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Algebraic spectral analysis of the locating degree matrix of a graph

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Abstract

In this article, we present a new graph matrix based on locating degree, we call it the locating degree matrix. The spectrum and energy related to this new matrix are established for certain classes of graphs. Bounds of the new related energy for a special type of regular graph are obtained.

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1. Introduction

There are many graph matrices that represent the graph. These matrices contains almost of the graph properties in a digital form. One of the famous matrices is the adjacency matrix. For a graph G, $A(G) = A = [a_{ij}]$ is defined as the $l \times l$ matrix with l vertices whereas $a_{ij} = 1$ if w_i is adjacent to w_j and $a_{ij} = 0$ furthermore. These roots of the polynomial $|\lambda I - A|$, with I denoting the identity matrix, represent the G eigenvalues. The eigenvalues of the adjacency matrix are all real numbers with the sum equal to zero since the matrix is symmetric. The list of different eigenvalues of G, $\lambda_1 > \lambda_2 > \ldots > \lambda_r$ with repetitions $m_1, m_2, \ldots, m_r$, will form the spectrum of the graph G and represented by

$$\mathrm{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

In [7], the graph energy has defined as the sum of the absolute values of eigenvalues:

$$E(G) = \sum_{i=1}^{r} |\gamma_i|.$$

The book [16], has information and details on graph energy theory, whereas [11] and [8] include information on its chemical applications. On graph energy, a lot of study has been done. in [1, 2, 3, 4, 10, 13, 14], For spectral analysis see [17].

Lemma 1.1: [5] *For the graphs* K_l, $K_{l,s}$ *and* C_l,

- $\mathrm{Spec}(K_l) = \begin{pmatrix} l-1 & -1 \\ 1 & l-1 \end{pmatrix}.$
- $\mathrm{Spec}(K_{l,s}) = \begin{pmatrix} \sqrt{ls} & -\sqrt{ls} & 0 \\ 1 & 1 & l+s-2 \end{pmatrix}.$
- $\mathrm{Spec}(C_l) = \begin{cases} \begin{pmatrix} 2 & 2\cos\frac{2\pi}{l} & \cdots & 2\cos\frac{2(l-1)\pi}{l} \\ 1 & 2 & \cdots & 2 \end{pmatrix}, & \text{if } l \text{ is odd;} \\ \begin{pmatrix} 2 & 2\cos\frac{2\pi}{l} & \cdots & 2\cos\frac{2(l-2)\pi}{l} & -2 \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}, & \text{if } l \text{ is even.} \end{cases}$

2. The Locating Degree Spectrum

The locating degree matrix (LDM for abbreviation) of a graph G which we introduce in this section was motivated by earlier work in [15] and [18].

Definition 2.1: For any connected graph G with vertex set $V(G)=\{w_1,w_2,...,w_l\}$. G's locating degree function $\mathbf{L}_s(G)$ from $V(G)$ to $(\mathbb{Z}\cup\{0\})^{s+1}$, where s is the diameter of the graph described as $\mathbf{L}_s(w_j)$ = $\vec{w}_j$ = $(\Gamma_0(w_j)$, $\Gamma_1(w_j)$, ..., $\Gamma_s(w_j))$, where $\Gamma_i(w_j)$ is the number of vertices in G that are at most i edges away from the vertex w_j. For the vertex w_j, the vector $\vec{w}_j$ is referred to as the locating degree vector. In a graph G, the product of two locating degree vectors w_i and w_j is represented by the symbol $Ld(w_i.w_j)$, and it is defined as:

$$Ld(\overrightarrow{w_i}.\overrightarrow{w_j})=\begin{cases}\overrightarrow{w_i}.\overrightarrow{w_j}, & \text{if } i \text{ does not equal } j \text{ and } w_i \text{ adjacent to } w_j;\\ 0 & \text{otherwise,}\end{cases}$$

such that $\overrightarrow{w_i}.\overrightarrow{w_j}$ vector dot product vectors $\overrightarrow{w_i}$ with $\overrightarrow{w_j}$ in $(\mathbb{Z}\cup\{0\})^s$ of dimension s.

We denote to The LDM of the graph G is by $\mathbf{LD}=\mathbf{Ld}(G)=[a_{ij}]$, such that

$$a_{ij}=Ld(\overrightarrow{w_i}.\overrightarrow{w_j}).$$

You may write $Ld-$ **characteristic polynomial** of G as $det(\gamma I-\mathbf{LD})$ and denoted by $P_{Ld}(G)=\sum_{i=0}^{l}a_i\gamma^{l-i}$. The $Ld-$eigenvalues of G are the eigenvalues of $\mathbf{LD}$, the $Spec_{Ld}(G)$ is expressed as the spectrum of the matrix $\mathbf{LD}$. In the case when the new eigenvalues of G are $\gamma_1,\gamma_2,...,\gamma_m$, with multiplicities $t_1,t_2,...,t_m$, then $Spec_Ld(G)$ may be expressed as:

$$\begin{pmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_m\\ t_1 & t_2 & \cdots & t_m\end{pmatrix}.$$

Obviously the LDM is an $m\times m$ real symmetric matrix. This means that $\gamma_1,\gamma_2,\ldots,\gamma_m$ (its eigenvalues) are all real numbers. In the same way, since the $trace(\mathbf{Ld}(G))=0$, then $\sum_{i=1}^{m}\gamma_i=0$.

