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Balancing and Lucas-Balancing hybrid numbers and some identities

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Abstract

In this paper, we introduce Balancing and Lucas-Balancing hybrid numbers. We examine some identities of Balancing and Lucas-Balancing hybrid numbers. We give some basic definitions and properties related to them. In addition, we find Binet's Formula, Cassini's identity, Catalan's identity, d'Ocagne identity, generating functions, exponential generating functions of the Balancing and Lucas-Balancing hybrid numbers.

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1. Introduction

Number sequences have been studied by researchers for many years. Number sequences, especially Fibonacci numbers, have been studied and many generalizations have been made. [4,6,8,11,16,20-23]

The notion of Balancing numbers $\{B_n\}_{n=0}^{\infty}$ has been introduced by Behera and Panda [2]. Balancing numbers have been studied extensively in the last twenty years.

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In [2], they defined a balancing number n as the solution of the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

calling r as the balancer corresponding to n. The first few balancing numbers are 1, 6, 35, 204 and 1189.

The sequence of balancing numbers has been studied extensively and many generalizations of these numbers have been made [5, 9, 10, 12, 13, 17, 24-26, 28, 29].

Liptai et al. generalized the concept of balancing numbers in [18].

The Balancing numbers sequence is denoted by $\{B_n\}_{n=0}^{\infty}$. We can define the reccurence relation of Balancing numbers by

$$B_n=6B_{n-1}-B_{n-2}, n\geq 2$$

with the initial terms, $B_0=0,\ \ B_1=1$. [1]

Binet formula for Balancing numbers as follows

$$B_n=\frac{{\lambda_1}^n-{\lambda_2}^n}{\lambda_1-\lambda_2}. \text{ [2]}$$

Note that,

- $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$,
- $\lambda_1\lambda_2=1$,
- $\lambda_1+\lambda_2=6$.

The numbers $C_n=\sqrt{8{B_n}^2+1}$ is called the n th Lucas-balancing number. [24]

The Lucas-balancing numbers $\{C_n\}_{n=0}^{\infty}$ satisfy the recurrence relation

$$C_n=6C_{n-1}-C_{n-2}, n\geq 2$$

with the initial terms, $C_0=1,\ \ C_1=3$.

Binet formula for Lucas-Balancing numbers as follows

$$B_n=\frac{{\lambda_1}^n-{\lambda_2}^n}{2}.$$

We know the following properties for B_n

$$B_{-n}=-B_n \text{ and } C_{-n}=C_n. \text{ [28]}$$

Theorem 1.1: [24] *Let* n *be a natural number then we have*

i) $B_1+B_3+\cdots+B_{2n-1}={B_n}^2$,

ii) $B_2+B_4+\cdots+B_{2n}=B_nB_{n+1}$,

iii) $B_{2n+1}={B_{n+1}}^2-{B_n}^2$,

iv) ${B_n}^2=1+B_{n-1}B_{n+1}$,

v) $B_{2n+2}=B_{n+2}B_{n+1}-B_{n+1}B_n$.

Özdemir introduced the hybrid numbers and give some identities of them [18]. The set of hybrid numbers is $K=\{a+bi+c\varepsilon+dh : a,b,c,d\in\mathbb{R}\}$. Let be $z_1 = a_1 + b_1 i + c_1 \varepsilon + d_1 h,\ z_2 = a_2 + b_2 i + c_2 \varepsilon + d_2 h$ any two hybrid numbers. Then we have the following properties.

1. $z_1 = z_2 \Leftrightarrow a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2$ and $d_1 = d_2$.
2. $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)\varepsilon + (d_1 + d_2)h$
3. $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)\varepsilon + (d_1 - d_2)h$
4. $k.z_1 = ka_1 + kb_1 i + kc_1 \varepsilon + kd_1 h$, where $k \in \mathbb{R}$ **.**

Some key features used for hybrid counts are given by the following definition.

Definition 1.2: There are the following definitions where z is any hybrid number such as

$$z = a + bi + c\,\varepsilon + dh \text{ [19].}$$

1. The conjugate of z is

$$\bar{z} = a - bi - c\varepsilon - dh.$$

2. The character of z is

$$C(z) = z\bar{z} = a^2 + (b - c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2.$$

3. The norm for z has the form

$$\|z\| = N(z) = \sqrt{C(z)}.$$

4. The vector representation for z is

$$V_z = (a, b - c, c, d).$$

5. The scalar section of z is

$$S(z) = a.$$

6. The vector section of z is

$$V(z) = bi + c\varepsilon + dh.$$

The product rule for hybrid units is shown in Table 1.1.

