Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 45 (2024), No. 5, pp. 1305–1316
DOI : 10.47974/JIOS-1403

Optimization for a parabolic system involving Schrödinger operator with coefficients of variables beneath conjugation conditions

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Abstract

This article is dedicated to the optimization for N×N parabolic system with mixed boundary conditions beneath conjugation conditions and including Schrödinger operator according to the maximum principle. Through Green's formula, the properties of the first eigenvalues and the Lax-Milgram lemma, we provide evidence for the solutions to these systems' existence and uniqueness. The maximum principle and the existence of a solution for a given linear parabolic system with variable coefficients are also topics of study.

Subject Classification: *Primary 35K40, Secondary 30C80.*

Keywords: *Parabolic systems, Maximum principle.*

1. Introduction

Optimum necessary and sufficient conditions for systems discip-lined by partial differential operators were investigated by Lions in [1-2]. Problems of Optimal control for distributed systems including partial differential equations have numerous mechanical origins and a different of technological and scientific applications. Many optimal control problems of systems with Schrödinger operators have been discussed in [3-7]. There are some problems on an infinite number of variables for sys-tem. Many papers discussed distributed and boundary control as in [10-18]. This article deals with the maximum principle and having positive solutions to a given parabolic system with the Schrödinger operator, apart from the conjugation conditions.

2. On distributed control of a parabolic system

2.1 *2×2 parabolic systems under conjugation condition*

Let 2×2 parabolic system being

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$$\begin{cases} (-\Delta + q_1)z_1 + \frac{\partial z_1}{\partial t} - \alpha(x)z_1 - \beta(x)z_2 = f_1 \text{ in } D_1 \\ (-\Delta + q_2)z_2 + \frac{\partial z_2}{\partial t} - \gamma(x)z_1 - \delta(x)z_2 = f_2 \text{ in } D_2 \\ z_1 = z_2 = 0\,, \frac{\partial z}{\partial v_A} = g_1, \frac{\partial z_2}{\partial v_A} = g_2 \text{ on } \sigma \end{cases} \quad (1)$$

where $\frac{\partial Z}{\partial S_A} = \left[\frac{\partial z_i}{\partial x_k}\cos(v, x_k) + \frac{\partial z_i}{\partial x_k}\cos(v, x_k)\right]$ on σ, $\cos(n, x_k)$ = n−cosine direction of s, s being the orthogonal to the σ outside to D and $g_1, g_2, \in H^{\frac{1}{2}}(\sigma)$, $k = 1,2\,, Z = \{z_1, z_2\}$, $f_1, f_2 \in L^2_{\frac{1}{g}}(\Omega)$, $\alpha(x),\delta(x) \leq \sqrt{\alpha(x)\delta(x)}$, $g(x)=\min\{\alpha(x), \beta(x), \gamma(x), \delta(x)\}$, $q= \max\{q_1, q_2\}$ and conjugation conditions(*)

$$\left[\frac{\partial z_1}{\partial S_A}\right]=\left[\frac{\partial y_1}{\partial x_1}\cos(n, x_k) + \frac{\partial z_1}{\partial x_2}\cos(n, x_k)\right]=[Z]=0,$$

$$\left\{\frac{\partial z_1}{\partial x_1}\cos(n, x_k) + \frac{\partial z_1}{\partial x_2}\cos(n, x_k)\right\}^{\pm} = r[z_1],$$

$$\left[\frac{\partial z_2}{\partial S_A}\right]=\left[\frac{\partial z_2}{\partial x_1}\cos(n, x_k) + \frac{\partial z_2}{\partial x_2}\cos(n, x_k)\right]=0,$$

$$\left\{\frac{\partial z_2}{\partial x_1}\cos(n, x_k) + \frac{\partial z_2}{\partial x_2}\cos(n, x_k)\right\}^{\pm} = r[z_2]\,,$$

where D is an open subset of $\mathbb{R}^n$ with smooth boundary σ, Schrodinger operator represented by $(-\Delta + q)$ is , $D = D_1 \cup D_2$, $k = 1,2, \sigma = (\partial D_1 \cup \partial D_2)\backslash \mu$, $\mu = \partial D_1 \cup \partial D_2 \neq \varphi$, $[\Lambda] = \Lambda^+ - \Lambda^-, \{\Lambda(\delta)\}^+ = \Lambda(\delta + n)$ and $\{\Lambda(\delta)\}^- = \Lambda(\delta - n)$. The presented parabolic equation is defined in differential and strictly Lipchitz domains in D, $D_2 \epsilon R^n$.

