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On the solution of two-person zero-sum matrix games

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Abstract

In this article, 0-sum 2-person matrix games are considered. Some basic concepts and results are recalled. The numerical solution of matrix games by dual problems of linear optimization is discussed. Illustrative numerical examples are also included.

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1. Introduction

In various human activities, conflict situations arise, which are characterized by opposite interests of individuals or teams, trying to achieve their goals to the detriment of each other (in antagonistic models) or in favor of each other (in cooperative models).

Each part formulates the goals, has active means to achieve them, evaluates strategies (approaches), makes reasonable (optimal) choice of behavior in the specific situation, that is, a kind of game with a rational opponent or competitor is led.

The part of Operations Research, connected with mathematical modeling of conditions of the conflict and search for an optimal solution (decision) on this basis, is called Game theory. Here, the concept of "game" is used in the sense of "mathematical model of conflict situation".

In Game theory, participants in the game, that is, opponents (partners, respectively), are called *players* (*participants in the game*). Each participant has a set – finite or infinite – opportunities for action, called *strategies*.

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Payoffs (or *payments, gains, results*) in a game are defined by functions which depend on each of strategies of the participants.

Matrix $0-sum$ $2-person\ game$ is a collection of rules, such that after their implementation, the sum of profits (income) v_1 and v_2 of the participants is equal to zero: $v_1+v_2=0,$ that is, the profit of the 1st participant is equal to the loss of the 2nd participant, and vice versa.

For such a game, it is enough to know the results in the form of payments for one of the participants, that is, to know the so-called *payoff (payment) matrix*.

Let among m possible moves (*pure strategies*) the 1st participant selects i-th move (i-th pure strategy), and 2nd participant, among n possible moves (pure strategies) selects j-th move (j-th pure strategy), and for this combination of moves, the 1st participant obtains a_{ij} units profit, and the 2nd participant obtains $(-a_{ij})$ units "profit" (interests are opposite and the game is 0-sum).

The $m\times n$ matrix

$$A=(a_{ij})_{m\times n}=\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}\in\mathbb{R}^{m.n}$$

is said to be a *payoff (payment, profit) matrix* for the 1st participant.

Because the interests are opposite, then $-A=(-a_{ij})_{m\times n}$ turns out to be payment matrix of the 2nd participant.

Example 1: ("Coin flipping", "Coin tossing", "Heads or tails"): Each of two participants, without knowing the strategy of another participant, bets on his coin "head" or "tail" above. In the case of matching names "head" or "tail", the 2nd participant pays the 1st participant a coin unit, and in the case when the names "head" or "tail" do not match, the 1st participant pays the 2nd participant a unit coin.

Each participant has two strategies in this game: "head" or "tail", which form the payment matrix for the 1st participant

$$A=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}_{2\times 2}.$$

In the rows of A, all possible profits of the 1st participant are entered, when he chooses "head" or "tail", respectively, and in the columns of A, all possible profits of the 1st participant are entered, when the 2nd participant chooses "head" or "tail", respectively.

Example 2. (Colonel Blotto Game): Two armies are fighting for two objects. The first army, commanded by Colonel Blotto, consists of 4 regiments, and the second army consists of 3 regiments. The army that sends more regiments to the object, conquers it, defeats all enemy regiments there, and receives a point for a conquered object and a point for each defeated regiment. If both armies send equal number of regiments to the same object, each of them receives 0 points, because the forces are equal. Colonel Blotto's goal is to distribute his regiments so that to get more points, that is, to get better score.

Find the payment matrix of this game.

Solution: Each strategy is presented as an ordered pair of nonnegative integers, such that the first nonnegative integer shows the number of regiments, sent to the first object, and the second one shows the number of regiments, sent to the second object by the corresponding army.

Strategies of the first army, which has 4 regiments, are $(4,0)$, $(0,4)$, $(3,1)$, $(1,3)$, $(2,2)$, and strategies of the second army, which has 3 regiments, are $(3,0)$, $(0,3)$, $(2,1)$, $(1,2)$.

Then payment matrix of the game is

I army/II army	(3,0)	(0, 3)	(2, 1)	(1, 2)
(4, 0)	4	0	2	1
(0, 4)	0	4	1	2
(3, 1)	1	–1	3	0
(1, 3)	–1	1	0	3
(2, 2)	–2	–2	2	2

Notes about obtaining elements of the first column of the payment matrix:

Concerning first strategy of the I-st army and first strategy of the II-nd army: the relationship between the forces at the first object is 4 : 3 regiments, therefore I-st army conquers this object and receives a point for conquering this object and a point for each of the three defeated regiments of the II-nd army there; since both armies sent 0 regiments to the second object, then both armies receive 0 points for it. Therefore I-st army receives $1+3=4$ points.

Concerning second strategy of I-st army and first strategy of II-nd army: the relationship between the forces at the first object is 0 : 3 regiments, therefore II-nd army conquers the first object and receives a point for conquering this object, therefore I-st army receives (-1) point; I-st army

conquers the second object because relationship between the forces at this object is 4 : 0 regiments, and it receives 1 point for conquering this object, and does not receive points for defeated regiments because II-nd army does not have regiments there. Hence, I-st army receives a total of $-1+1=0$ points.