Hybrid numbers have been studied by many scientists and generalizations of these numbers have been made. [1,3,7,14,15,19,27,30-33]

In this paper, we define Balancing hybrid numbers and Lucas-Balancing hybrid numbers and give some properties of them. We find Binet formula, generating functions, exponential generating functions, the sum formula of the Balancing hybrid numbers and the Lucas-Balancing hybrid numbers.

Table 1.1
Multiplication of hybrid unit

.	i	ε	h
i	-1	$1-h$	$\varepsilon+i$
ε	$1+h$	0	$-\varepsilon$
h	$-\varepsilon-i$	ε	1

2. Balancing Hybrid Numbers and Lucas-Balancing Hybrid Numbers

Definition 2.1: Let $n \geq 0$ be integer, the Balancing and the Lucas- Balancing hybrid numbers HB_n and HC_n are defined by

$$HB_n = B_n + iB_{n+1} + \varepsilon B_{n+2} + hB_{n+3} \tag{2.1}$$

$$HC_n = C_n + iC_{n+1} + \varepsilon C_{n+2} + hC_{n+3} \tag{2.2}$$

respectively, where B_n is nth Balancing numbers, C_n is nth Lucas- Balancing numbers and i, ε, h are hybrid units which satisfy multiplication rule as in Table 1.1.

In Table 2.1 and Table 2.2, a few terms of Balancing and Lucas- Balancing hybrid numbers are given.

Table 2.1
Same values of Balancing hybrid numbers

n	HB_n
0	$i + 6\varepsilon + 35h$
1	$1 + 6i + 35\varepsilon + 204h$
2	$6 + 35i + 204\varepsilon + 1189h$
3	$35 + 204i + 1189\varepsilon + 6930h$

Table 2.2
Same values of Lucas-Balancing hybrid numbers

n	HC_n
0	$1 + 3i + 17\varepsilon + 99h$
1	$3 + 17i + 99\varepsilon + 577h$
2	$17 + 99i + 577\varepsilon + 3363h$
3	$99 + 577i + 3363\varepsilon + 19601h$

Definition 2.2: The conjugate of Balancing hybrid number HB_n and HC_n is defined by

$$\overline{HB_n} = \overline{B_n + \imath B_{n+1} + \varepsilon B_{n+2} + hB_{n+3}} = B_n - iB_{n+1} - \varepsilon B_{n+2} - hB_{n+3},$$

$$\overline{HC_n} = \overline{C_n + \imath C_{n+1} + \varepsilon C_{n+2} + hC_{n+3}} = C_n - iC_{n+1} - \varepsilon C_{n+2} - hC_{n+3},$$

respectively.

Theorem 2.3: *The character of Balancing and Lucas-Balancing numbers are as follows, respectively,*

i) $C(HB_n) = 37{B_n}^2 - 1212{B_{n+1}}^2 + 422B_nB_{n+1}$,

ii) $C(HC_n) = 37{C_n}^2 - 1212{C_{n+1}}^2 + 422C_nC_{n+1}$.

Proof: We will only do the proof of i). Part ii) is done similarly.

i) $C(HB_n) = HB_n\overline{HB_n}$

$$= (B_n + iB_{n+1} + \varepsilon B_{n+2} + hB_{n+3})(B_n - iB_{n+1} - \varepsilon B_{n+2} - hB_{n+3})$$

$= {B_n}^2 - iB_nB_{n+1} - \varepsilon B_nB_{n+2} - hB_nB_{n+3} + iB_{n+1}B_n - i^2{B_{n+1}}^2 - i\varepsilon B_{n+1}B_{n+2}$
$-ihB_{n+1}B_{n+3} + \varepsilon B_{n+2}B_n - \varepsilon iB_{n+2}B_{n+1} - \varepsilon^2{B_{n+2}}^2 - \varepsilon hB_{n+2}B_{n+3}$
$hB_{n+3}B_n - hiB_{n+3}B_{n+1} - h\varepsilon B_{n+3}B_{n+2} - h^2{B_{n+3}}^2$
$= {B_n}^2 + {B_{n+1}}^2 - {B_{n+3}}^2 - (1-h)B_{n+1}B_{n+2} - (\varepsilon + i)B_{n+1}B_{n+3} - (h + 1)B_{n+2}B_{n+1} + \varepsilon B_{n+2}B_{n+3} + (\varepsilon + i)B_{n+3}B_{n+1} - \varepsilon B_{n+3}B_{n+2}$
$= {B_n}^2 + {B_{n+1}}^2 - {B_{n+3}}^2 - 2B_{n+1}B_{n+2},$
$= {B_n}^2 + {B_{n+1}}^2 - (35B_{n+1} - 6B_n)^2 - 2B_{n+1}(6B_{n+1} - B_n)$
$= 37{B_n}^2 - 1212{B_{n+1}}^2 + 422B_nB_{n+1}.$

Thus, the proof is obtained.