2.1.1 Existence and uniqueness of Solution

Operator equation $Ay = \lambda y$

To prove the existence of the state $Z = \{z_1, z_2\}$of system (1), we state briefly some results introduced in [8] concerning the Eigen value problem:

$$\begin{cases} (-\Delta + q)\varphi = \lambda g(x)\varphi \quad \text{in } D \\ y = 0 \\ \varphi = 0\,, \frac{\partial \varphi}{\partial S_A} = g \qquad \text{on } \sigma \end{cases}$$

Since $H_0^1(D) \subseteq L^2(D) \subseteq H^{-1}(D)$,

Then we have chain of the form

$$[H_0^1(D)]^2 \subseteq [L_2(D)]^2 \subseteq [H^{-1}(D)]^2$$

we define the following bilinear form on $[H_0^1(D)]^2 \times [H_0^1(D)]^2$ $a(z, \varphi)=$

$$-\int_D \{(\Delta z_1 + q_1)\varphi_1 + (\Delta z_2 + q_2)\varphi_2\}\,dx + \int_\mu \{r[z_1][\varphi_1] + r[z_2][\varphi_2]\,d\mu$$

$$-\int_D (\alpha(x)z_1\varphi_1 + \delta(x)z_2\varphi_2 + \beta(x)\, z_1\varphi_2 + \gamma(x)\, z_2\varphi_1)dx$$

$$L(\varphi)=\int_D (f_1\varphi_1 + f_2\varphi_2)dx + \int_\sigma (g_1\varphi_1 + g_2\varphi_2)\,d\sigma$$

Lemma 1: *Let* $f_1, f_2 \in L_2(D)$, $q= max\{q_1, q_2\}$, $f_1, f_2 > 0$ *in system (1) satisfy* $\alpha < \lambda^+_{q\,1}$, $\beta < \lambda^+_{q_2}(\lambda^+_{q\,1} - \alpha)(\lambda^+_{q\,2} - \delta) > \beta\gamma$.

Lemma 2: *First Eigen values* λ^+_q *characterized by* $\lambda^+_q \int_D |z_1|^2\, dx \leq \int_D (|\nabla z|^2 + q|z|^2)dx$

Theorem 1: *For* $F = (f_1, f_2) \in [L_2(D)]^2$, *there exist a unique solution* $Z \in [H^1_0(D)]^2$ *of system*(1).

Proof: The bilinear form is denoted as

$$b(z,\Psi) = -\int_D \{(\Delta+q_1)z_1)\Psi_1 + (\Delta + q_2)z_2)\Psi_2\}\,dx + \int_\mu \{r[z_1][\Psi_1] + r[z_2][\Psi_2]d\mu - \int_D \alpha(x)z_1\Psi_1 + \delta(x)\,z_2\Psi_2 + \beta(x)z_1\Psi_2 + \gamma(x)\,z_2\Psi_1)dx + \int_\sigma \{F(z_1,\Psi_1) + F(z_2,\Psi_2)\}d\sigma$$

$$= -\int_D \{(\Delta+q_1)z_1)\Psi_1 + (\Delta+q_2)z_2)\Psi_2\}\,dx + \int_\mu \{r[z_1][z_2\Psi_1] + r[z_2][y_2\Psi_2]d\mu - \int_D \alpha(x)z_1\Psi_1 + \delta(x)z_2\Psi_2 + \beta(x)\,z_1\Psi_2 + \gamma(x)\,z_2\Psi_1)dx + \int_\sigma \left(\Psi_1\frac{\partial z_1}{\partial v_A} + \Psi_2\frac{\partial z}{\partial v_A}\right) d\sigma$$

For $\alpha + s > 0$ and $\delta + n > 0$ and the equivalent norm $\|z\|^2_{q,n} = \int_D (|\nabla z|^2 + (q+n)|z|^2)dx$, select a large enough of s

$$b(z,z) = \int_D (|\nabla z_1|^2 + (q_1 + \alpha(x))|z_1|^2 + |\nabla z_2|^2 + (q_2 + \delta(x))|z_2|^2)dx + \int_\gamma \{r[z_1]^2 + r[z_2]^2 d\mu - 2\int_\Omega \alpha(x)|z_1|^2\,dx - 2\int_\Omega \delta(x)|y_2|^2\,dx - \int_\Omega (\beta(x) + \gamma(x))\,z_1z_2\,dx$$

By using $\alpha(x), \gamma(x) \leq \sqrt{\alpha(x)\delta(x)}$

$$b(z,z) \geq \int_D \{(|\nabla z_1|^2 + (q_1 + \alpha(x))|z_1|^2) + (|\nabla z_2|^2 + (q_2 + \delta(x))|z_2|^2)\}dx + \int_\gamma \{r[z_1]^2 + r[z]^2 d\mu - 2\int_\Omega \alpha(x)|z_1|^2\,dx - 2\int_D \delta(x)|z_2|^2\,dx - 2\int_D \sqrt{\alpha(x)\delta(x)}z_1z_2\,dx$$

By using Cauchy Schwartz and lemma 2 in the paper

$$b(z,z) \geq \left(1 - \frac{2}{(\lambda^+_{q_1}(\alpha)+1)}\right)z^2_{q_1} + \left(1 - \frac{2}{(\lambda^+_{q_2}(\delta)+1)}\right)z^2_{q_2} - \frac{2\sqrt{(|\nabla z_1|^2 + (q_1+\alpha(x))|z_1|^2}\sqrt{(|\nabla z_{21}|^2 + (q_2+\delta(x))|z_2|^2}}{\sqrt{(\lambda^+_{q_2}(\delta)+1)(\lambda^+_{q_1}(\alpha)+1)}}$$