Concerning third strategy of I-st army and first strategy of II-army: since both armies sent 3 regiments to the first object (equal forces 3 : 3), they receive 0 points for this object; I-st army conquers the second object (relationship between the forces at this object is 1 : 0) and it receives 1 point for conquering this object, and does not receive points for defeated regiments because II-nd army does not have regiments there. Therefore, I-st army receives a total of $1+0=1$ point.

Regarding fourth strategy of I-st army and first strategy of II-nd army: II-nd army conquers the first object because it sends 3 regiments whereas I-st army sends 1 regiment there, and defeats 1 regiment of the I-st army, therefore I-st army receives (-1) point for the loss of this object and (-1) for the lost regiment; I-st army conquers the second object (relationship between the forces at this object is 3 : 0), it receives 1 point for conquering this object, and since there are no regiments of II-nd army, then I-st army receives 0 points for defeated regiments. Therefore, I-st army receives a total of $-1-1+1=-1$ point.

Concerning fifth strategy of I-st army and first strategy of II-nd army: II-nd army conquers the first object (relationship between the forces at this object is 2 : 3), hence I-st army receives (-1) point; II-nd army defeats both regiments of I-st army at the first object, therefore I-st army receives (-2) points; I-st army conquers the second object (relationship between the forces is 2 : 0) and receives 1 point for this conquered object, and does not receive points for defeated regiments because II-nd army does not have regiments there. Therefore, I-st army receives a total of $-1-2+1=-2$ points.

Elements of remaining columns in the required payment matrix, presented by a tableau, are calculated similarly.

The topics, considered in this article, are studied, for example, in [1–11].

2. Solution of matrix games $m \times n$

The following property holds true.

Property 1: If the value of a matrix game with payment matrix $A=(a_{ij})_{m\times n}$ is v, then the value of a matrix game with payment matrix $A_c=(a_{ij}+c)_{m\times n}$, where $c=const.$, is $v+c$.

Indeed, we have

$$E'(x,y)=\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+c)x_iy_j=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_iy_j+c\left(\sum_{i=1}^{m}x_i\right)\left(\sum_{j=1}^{n}y_j\right)=E(x,y)+c,$$

$$E'(x^*,y^*)=\max_{x\in X}(\min_{y\in Y}E'(x,y))=\max_{x\in X}(\min_{y\in Y}E(x,y))+c=E(x^*,y^*)+c=v+c,$$

where x_1, …, x_m are the probabilities (frequencies), with which 1st participant selects the 1-st, …, m-th pure strategy, respectively, y_1, …, y_n are probabilities (frequencies), with which 2nd participant selects the 1-st, …, n-th pure strategies, and

$$E(x,y)=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_iy_j$$

is the *mathematical expectation* for profit of 1st participant.

Consider 0-sum 2-person matrix game with payment matrix

$$A=\begin{pmatrix}a_{11} & \dots & a_{1n}\\ \dots & \dots & \dots\\ a_{m1} & \dots & a_{mn}\end{pmatrix}, \tag{1}$$

where $a_{ij}\geq 0$ for each $i=1,\dots,m$ and $j=1,\dots,n$.

If the condition $a_{ij}\geq 0$ for each $i=1,\dots,m$ and $j=1,\dots,n$, is not satisfied, consider a matrix game with payment matrix $A_c=(a_{ij}+c)_{m\times n}$, where c is an appropriately chosen number, such that $a_{ij}+c\geq 0$ for each $i=1,\dots,m$ and $j=1,\dots,n$, and apply Property 1.

The following linear program can be associated with 1st participant I.

$$\min\left\{w(\mathrm{u})=\sum_{i=1}^{m}u_i\right\}$$

s.t.

$$\sum_{i=1}^{m}a_{ij}u_i\geq 1,\quad j=1,\dots,n,$$

$$u_i\geq 0,\quad i=1,\dots,m,$$

and the following linear program can be associated with 2nd participant

II.

$$\max\left\{z(\xi)=\sum_{j=1}^{n}\xi_j\right\}$$

s.t.

$$\sum_{j=1}^{n}a_{ij}\xi_j\le 1,\quad i=1,\dots,m,$$

$$\xi_j\ge 0,\quad j=1,\dots,n.$$

Linear optimization programs I. and II. are pair of a primal program and a dual program. Matrices of coefficients of constraints of program I. and program II. are A^T and A, respectively.

Programs I. and II. are solved by the simplex method (or by the so-called Big M-method, if necessary).