Definition 2.4: The norm of Balancing and Lucas-Balancing numbers are as follows, respectively,

i) $\|HB_n\| = N(HB_n) = \sqrt{37{B_n}^2 - 1212{B_{n+1}}^2 + 422B_nB_{n+1}},$

ii) $\|HC_n\| = N(HC_n) = \sqrt{37{C_n}^2 - 1212{C_{n+1}}^2 + 422C_nC_{n+1}}.$

Definition 2.5: The vector representation for Balancing and Lucas-Balancing numbers are as follows, respectively,

i) $V_{HB_n} = (B_n, B_{n+1} - B_{n+2}, B_{n+2}, B_{n+3}),$

ii) $V_{HC_n} = (C_n, C_{n+1} - C_{n+2}, C_{n+2}, C_{n+3}).$

Note that, the scaler section of Balancing hybrid numbers is

$$S(HB_n) = B_n$$

and the vector section of Balancing hybrid numbers is

$$V(HB_n) = iB_{n+1} + \varepsilon B_{n+2} + hB_{n+3}.$$

Similarly, we get

$$S(HB_n) = B_n \text{ and } V(HC_n) = iC_{n+1} + \varepsilon C_{n+2} + hC_{n+3}.$$

Theorem 2.4: *For* $n \geq 2$, *we have*

i) $HB_{n+1} = 6HB_n - HB_{n-1},$

ii) $HC_{n+1} = 6HC_n - HC_{n-1},$

Proof:

i) $6HB_n - HB_{n-1} = 6B_n + 6iB_{n+1} + 6\varepsilon B_{n+2} + 6hB_{n+3} - (B_{n-1} + iB_n + \varepsilon B_{n+1} + hB_{n+2})$
$= (6B_n - B_{n-1}) + i(6B_{n+1} - B_n) + \varepsilon(6B_{n+2} - B_{n+1}) + h(B_{n+3} - B_{n+2})$
$= B_{n+1} + iB_{n+2} + \varepsilon B_{n+3} + hB_{n+4} = HB_{n+1}.$

The proof of $ii)$ equality is done in a similar way to $i)$.

Thus, the proof is completed.

Theorem 2.5: *For all non-negative integer* m, n, *we have*

i) $HB_{n+2} - HC_n = 2HC_{n+1},$

ii) $3HB_m \pm HC_n = 3HB_{m+n}$,
iii) $3HC_{n+2} - HC_n = 16HB_{n+1}$

Proof:

i) $HB_{n+2} - HC_n = (B_{n+2} + iB_{n+3} + \varepsilon B_{n+4} + hB_{n+5}) - (C_n + iC_{n+1} + \varepsilon C_{n+2} + hC_{n+3})$
$= (B_{n+2} - C_n) + i(B_{n+3} - C_{n+1}) + \varepsilon(B_{n+4} - C_{n+2}) + h(hB_{n+5} - C_{n+3})$
From $B_{n+2} - C_n = 2C_{n+1}$, we have
$HB_{n+2} - HC_n = 2C_{n+1} + 2iC_{n+2} + 2\varepsilon C_{n+3} + 2hC_{n+4} = 2HC_{n+1}$.
The proofs of ii) and i) equalities are done in a similar way to i).

Theorem 2.6: *(Binet's Formula) For* $n \geq 0$, *we have*

i) $$HB_n = \frac{\lambda_1{}^n \hat{\lambda}_1 - \lambda_2{}^n \hat{\lambda}_2}{\lambda_1 - \lambda_2}, \tag{2.3}$$

ii) $$HC_n = \frac{\lambda_1{}^n \hat{\lambda}_1 + \lambda_2{}^n \hat{\lambda}_2}{2}. \tag{2.4}$$

where

$$\hat{\lambda}_1 = \left(1 + i\lambda_1 + \varepsilon\lambda_1{}^2 + h\lambda_1{}^3\right),$$
$$\hat{\lambda}_2 = \left(1 + i\lambda_2 + \varepsilon\lambda_2{}^2 + h\lambda_2{}^3\right).$$