$$b(z,z) \geq \left(\frac{(\lambda^+_{q_1}(\alpha)-2)}{(\lambda^+_{q_1}(\alpha)+1)}\|z_1\|^2_{q_1} + \frac{(\lambda^+_{q_2}(d)-2)}{(\lambda^+_{q_2}(d)+1)}\|z_2\|^2_{q_2}\right)$$

$$a(z,z) \geq c(\|y_1\|^2_{q_1} + \|y_2\|^2_{q_2}),\ c = \min\left(\frac{(\lambda^+_{q_1}(\alpha)-2)}{(\lambda^+_{q_1}(\alpha)+1)}, \frac{(\lambda^+_{q_2}(\delta)-2)}{(\lambda^+_{q_2}(\delta)+1)}\right)$$

$$z = (z_1, z_2) \in V_{q_1} \times V_{q_2}, r[z_1]^2, r[z_2]^2 \geq 0.$$

$$b(z,z) \geq \|z\|^2_{\left(H_0^1(D)\right)^2} \geq \|z\|^2_{[H^1(D)]^2} \quad \text{[Coerciveness]} \quad \forall z \in H_0^1(D) \tag{2}$$

$$c = \text{const}, z = (z_1, z_2)\ |b(y,Z)| \leq c_{1\|\Psi\|_{H_0^1(D)}\|y\|_{H_0^1(D)}} c_1 = \text{const}\ , \forall y, Z \in H_0^1(D)$$

and $L(Z)$ is continuous on $[H_0^1(D)]^2$ then through Lax-Milgram lemma $\exists$ the unique solution $y \in H_0^1(D)$, s. t. $b(y,Z) = L(Z)$ for all $Z \in [H_0^1(D)]^2, y = (y_1, y_2)$.

2.1.2 Control problem formulation

Suppose $[L^2(D)]^2$ is given the space of controls.

The energy functional

$$\varphi(v) = b(y,v) - 2L(v) \tag{3}$$

A unique state $y(u) \in \{v | D_l \in W_2^1(D_l): l = 1,2; v|\sigma = 0\}$,

where $W_2^1(D)$ is a set of the sobolev functions are specified on domain D_l to every control $u \in [L_2(D)]^2$. For a control $v = (v_1, v_2) \in [L_2(D)]^2$ the state of the system $y(u) = \{y_1(u), y_2(u)\}$ is given by the solution of the following system:-

$$\begin{cases} (-\Delta + q_1)y_1 - \alpha(x)y_1 - \beta(x)y_2 + \frac{\partial y_1}{\partial t} = f_1 + v_1 \text{ in } D_1 \\ (-\Delta + q_2)y_2 - \gamma(x)y_1 - \delta(x)y_2 + \frac{\partial y_2}{\partial t} = f_2 + v_2 \text{ in } D_2 \\ y_1 = y_2 = 0\ , \frac{\partial y_1}{\partial s_A} = g_1, \frac{\partial y_2}{\partial s_A} = g_2 \text{ on } \sigma \end{cases} \tag{4}$$

Observation equation is denoted by $z(u) = \{z_1(u), z_2(u)\} \equiv y(u)$.

Cost functional is expressed by:

$$J(u) = (\varpi u, u)_{[L_2(D)]^2} + \|y(u) - z_d\|^2_H, H \subset L_2(D) \tag{5}$$

$$= \int_D (y_1(u) - z_{1d})^2\, dx + \int_D (y_2(u) - z_{2d})^2\, dx + (\varpi v, v)_{[L_2(D)]^2},$$

Where $Z_d = \{Z_{1d}, Z_{2d}\}$ in $[L_2(D)] \times [L_2(D)]$

ϖ is a positive Hermitian definite operator defined on $[L_2(D)]^2$ such that:

$$(\varpi v, v)_{[L_2(D)]^2} \geq \rho \|u\|^2_{[L_2(D)]^2}\ , \rho > 0 \tag{6}$$

The control problem then is to find:

$\begin{cases} v \in U_{ad} \text{such that} \\ J(v) = \inf J(v) \text{ for all } v \in U_{ad} \end{cases}$, where U_{ad} is a closed convex subset from$[L_2(D)]^2$, since the cost function (5) can be written as

$$J(u) = \int_D (y_1(u) - y_1(0)) + (y_1(0) - z_{1d}))^2\, dx + \int_D (y_1(u) - y_1(0)) + (y_1(0) - z_{1d}))^2\, dx + (\varpi u, u)_{[L_2(D)]^2}$$

$$= \pi(u,u) - 2L(u) + \|z_{id} - y_i(0)\|^2\ , i = 1,2,$$

$$\text{Where } \pi(v,u) = (\varpi v,u)_{[L_2(D)]^2} + \left(y_1(v)-y_1(0), y_1(u)-y_1(0)\right)_{L_2(D)} + \left(y_2(v)-y_2(0), y_2(u)-y_2(0)\right)_{L_2(D)},$$

$$L(u) = (z_{id}-y_i(0), y_i(u)-y_i(0))_{L_2(D)}, i = 1,2.$$

Assume that (2), (6) hold. The cost function being given (5), necessary and sufficient for u to be an optimal control is that the following equations and inequalities be satisfied

