that $v_0x, yv_d \in E(G)$. Set $x = v_1$ and $y = v_{d-1}$, other sub-cases are treated likewise. Then $P = P' \cup \{v_0v_1, v_{d-1}v_d\}$ is a diametrical path of T. If either v_0 or v_d has at least $\frac{1}{3}\delta$ neighbours not in T', then

$$\gamma(G) \leq n - |N(A)| - \frac{1}{3}\delta \leq n - \frac{2}{3}\left(r - \frac{1}{2}\right)\delta.$$

Assume neither v_0 nor v_d has greater than or equal to $\frac{1}{3}\delta$ neighbours in $V(G-T')$. We note first that common neighbours of v_0 and v_d can only be found in $V(G-T')$ if they exist, otherwise, we contradict the fact that $d_G(v_0, v_r) = r$ or $d_G(v_{r-1}, v_d) = r$. Thus by our assumption, $|N_G(v_0) \cap N_G(v_d)| < \frac{1}{3}\delta$. Let $B = \{v_0, v_{3i}, v_d : 1 \leq i \leq \frac{1}{3}(d-4)\}$. Then $B \setminus \{v_0, v_d\}$ is an independent set and none of its elements share a neigbhour, otherwise, we find a a spanning tree of G of diameter less than $2r-1$ thereby contradicting

Fact 1 or we contradict the fact that $d_G(v_0,v_r) = r = d_G(v_{r-1},v_d)$. By the same argument, for all vertices in B, only v_0 and v_d can possibly be adjacent or share a neighbour in G. Thus,

$$\begin{aligned}|N(B)| &= (|N(v_0)|-1)+|N(v_3)|+|N(v_6)|+\cdots+|N(v_{d-4})|+(|N(v_d)|-1)\\ &\quad -|N(v_0)\cap N(v_d)|\\ &> \tfrac{2r-5}{3}\delta+2\delta-2-\tfrac{1}{3}\delta\\ &= \tfrac{2}{3}r\delta-2.\end{aligned}$$

Now $\gamma(G)\le n-|N(B)|$ and the result follows for all $\delta\ge 6$. For $\delta=3$, $N(v_0)\cap N(v_d)=\varnothing$ and the result follows by the previous argument. For $\delta=4$ and $\delta=5$, $|N(v_0)\cap N(v_d)|\le 1$. So, the result holds for $\delta=5$. Consider $\delta=4$, then the result holds when $|N(v_0)\cap N(v_d)|=0$. Assume that $|N(v_0)\cap N(v_d)|=1$. Then $v_0v_d\notin E(G)$, otherwise v_0 has $2\ge\frac{1}{3}\delta$ neighbours in $V(G-T')$ and we are already done. Now

$$\begin{aligned}|N(B)| &= |N(v_0)|+|N(v_3)|+|N(v_6)|+\cdots+|N(v_{d-4})|+|N(v_d)|\\ &\quad -|N(v_0)\cap N(v_d)|\\ &\ge \tfrac{2r-5}{3}\delta+2\delta-1\\ &= \tfrac{2r+1}{3}\delta-1\\ &= 4\tfrac{(2r+1)}{3}-1\\ &> \tfrac{8}{3}r-\tfrac{4}{3} \text{ as desired.}\end{aligned}$$

Therefore, the second bound holds for all $\delta\ge 2$.

To establish the third bound, we note first that $\operatorname{diam}(T')\equiv 2 \text{ modulo } 3$, by our construction. Hence, for $r\equiv 0 \text{ modulo } 3$, $\operatorname{diam}(T')\ne 2r-3$ and $\operatorname{diam}(T')\ne 2r-2$, otherwise, we contradict the fact that $\operatorname{diam}(T')\equiv 2 \text{ modulo } 3$. Hence by following Fact 1, $\operatorname{diam}(T')\ge 2r-1$ and the result follows.

For each even $\delta\ge 2$, the graph $G(\delta,\delta+2)$ of order $\delta+2$ with every vertex of degree δ shows that the first and second bounds are sharp. Also, look at the following family:

Consider $d\equiv 2 \, mod\, 3$, let $G''_{\delta,d}$ be such that $V(G''_{\delta,d})=\bigcup_{i=0}^{d}V_i$:

$$|V_i| = \begin{cases}\delta-1 & \text{if } i\equiv 1 \bmod 3,\, i\ne 1 \text{ and } i\ne d-1\\ \delta & \text{if } i=1 \text{ and } i=v_{d-1}\\ 1, & \text{otherwise}\end{cases}$$

with $x \in V_i$ and $y \in V_j$ implying that $xy \in E(G''_{\delta,d})$ if and only if $|i-j| \le 1$.

Also consider the family $H_{\delta,r}$ a sub-family of the graphs by Griggs and Wu [15], which is such that $V(H_{\delta,r}) = \bigcup_{i=0}^{2r-1} V_i$ with

$$|V_i| + |V_{i+1}| + |V_{i+2}| = \delta + 1$$

for all *i*. Further, for $u_i \in V_i$ and $u_j \in V_j$, $u_i u_j \in E(G_{\delta,r})$ if and only if $|i-j| \le 1$. In addition, for $x \in V_0$ and $y \in V_{2r-1}$, $xy \in E(G_{\delta,r})$. One would easily see for instance that for $r \equiv 0$ modulo 3 and $\delta \equiv 2$ modulo 3, the bound is sharp when $|V_i| = \frac{\delta+1}{3}$ for all *i*. In this case $\gamma(H_{\delta,r}) = n - \frac{2}{3} r\delta$. We note that $H_{\delta,r}$ contains highly connected families. However, several 1-connected and 2-connected graphs can be constructed to show asymptotic sharpness of the bound.