Proof: Let's expand the right side of equation (2.3) according to the Binet form of the balancing numbers.

$$HB_n = B_n + iB_{n+1} + \varepsilon B_{n+2} + hB_{n+3}$$
$$= \frac{\lambda_1{}^n - \lambda_2{}^n}{\lambda_1 - \lambda_2} + i\frac{\lambda_1{}^{n+1} - \lambda_2{}^{n+1}}{\lambda_1 - \lambda_2} + \varepsilon\frac{\lambda_1{}^{n+2} - \lambda_2{}^{n+2}}{\lambda_1 - \lambda_2} + h\frac{\lambda_1{}^{n+3} - \lambda_2{}^{n+3}}{\lambda_1 - \lambda_2}$$
$$= \frac{\lambda_1{}^n}{\lambda_1 - \lambda_2}\left(1 + i\lambda_1 + \varepsilon\lambda_1{}^2 + h\lambda_1{}^3\right) - \frac{\lambda_2{}^n}{\lambda_1 - \lambda_2}\left(1 + i\lambda_2 + \varepsilon\lambda_2{}^2 + h\lambda_2{}^3\right)$$
$$= \frac{\lambda_1{}^n \hat{\lambda}_1 - \lambda_2{}^n \hat{\lambda}_2}{\lambda_1 - \lambda_2}.$$

Equation (2.4) is shown similarly.

Theorem 2.7: *We have*

i) $\hat{\lambda}_1 \hat{\lambda}_2 = 7 + i(7\lambda_2 - 5\lambda_1) + \varepsilon\left(2\lambda_2{}^2 - \lambda_2 + \lambda_1\right) + h\left(\lambda_2{}^3 + \lambda_1{}^3 - \lambda_2 + \lambda_1\right)$,
ii) $\hat{\lambda}_2 \hat{\lambda}_1 = 7 + i(7\lambda_1 - 5\lambda_2) + \varepsilon\left(2\lambda_1{}^2 + \lambda_2 - \lambda_1\right) + h\left(\lambda_1{}^3 + \lambda_2{}^3 - \lambda_1 + \lambda_2\right)$,
iii) $\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_2 \hat{\lambda}_1 = 14 + 12i + 68\varepsilon + 396h$,
iv) $\lambda_2 \hat{\lambda}_1 \hat{\lambda}_2 - \lambda_1 \hat{\lambda}_2 \hat{\lambda}_1 = (\lambda_2 - \lambda_1)(7 + 42i + 64\varepsilon + 192h)$.

Theorem 2.8: *Let* $S(HB_n)$ *be the sum of the Balancing and* $S(HC_n)$ *be the sum of the Lucas-Balancing hybrid numbers. Then we have*

i) $S(HB_n) = \sum_{k=0}^{n} HB_k = \frac{1}{4}[HC_{n+1} - 2HB_{n+1} - 1 - i - 5\varepsilon - 29h]$,
ii) $S(HC_n) = \sum_{k=0}^{n} HC_k = \frac{1}{4}[HC_{n+1} - HC_n + 2 - 2i - 14\varepsilon - 84h]$.

Proof:

i) $S(HB_n) = \sum_{i=0}^{n} HB_i = \sum_{k=0}^{n} B_k + i\sum_{k=0}^{n} B_{k+1} + \varepsilon \sum_{k=0}^{n} B_{k+2} + h\sum_{k=0}^{n} B_{k+3}$

$$= \left(\frac{C_{n+1}-2B_{n+1}-1}{4}\right) + i\left(\frac{C_{n+2}-2B_{n+2}-1}{4}\right) + \varepsilon\left(\frac{C_{n+3}-2B_{n+3}-1}{4} - 1\right) + h\left(\frac{C_{n+4}-2B_{n+4}-1}{4} - 7\right)$$

$$= \frac{1}{4}[HC_{n+1} - 2HB_{n+1} - 1 - i - 5\varepsilon - 29h]$$

The proof of $ii)$ equality is done in a similar way to $i)$.

Thus, the proof is obtained.