$$\begin{cases} \frac{\partial h_1}{\partial t} + (-\Delta + q_1)h_1(v) - \alpha(x)h_1(v) - \gamma(x)h_2(v) = y_1(v) - z_{1d} & \text{in } D_1 \\ \frac{\partial h_2}{\partial t} + (-\Delta + q_2)h_2(v) - \beta(x)h_1(v) - \delta(x)h_2(v) = y_2(v) - z_{2d} & \text{in } D_2 \\ h_1(v) = 0, h_2(v) = 0, \frac{\partial h_1}{\partial s_{A^*}} = \frac{\partial h_2}{\partial s_{A^*}} = 0 & , x\epsilon\sigma \\ \left[\frac{\partial h_1}{\partial x_1}\cos(n,x_k) + \frac{\partial h_2}{\partial x_2}\cos(n,x_k)\right] = \left[\frac{\partial h_1}{\partial s_{A^*}}\right] = [h] = 0 & , x\epsilon\mu \\ \left[\frac{\partial h_2}{\partial x_1}\cos(n,x_k) + \frac{\partial h_2}{\partial x_2}\cos(n,x_k)\right] = \left[\frac{\partial h_2}{\partial s_{A^*}}\right] = 0, x \in \mu \\ \left\{\frac{\partial h_1}{\partial x_1}\cos(n,x_k) + \frac{\partial h_1}{\partial x_2}\cos(n,x_k)\right\}^{\pm} = r[h_1], x \in \mu \\ \left\{\frac{\partial h_2}{\partial x_1}\cos(n,x_k) + \frac{\partial h_2}{\partial x_2}\cos(n,x_k)\right\}^{\pm} = r[h_2], x\epsilon\mu \end{cases} \tag{7}$$

h (v) $= \{h_1(v), h_2(v)\}$ is the adjoint state.

Proof overview

Since U_{ad} is constrained, the cost function J(v) is differentiable, then the optimal control u is distinguish by $J'(v)(u-v) \geq 0 \ \forall\, v \in U_{ad}$ which is equivalent to

$$(\varpi u, u-v)_{[L_2 D]^2} + \left(y_i(v)-z_{id}, y_i(u)-y_i(v)\right) \geq 0 \tag{8}$$

Since $(A^*h, y) = (h, Ay)$ where A is defined by:

$$A\varphi = A\{y_1, y_2\}$$

$= (-\Delta+q_1)y_1 - \alpha(x)y_1 - \beta(x)y_2, (-\Delta+q_2)y_2 - \gamma(x)y_1 - \delta(x)y_2)$, then

$(h, Ay) = ((h_1, (-\Delta+q_1)y_1\alpha(x)y_1 - \beta(x)y_2), (h_2, (-\Delta+q_2)y_2 - \gamma(x)y_1 - \delta(x)y_2))$

Through Green's formula or by the derivative in the sense of distribution

$$A\varphi = \left((-\Delta+q_1)h_1 - \alpha(x)h_1 - \gamma(x)h_2, y_1\right) + \left((-\Delta+q_2)h_2 - \beta(x)h_1 - \delta(x)h_2, y_2\right)$$

$= (A^*h, y)$, where $A^*h = A^*\{h_1(u), h_2(u)\} = y(u) - z_d$ then

$Ah(v) + \frac{\partial h}{\partial t} + M^T h(v) = Y(v) - Z_d$, M^T is transpose of M, h (v) adjoint state $A = -\Delta + q, M = a_{ij}(x), i, j = 1,2$, (8) is equivalent to

$$\left(-\Delta+q_1)h_1-\alpha(x)h_1-\gamma(x)h_2-\frac{\partial h}{\partial t},y_1(u)-y_1(v)\right)+(\frac{\partial h_2}{\partial t}+$$
$$(-\Delta+q_2)h_2-\beta(x)h_1-\delta(x)h_2,y_2(u)-y_2(v))+(\varpi\,u,u-v)_{[L_2(D)]^2}\ \geq 0.$$
$$(h_1,\frac{\partial}{\partial t}(y_1(u)-y_1(v))+(-\Delta+q_1)(y_1(u)-y_1(v)))+(h_2,\frac{\partial}{\partial t}(y_2(u)-$$
$$y_2(v))+(-\Delta+q_2)(y_2(u)-y_2(v)))+(h_1,\frac{\partial}{\partial v_A}(y_1(u)-y_1(v))+$$
$$+\left(h_2,\frac{\partial}{\partial S_A}(y_2(u)-y_2(v)\right)+\left(\frac{\partial h_2}{\partial S_A},(y_2(u)-y_2(v)\right)+(\frac{\partial h_1}{\partial S_A},(y_1(u)-$$
$$y_1(v))+(-\beta(x)h_1-\delta(x)h_2,y_2(u)-y_2(v))+(-\alpha(x)h_1-\gamma(x)h_2,y_1(u)-$$
$$y_1(v))+(\varpi u,u-v)_{[L_2(D)]^2}\ \geq 0$$
$$\int_D\{\frac{\partial h_1}{\partial S_A}g(x)(u_1-v_1)+\frac{\partial h_2}{\partial S_A}g(x)(u_2-v_2)\}dx\ +(\varpi u,u)_{[L_2(D)]^2}\ \geq 0.$$