Let $\mathrm{u}^*=(u_1^*,\dots,u_m^*)$ be optimal solution of program I. and $\xi^*=(\xi_1^*,\dots,\xi_n^*)$ be optimal solution of program II. According to Duality theorem,

$$w(\mathrm{u}^*)=z(\xi^*)\overset{\text{def}}{=}\frac{1}{v}.$$

Then v is value of the considered matrix game (or the matrix game with payment matrix A_c), and optimal solutions (optimal strategies) of 1st participant and 2nd participant are $\mathrm{x}^*=v\mathrm{u}^*$ and $\mathrm{y}^*=v\xi^*$, respectively.

Example 3 below illustrates the approach described above.

Example 3: Given the matrix game with payment matrix

$$A=\begin{pmatrix}1 & -4 & 3\\ -4 & 3 & -6\\ 3 & -6 & 5\end{pmatrix}.$$

Find optimal strategies of both participants.

Solution: Find minimal elements per rows and maximal elements per columns of the payment matrix:

$$A=\begin{pmatrix}1 & -4 & 3 & -4\\ -4 & 3 & -6 & -6\\ 3 & -6 & 5 & -6\\ 3 & 3 & 5 & \end{pmatrix}$$

and calculate

$$\alpha = \max_i \min_j a_{ij} = \max\{-4,-6,-6\} = -4, \quad \beta = \min_j \max_i a_{ij} = \min\{3,3,5\} = 3.$$

Since $\alpha \neq \beta$, this game has no saddle point.

Since there are negative a_{ij}'s and $\min_{i,j} a_{ij} = -6$, we choose, for example, $c = 7$, and obtain the following payment matrix with positive entries

$$A_c = (a_{ij} + c)_{3\times 3} = \begin{pmatrix} 8 & 3 & 10 \\ 3 & 10 & 1 \\ 10 & 1 & 12 \end{pmatrix}.$$

The linear optimization problem, associated with the 1st participant of the matrix game with payment matrix A_c, is

I.

$$\min\{w(\mathrm{u}) = u_1 + u_2 + u_3\}$$

s.t.

$$8u_1 + 3u_2 + 10u_3 \geq 1$$

$$3u_1 + 10u_2 + u_3 \geq 1$$

$$10u_1 + u_2 + 12u_3 \geq 1$$

$$u_1 \geq 0, u_2 \geq 0, u_3 \geq 0,$$

and the linear optimization problem, associated with 2nd participant of the matrix game with payment matrix A_c, is

II.

$$\max\{z(\xi) = \xi_1 + \xi_2 + \xi_3\}$$

s.t.

$$8\xi_1 + 3\xi_2 + 10\xi_3 \leq 1$$

$$3\xi_1 + 10\xi_2 + \xi_3 \leq 1$$

$$10\xi_1 + \xi_2 + 12\xi_3 \leq 1$$

$$\xi_1 \geq 0, \xi_2 \geq 0, \xi_3 \geq 0.$$

As we know, I. and II. are pair of primal and dual linear optimization problems.

The optimal solution of program I. is obtained by simplex method

$$u_1^* = \frac{1}{24}, u_2^* = \frac{1}{12}, u_3^* = \frac{1}{24},$$

and the optimal value of the objective function of this program is $w(\mathrm{u}^*) = \frac{1}{24} + \frac{1}{12} + \frac{1}{24} = \frac{1}{6}$.

Similarly, the optimal solution to program II. is

$$\xi_1^* = \frac{1}{24}, \xi_2^* = \frac{1}{12}, \xi_3^* = \frac{1}{24},$$

and the optimal value of the objective function of this program is $z(\xi^*) = \frac{1}{24} + \frac{1}{12} + \frac{1}{24} = \frac{1}{6}$.

Therefore $v_c = \frac{1}{w(\mathrm{u}^*)} = \frac{1}{z(\xi^*)} = \frac{1}{\frac{1}{6}} = 6$ is the value of matrix game with payment matrix A_c and

$$x_1^* = v_c u_1^* = 6.\frac{1}{24} = \frac{1}{4}, \quad x_2^* = v_c u_2^* = 6.\frac{1}{12} = \frac{1}{2}, \quad x_3^* = v_c u_3^* = 6.\frac{1}{24} = \frac{1}{4}.$$

Similarly,

$$y_1^* = v_c \xi_1^* = 6.\frac{1}{24} = \frac{1}{4}, \quad y_2^* = v_c \xi_2^* = 6.\frac{1}{12} = \frac{1}{2}, \quad y_3^* = v_c \xi_3^* = 6.\frac{1}{24} = \frac{1}{4}.$$

Therefore optimal mixed strategies of 1st participant and 2nd participant are

$$S_{\mathrm{I}}^* = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\} \quad \text{and} \quad S_{\mathrm{II}}^* = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\},$$

respectively, and value of the original game with payment matrix A is $v = v_c - c = 6 - 7 = -1$.

Due to the symmetry of the given matrix A (and, hence, symmetry of A_c), optimal strategies of the 1st participant and the 2nd participant are the same.

3. Conclusion

In this article, the 0-sum 2-person matrix games are considered. Some basic concepts and results are recalled. The numerical solution of matrix games by a pair of primal problem and dual problem is discussed. Some illustrative numerical examples are also included.

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