Theorem 2.9: *Let* n *be a natural number, then we get*

i) $HB_1 + HB_3 + \cdots + HB_{2n-1} = B_{n+1}HB_{n-1} + 1 - \varepsilon - 6h.$

ii) $HB_2 + HB_4 + \cdots + HB_{2n} = B_{n+1}HB_n + 1 - 6\varepsilon - 35h.$

Proof: i) $HB_1 + HB_3 + \cdots + HB_{2n-1}$

$$= (B_1 + iB_2 + \varepsilon B_3 + hB_4) + (B_3 + iB_4 + \varepsilon B_5 + hB_6) + \cdots + (B_{2n-1} + iB_{2n} + \varepsilon B_{2n+1} + hB_{2n+2})$$

$$= (B_1 + B_3 + \cdots + B_{2n-1}) + i(B_2 + B_4 + \cdots + B_{2n}) + \varepsilon(B_3 + iB_5 + \cdots + B_{2n+1}) + h(B_4 + B_6 + \cdots + B_{2n+2})$$

From Theorem 1.1. $i)$ and $ii)$, we have

$HB_1 + HB_3 + \cdots + HB_{2n-1}$

$$= {B_n}^2 + iB_nB_{n+1} + \varepsilon\left({B_n}^2 - 1 + B_{2n+1}\right) + h(B_nB_{n+1} - 6 + B_{2n+2})$$

From Theorem 1.1. $iii), iv)$ and $v)$, we have

$HB_1 + HB_3 + \cdots + HB_{2n-1}$

$$= 1 + B_{n-1}B_{n+1} + iB_nB_{n+1} + \varepsilon\left({B_{n+1}}^2 - 1\right) + h(B_{n+1}B_{n+2} - 6)$$

$$= B_{n+1}(B_{n-1} + iB_n + \varepsilon B_{n+1} + hB_{n+2}) + 1 - \varepsilon - 6h$$

$$= B_{n+1}HB_{n-1} + 1 - \varepsilon - 6h.$$

The proof of $ii)$ equality is done in a similar way to $i)$.

Thus, the proof is obtained.

Theorem 2.10: *The generating functions for the Balancing hybrid numbers and Lucas-Balancing hybrid numbers are given, respectively, by*

i) $GHB_n(x) = \sum_{n=1}^{\infty} HB_n x^n = \frac{i+6\varepsilon+35h+x(1-\varepsilon-6h)}{1-6x+x^2}$,

ii) $GHC_n(x) = \sum_{n=1}^{\infty} Hc_n x^n = \frac{1+3i+17\varepsilon+99h-x(3+i+3\varepsilon+17h)}{1-6x+x^2}$.

Proof: i)

$$GHB_n(x) = HB_0 + HB_1x + HB_2x^2 + HB_3x^3 + \cdots + HB_nx^n + \cdots \quad (2.5)$$

Now, multiplying the equation (2.5) by $1, 6x$ and x^2, respectively, we get

$$GHB_n(x) = HB_0 + HB_1x + HB_2x^2 + HB_3x^3 + \cdots + HB_nx^n + \cdots$$

$$6xGHB_n(x) = 6xHB_0 + 6x^2HB_1 + 6x^3HB_2 + 6x^4HB_3 + \cdots + 6x^{n+1}\,HB_n + \cdots$$
$$x^2GHB_n(x) = \ x^2HB_0 + \ x^3HB_1 + x^4HB_2 + x^5HB_3 + \cdots + x^{n+2}\,HB_n + \cdots$$

After necessary calculations and using the recurrence relation, we obtain

$$GHB_n(x)(1-6x+x^2) = HB_0 + HB_1x - 6xHB_0,$$
$$GHB_n(x) = \frac{HB_0 + x(HB_1 - 6HB_0)}{1-6x+x^2}$$
$$= \frac{i + 6\varepsilon + 35h + x(1-\varepsilon-6h)}{1-6x+x^2}.$$

The proof of (ii) is similar to that of (i). So, we omit them.

Theorem 2.11: *The exponential generating functions for the Balancing hybrid numbers and Lucas-Balancing hybrid numbers are given by*

i) $\sum_{n=1}^{\infty} HB_n \frac{x^n}{n!} = \frac{\hat{\lambda}_1 e^{\lambda_1 x} - \hat{\lambda}_2 e^{\lambda_2 x}}{\lambda_1 - \lambda_2}$

ii) $\sum_{n=1}^{\infty} HC_n \frac{x^n}{n!} = \frac{\hat{\lambda}_1 e^{\lambda_1 x} + \hat{\lambda}_2 e^{\lambda_2 x}}{2}$

respectively.