Remark 1: Generalization to n × n systems

$$AY+\frac{\partial Y}{\partial t}+MY=F \qquad \text{in D}$$
$$Y=0,\frac{\partial Y}{\partial S_A}=g \qquad \text{on } \sigma$$

and conjugation conditions

$$\left[\frac{\partial Y}{\partial S_A}\right]=\left[\sum_{i,k=1}^{n}\frac{\partial y_i}{\partial x_k}\cos(n,x_k)\right]=[Z]=0$$
$$\left\{\sum_{i,k=1}^{n}\frac{\partial y_i}{\partial x_k}\cos(n,x_k)\right\}^{\pm}=r[z]$$

where D is an open subset of $\mathbb{R}^n$ with smooth boundary σ, N×N diagonal matrix of Schrödinger operator represented by $(-\Delta+q)$ is, $D=D_1\cup D_2, k=1,2,$ $\sigma=(\partial D_1\cup\partial D_2)\backslash\mu$, $\mu=\partial D_1\cup\partial D_2\neq\varphi$, $[\Lambda]=\Lambda^+-\Lambda^-$,$\{\Lambda(\delta)\}^+=\Lambda(\delta+n)$ and $\{\Lambda(\delta)\}^-=\Lambda(\delta-n)$.

In this case, the bilinear form is given by:

$$a(y,\Psi)=$$
$$\sum_{i=1}^{n}\int_D\frac{\partial y_i}{\partial t}+(-\Delta+q_i)y_i)\Psi_i\,dx+\int_\mu\ r[y_i][\Psi_i]d\mu+\sum_{i,j=1}^{n}\int_\sigma\ F(y_i,\Psi_i)\,d\sigma$$
$$-\sum_{i,j=1}^{n}\int_D\ a_{ij}(x)\,y_j\Psi_i\,dx.$$

The linear form is given by:

$$L(\Psi)=\sum_{i=1}^{n}(\int_D\ f_i\Psi_i dx+\int_\sigma\ g_i\Psi_i d\sigma)$$

The cost functional is denoted as:

$$J(v)=(\bar{a}v,v)_{[L^2(D)]^n}+\|y(v)-z_d\|_H^2$$

$$J(v)=\sum_{i=1}^{n}\int_D\ (y_i(v)-z_{id})^2\,dx+(\varpi v,v)_{[L^2(D)]^2}$$

where $Z_d=\{Z_{1d},Z_{2d},\dots\dots,Z_{nd}\}$ in $[L^2(\sigma)]^n$, $H\subset L_2(D)$, ϖ is a Hermitian positive determinant defined on $[L^2(D)]^2$ such that:

$(\varpi v,v)_{[L^2(D)]^2}\ \geq\ C\|u\|^2_{[L^2(D)]^2}\ C>0$. In this case, it is necessary and sufficient that u is an optimal control of the following equations and inequalities.

$$\begin{cases} Ah(v) + \frac{\partial h}{\partial t} + M^T h(v) = Y(v) - Z_d \ \text{ in } \ D \\ h(v) = 0\,, \frac{\partial h_i}{\partial s_{A^*}} = 0 \qquad\qquad \text{on } \sigma \\ \left[\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k} \cos(n, x_k)\right] = [\sum_{i=1}^{n} h_i] = 0 \ , \left\{\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k} \cos(n, x_k)\right\}^{\pm} = r[h] \end{cases}$$

$$M = \left(a_{ij}(x)\right)_{i,j=1}^{n} \ , \ M^T = \sum_{j=1}^{n} a_{ji}(x) \text{ is transpose of M.}$$

If constraints are absent i.e. when $\mu_\partial = \mu$ the equality $p(u) + \varpi v = 0$ i. e. $(-\frac{h(v)}{\varpi} = v)$ then the differential problem of finding the vector-function

$$(y, h)^T \begin{cases} Ay(v) + \frac{\partial y}{\partial t} + \frac{h(v)}{\varpi} + My(v) = F \qquad \text{in D} \\ Ah(v) + \frac{\partial h}{\partial t} + M^T h(v) - y(v) = -z_d \ \text{ in D} \\ h(v) = y(v) = 0, \frac{\partial z}{\partial s_A} = g\,, \frac{\partial h}{\partial s_{A^*}} = 0 \quad \text{on } \sigma \\ \left[\sum_{i,k=1}^{n} \frac{\partial y_i}{\partial x_k} \cos(n, x_k)\right] = 0 \ , \left\{\sum_{i,k=1}^{n} \frac{\partial y_i}{\partial x_k} \cos(n, x_k)\right\}^{\pm} = r[y], x \epsilon \mu \\ \left\{\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k} \cos(n, x_k)\right\}^{\pm} = 0 \ , \left\{\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k} \cos(n, x_k)\right\}^{\pm} = r[h]\,, x \epsilon \mu \end{cases} \tag{9}$$

We can find last relations at n= 2 as the same in the first of paper.

3. Boundary control of parabolic system with conjugation conditions involving Schrödinger operator

3.1.1 Control problem formulation

Let $[L^2(\sigma)] \times [L^2(\sigma)]$ being the space of controls.