We also note that for $r \equiv 2$ modulo 3, $\delta = 2$ and $|V_i| = \frac{\delta+1}{3} = 1$ for each *i*, $H_{\delta,r}$ yields a cycle C_{2r} of order $2r$ and $\gamma(C_{2r}) = \frac{2}{3}(r+1)$. It is interesting that this is the same family C_n for $n \equiv 4$ modulo 6 stated in [18] and attains the second bound of Theorem 3.

Similar ideas as in the proof of Theorem 3, yields Theorem 4.

Theorem 4: *Let G be C_3-free and connected with $\delta \ge 3$ and $\mathrm{rad}(G) = r$. Then, $\gamma(G) \le n - \frac{1}{2}(\mathrm{diam}(G)+1)(\delta-1)$ and $\gamma(G) \le n-(r-1)(\delta-1)$.*

Proof: To establish the first bound, define the diametrical path $P(u_0, u_d)$ as in Theorem 3 and consider the set $A_1 = \{u_{4i}u_{4i+1} : 0 \le i \le \frac{d-1}{4}\}$. Because $N(u) \cap N(v) = \varnothing, \forall u, v \in A_1$ and $|A_1| \ge \frac{1}{2}(\mathrm{diam}(G)+1)$, the result holds.

To prove the second bound, form a tree T'' using similar ideas as in the proof of the previous theorem. That is, let G' be a maximal matching that can be obtained in *G*. If there exist a K_1 that is a distance 3 from G', add it to G'. Repeat this procedure until every K_1 subgraph of *G* not in G' is within distance 2 of G'. By the construction and since *G* is C_3-free, for $x, y \in V(G')$, $N(x) \cap N(y) = \varnothing$. Define F' to be the forest with $V(F') = V(G') \cup \{N(x) : x \in V(G')\}$ and

$$E(F') = \{xy \in E(G')\} \cup \{xu : x \in V(G') \text{ and } u \in N(x)\}.$$

Set *s* to be the cardinality of components of G'. Then, there exist $s-1$ edges of *G* whose addition to F' yields a tree. Define $T'' \le G$ to be a tree of minimal diameter amongst all trees that are formed by this procedure.

Then every vertex in $V(G-T'')$ is adjacent to a vertex in T'' and $\mathrm{diam}(T'') \geq 2r-3$, see Fact 1. Thus, the result holds.

Here, we show that there exist a construction that meets the bound in terms of the diameter. Take a look at $d \equiv 3$ modulo 4 and define $G'''_{\delta,d}$ such that $V(G'''_{\delta,d}) = \bigcup_{i=0}^{d} V_i$ [7]

$$|V_i| = \begin{cases} \delta - 1 & \text{if } i \equiv 1 \text{ or } 2 \bmod 4, i \neq 1 \text{ and } i \neq d-1 \\ \delta & \text{if } i = 1 \text{ and } i = v_{d-1} \\ 1, & \text{otherwise} \end{cases}$$

with vertices in V_i and V_j being neighbours if and only if $|i-j| = 1$ [8].

The family $\mathcal{F}_{2,3}$ [23] (see also [20]) shows that the second bound in Theorem 4 is sharp. In addition, the following highly connected sub-family of the family of graphs $\mathcal{G}$ [23] (see also[20]) reveals that the second bound is also sharp, apart from the additve constant multiples of δ. Moreover, it gives an implication that there is a chance for the second bound to be improved to $\gamma(G) \leq n - r(\delta - 1)$ when $r \equiv 0$ modulo 4:

Consider any triangle-free graph $G_{(\delta,2r-1)}$ of minimum degree δ such that $r \equiv 0$ modulo 4, $V(G_{(\delta,2r-1)}) = \bigcup_{i=0}^{2r-1} V_i$ with

$$|V_i| + |V_{i+1}| + |V_{i+2}| + |V_{i+3}| \geq 2\delta$$

for all i with two distinct vertices $u_i \in V_i$ and $u_j \in V_j$ implying that $u_i u_j \in E(G_{(\delta,2r-1)})$ provided $|i-j| = 1$. In addition, whenever $x \in V_0$ and $y \in V_{2r-1}$, then $xy \in E(G_{(\delta,2r-1)})$. An example of such a family is where $|V_i| = \frac{\delta}{2}$ for all i. Thus, if $\gamma(G) \leq n - r(\delta - 1)$ for $r \equiv 0$ modulo 4 is true, then it is sharp. It would be interesting if this bound can be confirmed.

For further study, one can choose to consider to improve Theorem 4 by generating sharp bounds for each integer $c \in \{0,1,2,3\}$, where $r \equiv c$ modulo 4. Apart from this, researchers can also consider graphs of given girth $g \geq 5$.

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