Proof: i)

$$\sum_{n=1}^{\infty} HB_n \frac{x^n}{n!} = \sum_{n=1}^{\infty} \left(\frac{{\lambda_1}^n \hat{\lambda}_1 - {\lambda_2}^n \hat{\lambda}_2}{\lambda_1 - \lambda_2}\right) \frac{x^n}{n!}$$
$$= \frac{\hat{\lambda}_1}{\lambda_1 - \lambda_2} \sum_{n=1}^{\infty} \frac{(\lambda_1 x)^n}{n!} - \frac{\hat{\lambda}_2}{\lambda_1 - \lambda_2} \sum_{n=1}^{\infty} \frac{(\lambda_2 x)^n}{n!}$$
$$= \frac{\hat{\lambda}_1 e^{\lambda_1 x} - \hat{\lambda}_2 e^{\lambda_2 x}}{\lambda_1 - \lambda_2}.$$

The proof of (ii) is similar to that of (i). So, the proof is completed.

Theorem 2.12: *(Catalan identity) We have,*

i) $HB_{n-r}HB_{n+r} - (HB_n)^2 = -B_r(7B_r + i\,(7B_{r+1} - 5B_{r-1}) + \varepsilon(2B_{r+2} - B_{r+1} - B_{r-1}) \ + (B_{r+3} + B_{r-3} - B_{r-1} - B_{r+1})h),$

ii) $HC_{n-r}HC_{n+r} - (HC_n)^2 = \frac{-C_r}{2}(7B_r + i\,(7B_{r+1} - 5B_{r-1}) + \varepsilon(2B_{r+2} - B_{r+1} - B_{r-1}) \ + (B_{r+3} + B_{r-3} - B_{r-1} - B_{r+1})h).$

Proof:

i)
$$HB_{n-r}HB_{n+r} - (HB_n)^2$$
$$= \left(\frac{{\lambda_1}^{n-r}\hat{\lambda}_1 - {\lambda_2}^{n-r}\hat{\lambda}_2}{\lambda_1 - \lambda_2}\right)\left(\frac{{\lambda_1}^{n+r}\hat{\lambda}_1 - {\lambda_2}^{n+r}\hat{\lambda}_2}{\lambda_1 - \lambda_2}\right) - \left(\frac{{\lambda_1}^{n}\hat{\lambda}_1 - {\lambda_2}^{n}\hat{\lambda}_2}{\lambda_1 - \lambda_2}\right)^2$$
$$= \frac{1}{(\lambda_1 - \lambda_2)^2}\left(-{\lambda_1}^{n-r}{\lambda_2}^{n+r}\hat{\lambda}_1\hat{\lambda}_2 - {\lambda_2}^{n-r}{\lambda_1}^{n+r}\hat{\lambda}_2\hat{\lambda}_1 + {\lambda_1}^n{\lambda_2}^n\hat{\lambda}_1\hat{\lambda}_2 + {\lambda_2}^n{\lambda_1}^n\hat{\lambda}_2\hat{\lambda}_1\right)$$
$$= \frac{1}{(\lambda_1 - \lambda_2)^2}\left({\lambda_1}^n{\lambda_2}^n\hat{\lambda}_1\hat{\lambda}_2\left(1 - \frac{{\lambda_2}^r}{{\lambda_1}^r}\right) + {\lambda_2}^n{\lambda_1}^n\hat{\lambda}_2\hat{\lambda}_1\left(1 - \frac{{\lambda_1}^r}{{\lambda_2}^r}\right)\right)$$
$$= \frac{({\lambda_1}^r - {\lambda_2}^r)}{(\lambda_1 - \lambda_2)^2}\left(\frac{\hat{\lambda}_1\hat{\lambda}_2}{{\lambda_1}^r} - \frac{\hat{\lambda}_2\hat{\lambda}_1}{{\lambda_2}^r}\right)$$

$$= \frac{({\lambda_1}^r - {\lambda_2}^r)}{(\lambda_1-\lambda_2)}\left({\lambda_2}^r \hat{\lambda}_1\hat{\lambda}_2 - {\lambda_1}^r \hat{\lambda}_2\hat{\lambda}_1\right)$$

$$= -\frac{({\lambda_1}^r - {\lambda_2}^r)}{(\lambda_1-\lambda_2)}(7B_r + i\,(7B_{r+1} - 5B_{r-1}) + \varepsilon(2B_{r+2} - B_{r+1} - B_{r-1}) + (B_{r+3} + B_{r-3} - B_{r-1} - B_{r+1})h)$$

The proof of (ii) is similar to that of (i). Thus, the proof is completed.