For a control $v = (v_1, v_2) \in [L_2(\sigma)]^2$ the state of the system y (v) = $\{y_1(v), y_2(v)\}$ is given by the solution of the following system:-

$$\begin{cases} (-\Delta + q_1)y_1 + \frac{\partial y_1}{\partial t} - \alpha(x)y_1 - \beta(x)y_2 = f_1 \text{ in } D_1 \\ (-\Delta + q_2)y_2 + \frac{\partial y_2}{\partial t} - \gamma(x)y_1 - \delta(x)y_2 = f_2 \text{ in } D_2 \\ y_1 = 0, y_2 = 0\,, \frac{\partial y_1}{\partial v_A} = g_1 + v_1, \frac{\partial y_2}{\partial v_A} = g_2 + v_2 \text{ on } \sigma \end{cases} \tag{10}$$

The observation equation is denoted by z (v) =$\{z_1(v), z_2(v)\} \equiv y(v)$.

Since the cost function (5) can be written as:

$$J(u) = \int_\Gamma (y_1(u) - y_1(0)) + (y_1(0) - z_{1d}))^2 \, dx$$

$$+ \int_\sigma (y_1(u) - y_1(0)) + (y_1(0) - z_{1d}))^2 \, dx + (\varpi u, u)_{[L^2(\sigma)]^2}$$

$$= \pi(u, u) - 2L(u) + \|z_{id} - y_i(0)\|^2 \,, i = 1, 2, \text{ where}$$

$$\pi(v, u) = (\varpi v, u)_{[L^2(\sigma)]^2} + \left(y_1(v) - y_1(0), y_1(u) - y_1(0)\right)_{L_2(\sigma)}$$

$$+ \left(y_2(v) - y_2(0), y_2(u) - y_2(0)\right)_{L_2(\sigma)} ,$$

$$L(u) = (z_{id} - y_i(0), y_i(u) - y_i(0))_{L_2(\sigma)}, i = 1,2$$

since L(v) is continuous on $[L_2(\sigma)] \times [L_2(\sigma)]$ then there exist a unique optimal control $v \in U_{ad}$ [1]. The function y is defined there, the energy functional minimizes [9].

Moreover we have the following theorem which gives the characterization of the optimal control v:

The necessary and sufficient for v to be an optimal control is that the following equations and inequalities be satisfied

$$\begin{cases} (-\Delta + q_1)h_1 + \frac{\partial h_1}{\partial t} - \alpha(x)h_1 - \gamma(x)h_2 = 0 & \text{in } D_1 \\ (-\Delta + q_2)h_2 + \frac{\partial h}{\partial t} - \beta(x)h_1 - \delta(x)h_2 = 0 & \text{in } D_2 \\ h_1(v) = 0, h_2(v) = 0\,, \frac{\partial h_1}{\partial S_{A^*}} = y_1(v) - z_{1d}\frac{\partial h_2}{\partial v_{A^*}} = y_2(v) - z_{2d} & , x\epsilon\sigma \\ \left[\frac{\partial h_1}{\partial x_1}\cos(n, x_k) + \frac{\partial h_2}{\partial x_2}\cos(n, x_k)\right] = [\frac{\partial h_1}{\partial S_{A^*}}] = 0 & , x\epsilon\mu \\ \left[\frac{\partial h_2}{\partial x_1}\cos(n, x_k) + \frac{\partial h_2}{\partial x_2}\cos(n, x_k)\right] = [\frac{\partial h}{\partial S_{A^*}}] = 0\,, x \in \mu & \\ \left\{\frac{\partial h_1}{\partial x_1}\cos(n, x_k) + \frac{\partial h_1}{\partial x_2}\cos(n, x_k)\right\}^{\pm} = r[h_1], x \in \mu & \\ \left\{\frac{\partial h_2}{\partial x_1}\cos(n, x_k) + \frac{\partial h_2}{\partial x_2}\cos(n, x_k)\right\}^{\pm} = r[h_2], x\epsilon\mu & \end{cases} \tag{11}$$

h (v)= $\{h_1(v), h_2(v)\}$ is the adjoint state.

Overview of the proof

Since U_{ad} is bounded, the cost function J(u) is differentiable then the optimal control u is characterized by $J'(v)(u - v) \geq 0 \ \forall v \in U_{ad}$ which is equivalent to

$$(\varpi v, u - v)_{[L_2(\sigma)]^2} + \left(y_i(v) - z_{id}, y_i(u) - y_i(v)\right) \geq 0, \tag{12}$$

Since $(A^*h, y) = (h, Ay)$ where A is defined by:

$A\varphi = A\{y_1, y_2\} = \left((-\Delta + q_1)y_1 - \alpha(x)y_1 - \beta(x)y_2, (-\Delta + q_2)y_2 - \gamma(x)y_1 - \delta(x)y_2\right)$

Then

$(h, Ay) = ((h_1, (-\Delta + q_1)y_1 - \alpha(x)y_1 - \beta(x)y_2), (h_2, (-\Delta + q_2)y_2 - \gamma(x)y_1 - \delta(x)y_2))$

By using Green's formula or the derivative in the sense of distribution

$$\left((-\Delta + q_1)h_1 - \alpha(x)h_1 - \gamma(x)h_2, y_1\right) + ((-\Delta + q_2)h_2 - \beta(x)p_1 - \delta(x)h_2, y_2)$$

$$= (A^*h, y), \text{ where } A^*p = A^*\{h_1(v), h_2(v)\} = 0$$

then $Ah(v) + \frac{\partial h}{\partial t} + M^T h(v) = 0$,

$Ah(v) - \frac{\partial h}{\partial S_{A^*}} + M^T h(v) = Y(v) - Z_d$, M^T is transpose of M, h (v) adjoint state

$A = -\Delta + q, M = a_{ij}(x), i, j = 1,2$, (12) is equivalent to

$$\left(-\Delta+q_1)h_1 - \alpha(x)h_1 - \gamma(x)h_2 - \frac{\partial h_1}{\partial t}, y_1(u) - y_1(v)\right) + (\frac{\partial h_2}{\partial t} + (-\Delta + q_2)h - \beta(x)h_1 - \delta(x)h_2, y_2(u) - y_2(v)) + (\varpi v, u - v)_{[L_2(\sigma)]^2} \geq 0.$$

$$(h_1, \frac{\partial}{\partial t}(y_1(u) - y_1(v)) + (-\Delta+q_1)(y_1(u) - y_1(v))) + (h_2, \frac{\partial}{\partial t}(y_2(u) - y_2(v)) + (-\Delta+q_2)(y_2(u) - y_2(v))) + (h_1, \frac{\partial}{\partial s_A}(y_1 u - y_1(v))$$

$$+\left(h_2, \frac{\partial}{\partial S_A}(y_2(u) - y_2(v)\right) + \left(\frac{\partial h_2}{\partial s_A}, (y_2(u) - y_2(v)\right)$$

$$+ (\frac{\partial h_1}{\partial s_A}, (y_1(u) - y_1(v))$$
$$+ (-\beta(x)h_1 - \delta(x)h_2, y_2(u) - y_2(v))$$
$$+ (-\alpha(x)h_1 - \gamma(x)h_2, y_1(u) - y_1(v)) + (\varpi v, u - v)_{[L^2(\sigma)]^2}$$
$$\geq 0$$

$$\int_\sigma \{\frac{\partial h_1}{\partial s_A} g(x)(u_1 - v_1) + \frac{\partial h_2}{\partial s_A} g(x)(u_2 - v_2)\}dx + (\varpi v, u - v)_{[L_2(\sigma)]^2} \geq 0 .$$

We can add the control in the part of Dirichlet condition, i.e.

$$\begin{cases} (-\Delta + q_1)y_1 + \frac{\partial y_1}{\partial t} - \alpha(x)y_1 - \beta(x)y_2 = f_1 \text{ in } D_1 \\ (-\Delta + q_2)y_2 + \frac{\partial y_2}{\partial t} - \gamma(x)y_1 - \delta(x)y_2 = f_2 \text{ in } D_2 \\ y_1 = v_1, y_2 = v_2 , \frac{\partial y_1}{\partial s_A} = g_1, \frac{\partial y_2}{\partial s_A} = g_2 \text{ on } \sigma \end{cases} \quad (13)$$

The difference appears in characterization

$$\begin{cases} (-\Delta + q_1)h_1 + \frac{\partial h_1}{\partial t} - \alpha(x)h_1 - \gamma(x)h_2 = 0 & \text{in } D_1 \\ (-\Delta + q_2)h_2 + \frac{\partial h_2}{\partial t} - \beta(x)h_1 - \delta(x)h_2 = 0 & \text{in } \sigma_2 \\ h_1(v) = y_1(v) - z_{1d}, h_2(v) = y_2(v) - z_{2d} , \frac{\partial h_1}{\partial s_{A^*}} = \frac{\partial h_2}{\partial s_{A^*}} = 0 & , x\epsilon\sigma \\ \left[\frac{\partial h_1}{\partial x_1}\cos(n, x_k) + \frac{\partial h_2}{\partial x_2}\cos(n, x_k)\right] = [\frac{\partial h_1}{\partial s_{A^*}}] = 0 & , x\epsilon\mu \\ \left[\frac{\partial h_2}{\partial x_1}\cos(n, x_k) + \frac{\partial h_2}{\partial x_2}\cos(n, x_k)\right] = [\frac{\partial h_2}{\partial s_{A^*}}] = 0 , x \in \mu \\ \left\{\frac{\partial h_1}{\partial x_1}\cos(n, x_k) + \frac{\partial h_1}{\partial x_2}\cos(n, x_k)\right\}^{\pm} = r[h_1], x \in \mu \\ \left\{\frac{\partial h_2}{\partial x_1}\cos(n, x_k) + \frac{\partial h_2}{\partial x_2}\cos(n, x_k)\right\}^{\pm} = r[h_2], x\epsilon\mu , \end{cases} \quad (14)$$

$$\int_\sigma \sum_{i=1}^n \frac{\partial h_i}{\partial v_A} g(x)(u_i - v_i)dx + (Nv, u - v)_{[L^2(\sigma)]^2} \geq 0 .$$