Theorem 2.13: *(Cassini identity) We have*

i) $HB_{n-1}HB_{n+1} - (HB_n)^2 = -(7 + 42i + 64\varepsilon + 192h)$,

ii) $HC_{n-1}HC_{n+1} - (HC_n)^2 = -\frac{17}{2}(7 + 42i + 64\varepsilon + 192h)$.

Proof: By using the Binet formula of the Balancing hybrid numbers we have,

i) $HB_{n-1}HB_{n+1} - (HB_n)^2$

$$= \left(\frac{{\lambda_1}^{n-1}\hat{\lambda}_1 - {\lambda_2}^{n-1}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)\left(\frac{{\lambda_1}^{n+1}\hat{\lambda}_1 - {\lambda_2}^{n+1}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right) - \left(\frac{{\lambda_1}^{n}\hat{\lambda}_1 - {\lambda_2}^{n}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)^2$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\Big({\lambda_1}^{2n}{\hat{\lambda}_1}^{2} - {\lambda_1}^{n-1}{\lambda_2}^{n+1}\hat{\lambda}_1\hat{\lambda}_2 - {\lambda_2}^{n-1}{\lambda_1}^{n+1}\hat{\lambda}_2\hat{\lambda}_1 + {\lambda_2}^{2n}{\hat{\lambda}_2}^{2}$$
$$-{\lambda_1}^{2n}{\hat{\lambda}_1}^{2} + {\lambda_1}^{n}{\lambda_2}^{n}\hat{\lambda}_1\hat{\lambda}_2 + {\lambda_2}^{n}{\lambda_1}^{n}\hat{\lambda}_2\hat{\lambda}_1 - {\lambda_2}^{2n}{\hat{\lambda}_2}^{2}\Big)$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\left({\lambda_1}^{n}{\lambda_2}^{n}\hat{\lambda}_1\hat{\lambda}_2\left(1 - \frac{\lambda_2}{\lambda_1}\right) + {\lambda_2}^{n}{\lambda_1}^{n}\hat{\lambda}_2\hat{\lambda}_1\left(1 - \frac{\lambda_1}{\lambda_2}\right)\right)$$

$$= \frac{(\lambda_1\lambda_2)^n(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)^2}\left(\frac{\hat{\lambda}_1\hat{\lambda}_2}{\lambda_1} - \frac{\hat{\lambda}_2\hat{\lambda}_1}{\lambda_2}\right)$$

$$= \frac{1}{(\lambda_1-\lambda_2)}\left(\lambda_2\hat{\lambda}_1\hat{\lambda}_2 - \lambda_1\hat{\lambda}_2\hat{\lambda}_1\right) = -(7 + 42i + 64\varepsilon + 192h)$$

$$= -(7B_1 + 7B_2 i + (2B_3 - B_2)\varepsilon + (B_4 - B_2 + B_{-2})h).$$

The proof of (ii) is similar to that of (i). Thus, the proof is completed.

Note that, if we get $r = 1$ in Catalan identity then, we obtain the Cassini identity.

Theorem 2.12: *(Vajda identity) We have*

i) $HB_{n+r}HB_{n+k} - HB_nHB_{n+r+k} = -B_r\big(7B_k + i\,(7B_{k+1} - 5B_{k-1}) + \varepsilon(2B_{k+2} - B_{k+1} - B_{k-1}) + h(B_{k+3} + B_{k-3} - B_{k-1} - B_{k+1})\big)$,

ii) $HC_{n+r}HC_{n+k} - HC_nHC_{n+r+k} = -\frac{C_r}{2}\big(7B_k + i\,(7B_{k+1} - 5B_{k-1}) + \varepsilon(2B_{k+2} - B_{k+1} - B_{k-1}) + h(B_{k+3} + B_{k-3} - B_{k-1} - B_{k+1})\big)$.

Proof: i)

$HB_{n+r}HB_{n+k} - HB_nHB_{n+r+k} =$

$$\left(\frac{{\lambda_1}^{n+r}\hat{\lambda}_1 - {\lambda_2}^{n+r}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)\left(\frac{{\lambda_1}^{n+k}\hat{\lambda}_1 - {\lambda_2}^{n+k}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right) - \left(\frac{{\lambda_1}^{n}\hat{\lambda}_1 - {\lambda_2}^{n}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)\left(\frac{{\lambda_1}^{n+r+k}\hat{\lambda}_1 - {\lambda_2}^{n+r+k}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)$$