Remark 2: Generalization to N × N systems

$$AY + \frac{\partial Y}{\partial t} + MY = F \qquad \text{in } D$$
$$Y = 0, \frac{\partial Y}{\partial S_A} = g \qquad \text{on } \sigma,$$

conjugation conditions

$$\left[\frac{\partial Y}{\partial S_A}\right]=\left[\sum_{i,k=1}^{n} \frac{\partial y_i}{\partial x_k}\cos(n, x_k)\right] =0$$
$$\left\{\sum_{i,k=1}^{n} \frac{\partial y_i}{\partial x_k}\cos(n, x_k)\right\}^{\pm}=r[y]$$

where D is an open subset of $\mathbb{R}^n$ with smooth boundary σ, N×N diagonal matrix of Schrödinger operator represented by $(-\Delta + q)$ is, $D = D_1 \cup D_2$, $k = 1,2, \sigma = (\partial D_1 \cup \partial D_2)\backslash\mu$, $\mu = \partial D_1 \cup \partial D_2 \neq \varphi, [\Lambda] = \Lambda^+ - \Lambda^-, \{\Lambda(\delta)\}^+ = \Lambda(\delta + n)$ and $\{\Lambda(\delta)\}^- = \Lambda(\delta - n)$.

The bilinear form is expressed by:

$$b(y, \Psi)= \sum_{i=1}^{n} \int_{\Omega} (-\Delta + q_i)y_i)\Psi_i \, dx + \int_{\gamma} r[y_i][\Psi_i] \, d\gamma$$
$$+ \sum_{i,j=1}^{n} \int_{\sigma} F(y_i, \Psi_i) \, d\sigma -$$
$$\sum_{i,j=1}^{n} \int_{D} a_{ij}(x) \, y_j \Psi_i \, dx + \sum_{i=1}^{n} \frac{\partial y_i}{\partial t} \Psi_i \, dx.$$

The linear form is denoted by:

$$L(\Psi) = \sum_{i=1}^{n} \left(\int_{D} f_i \Psi_i dx + \int_{\Gamma} g_i \Psi_i d\sigma\right).$$

The cost functional can be written by:

$$J(v) = (\varpi v, v)_{[L^2(\sigma)]^n} + \|y(v) - z_d\|_H^2$$

$$J(u) = \sum_{i=1}^{n} \int_{\sigma} (y_i(v) - z_{id})^2 \, dx + (\varpi v, v)_{[L_2(\sigma)]^2}, \text{ where}$$

$$Z_d = \{Z_{1d}, Z_{2d}, \ldots\ldots, Z_{nd}\} \text{ in } [L_2(\sigma)]^n, \; H \subset L_2(\sigma), \varpi$$

is a positive Hermitian operator, defined on $[L_2(\sigma)]^2$,

$$(\varpi v, v)_{[L_2(\sigma)]^2} \geq \varsigma \|v\|^2_{[L_2(\sigma)]^2}, \varsigma > 0.$$

In this case the necessary and sufficient that v to is the optimal control for the following equations and inequalities be examined.

$$\begin{cases} Ah(u) + \frac{\partial p}{\partial t} + M^T h(u) = 0 \text{ in } D \\ h(u) = 0, \frac{\partial h_i}{\partial v_{A^*}} = Y(v) - Z_d \qquad \text{on } \sigma \\ \left[\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k}\cos(n, x_k)\right] = 0, \left\{\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k}\cos(n, x_k)\right\}^{\pm} = r[h] \end{cases} \tag{15}$$

$M = \left(a_{ij}(x)\right)_{i,j=1}^{n}$, M^T= transpose of M.

If constraints are absent i.e. when $\mu_\partial = \mu$ the equality $h(v) + \bar{a}v = 0$ i. e. $\left(-\frac{h(v)}{\bar{a}} = v\right)$ then the differential problem of finding the vector-function $(y, h)^T$, that satisfies the relations

$$\begin{cases} Ay(u) + \frac{\partial y}{\partial t} + My(u) = F \qquad \text{in } D \\ Ah(u) + \frac{\partial h}{\partial t} + M^{T}h = 0 \qquad \text{in } D \\ h(v) = y(v) = 0, \frac{\partial y}{\partial s_A} = g - \frac{h(u)}{\bar{a}}, \frac{\partial h}{\partial s_{A^*}} - y(v) = -z_d \qquad \text{on } \sigma \\ \left[\sum_{i,k=1}^{n} \frac{\partial y_i}{\partial x_k} \cos(n, x_k)\right] = 0 \ , \left\{\sum_{i,k=1}^{n} \frac{\partial y_i}{\partial x_k} \cos(n, x_k)\right\}^{\pm} = r[y], x \epsilon \mu \\ \left\{\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k} \cos(n, x_k)\right\}^{\pm} = 0 \ , \left\{\sum_{i,k=1}^{n} \frac{\partial h_i}{\partial x_k} \cos(n, x_k)\right\}^{\pm} = r[h] \ , x \epsilon \mu \end{cases} \tag{16}$$

We can find last relations at $n = 2$.

4. Conclusions

Existence and uniqueness of the solution (state of the system) and necessary and sufficient condition for the optimality of the system in 2×2 and generalization to N×N in the cases of the control.

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Received September, 2022