$= \frac{1}{(\lambda_1-\lambda_2)^2}\left(-{\lambda_1}^{n+r}{\lambda_2}^{n+k}\hat{\lambda}_1\hat{\lambda}_2 - {\lambda_2}^{n+r}{\lambda_1}^{n+k}\hat{\lambda}_2\hat{\lambda}_1 + {\lambda_1}^{n}{\lambda_2}^{n+r+k}\hat{\lambda}_1\hat{\lambda}_2 + {\lambda_2}^{n}{\lambda_1}^{n+r+k}\hat{\lambda}_2\hat{\lambda}_1\right)$

$= \frac{1}{(\lambda_1-\lambda_2)^2}\left({\lambda_1}^{n}{\lambda_2}^{n+k}\hat{\lambda}_1\hat{\lambda}_2({\lambda_2}^{r} - {\lambda_1}^{r}) + {\lambda_2}^{n}{\lambda_1}^{n+k}\hat{\lambda}_2\hat{\lambda}_1({\lambda_1}^{r} - {\lambda_2}^{r})\right)$

$= \frac{({\lambda_2}^{r}-{\lambda_1}^{r})}{(\lambda_1-\lambda_2)}\left({\lambda_2}^{k}\hat{\lambda}_1\hat{\lambda}_2 - {\lambda_1}^{k}\hat{\lambda}_2\hat{\lambda}_1\right)$

$= -B_r(7B_k + i\,(7B_{k+1} - 5B_{k-1}) + \varepsilon(2B_{k+2} - B_{k+1} - B_{k-1}) + (B_{k+3} + B_{k-3} - B_{k-1} - B_{k+1})h).$

The proof of ii) is similar to that of i). So, we omit the proof.

Theorem 2.13: *(D' ocagne identity) For* $n \leq m$, *we have*

i) $HB_mHB_{n+1} - HB_nHB_{m+1} = -B_{m-n}(7 + 42i + 64\varepsilon + 192h)$,

ii) $HC_mHC_{n+1} - HC_nHC_{m+1} = -\frac{C_{m-n}}{2}(7 + 42i + 64\varepsilon + 192h)$.

Proof: i)

$HB_mHB_{n+1} - HB_nHB_{m+1}$

$= \left(\frac{{\lambda_1}^{m}\hat{\lambda}_1-{\lambda_2}^{m}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)\left(\frac{{\lambda_1}^{n+1}\hat{\lambda}_1-{\lambda_2}^{n+1}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right) - \left(\frac{{\lambda_1}^{n}\hat{\lambda}_1-{\lambda_2}^{n}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)\left(\frac{{\lambda_1}^{m+1}\hat{\lambda}_1-{\lambda_2}^{m+1}\hat{\lambda}_2}{\lambda_1-\lambda_2}\right)$

$= \frac{1}{(\lambda_1-\lambda_2)^2}\left(-{\lambda_1}^{m}{\lambda_2}^{n+1}\hat{\lambda}_1\hat{\lambda}_2 - {\lambda_2}^{m}{\lambda_1}^{n+1}\hat{\lambda}_2\hat{\lambda}_1 + {\lambda_1}^{n}{\lambda_2}^{m+41}\hat{\lambda}_1\hat{\lambda}_2 + {\lambda_2}^{n}{\lambda_1}^{m+1}\hat{\lambda}_2\hat{\lambda}_1\right)$

$\frac{1}{(\lambda_1-\lambda_2)^2}\left({\lambda_1}^{m}{\lambda_2}^{n}(\lambda_1\hat{\lambda}_2\hat{\lambda}_1 - \lambda_2\hat{\lambda}_1\hat{\lambda}_2) + {\lambda_1}^{n}{\lambda_2}^{m}(\lambda_2\hat{\lambda}_1\hat{\lambda}_2 - \lambda_1\hat{\lambda}_2\hat{\lambda}_1)\right)$

$= \frac{1}{(\lambda_1-\lambda_2)^2}(\lambda_1\hat{\lambda}_2\hat{\lambda}_1 - \lambda_2\hat{\lambda}_1\hat{\lambda}_2)({\lambda_1}^{m}{\lambda_2}^{n} - {\lambda_1}^{n}{\lambda_2}^{m})$

$= \frac{1}{(\lambda_1-\lambda_2)^2}(\lambda_2 - \lambda_1)(7 + 42i + 64\varepsilon + 192h){\lambda_1}^{n}{\lambda_2}^{n}({\lambda_1}^{m-n} - {\lambda_2}^{m-n})$

$= -(7 + 42i + 64\varepsilon + 192h)B_{m-n}$

The proof of ii) is similar to that of i). So, we omit the proof.

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