In whole paper the word graph means an undirected connected and simple graph and $\overrightarrow{w_i}.\overrightarrow{w_j}$, denote to the locating product of the vectors $\overrightarrow{w_i}$ with $\overrightarrow{w_j}$. For notions and terminologies of graph theory,the reader refer to [16].

Lemma 2.2: *For any graph with m vertices and with $Ld-$eigenvalues, $\gamma_1, \gamma_2, \ldots, \gamma_m$, we have*

1. $\sum_{i=1}^{m} \gamma_i = 0$
2. $\sum_{i=1}^{m} \gamma_i^2 = 2\sum_{1\le i<j\le m} (\overrightarrow{w_i}.\overrightarrow{w_j})^2$

Proof: (1) $\sum_{i=1}^{m} \gamma_i = trace(Ld(G)) = \sum_{i=1}^{m} a_{ii} = 0.$.

(2) In the case when $1 \le i \le m$, the sum of the entries of the matrix $(Ld(G))^2$ is the same as $trace(Ld(G))^2$.

$$trace[Ld(G)]^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} (\overrightarrow{w_i}.\overrightarrow{w_j})^2$$
$$= 2\sum_{1\le i<j\le m} (\overrightarrow{w_i}.\overrightarrow{w_j})^2.$$

Definition 2.3: The **Locating degree energy** of the graph G is

$$E_{Ld} = E_{Ld}(G) = \sum_{i=1}^{m} |\gamma_i|.$$

Example 2.4: For any path P_3, where $V(G) = \{w_1, w_2, w_3\}$. Then

$\overrightarrow{w_1} = (1,1,1), \overrightarrow{w_2} = (1,2,0), \overrightarrow{w_3} = (1,1,1)$ and

$$\mathbf{LD} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix},$$

the $Ld-$characteristic polynomial is $P_{Ld}(G) = \lambda^3 - 18\lambda$, the $Ld-$eigenvalues are $0, 3\sqrt{2}$ and $-3\sqrt{2}$ and the locating degree energy of G is

$$E_{Ld}(G) = 6\sqrt{2}.$$

Theorem 2.5: *For any complete graph K_l of $l \ge 2$ vertices,*

$$Spec_{Ld}(K_l) = \begin{pmatrix} 2l-l^2-2 & (l-1)(l^2-2l+2) \\ l-1 & 1 \end{pmatrix},$$

and $E_{Ld}(K_l) = 2(l-1)(l^2-2l+2).$

Proof: For each complete graph K_l with vertices $w_1, w_2, \ldots, w_l$, we denote each vertex by its degree vector $\overrightarrow{w_i}$, where $i = 1, \ldots, l$. Then $\overrightarrow{w_i} = (1, l-1)$. As a result, for every pair of vectors $\overrightarrow{w_i}, \overrightarrow{w_j}$,

$$\overrightarrow{w_i}.\overrightarrow{w_j} = l^2 - 2l + 2.$$

holds. Therefore, $\mathbf{Ld}(K_l) = (l^2 - 2l + 2)\mathbf{A}(K_l)$, where $A(K_l)$ is the adjacency matrix of K_l, and by using 1.1 it is not difficult to see that

$$Spec_{Ld}(K_l) = \begin{pmatrix} 2l-l^2-2 & (l-1)(l^2-2l+2) \\ l-1 & 1 \end{pmatrix},$$

Hence, $E_{Ld}(K_l) = 2(l-1)(l^2 - 2l + 2)$.

In the following result the $LD-$ energy and $LD-$ spectra of the $l-$ vertex cycle will be calculated.

Theorem 2.6: *For any even integer $l \geq 4$ for the cycle C_l, we have*

$$Spec_{Ld}(C_l) = \begin{pmatrix} 4(l-1) & 4(l-1)\cos\frac{2\pi}{l} & \cdots & 4(l-1)\cos\frac{2(l-2)\pi}{l} & -4(l-1) \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Further $E_{Ld}(C_l) = 2(l-1)E(C_l)$, for which $E(C_l)$ stands for the energy of C_l.

Proof: In the counterclockwise direction, we may identify the C_l cycle's vertices by labeling them as $\{w_1, w_2, \ldots, w_l\}$, it's not hard to figure out that as w_i is a vertex,

$$\overrightarrow{w_i} = \left(1, \underbrace{2, 2, 2, \ldots, 2, 2}_{(l/2-1) times}, , 1\right)$$

After this, based on both vertices w_i and w_j,

$$\begin{aligned} \overrightarrow{w_i}.\overrightarrow{w_j} &= ((1)(1) + (2)(2) + (2)(2) + \ldots + (2)(2) + (1)(1)) \\ &= 2(l-1). \end{aligned}$$

Therefore, $\mathbf{Ld}(C_l) = 2(l-1)\mathbf{A}(C_l)$.

Hence, by using the result 1.1, we have,

$$Spec_{Ld}(C_l) = \begin{pmatrix} 4(l-1) & 4(l-1)\cos\frac{2\pi}{l} & \cdots & 4(l-1)\cos\frac{2(l-2)\pi}{l} & -4(l-1) \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

So, it is easy to understand that

$$E_{Ld}(C_l) = 2(l-1), E(C_l),$$

where $E(C_l)$ denote to the energy of the cycle C_l.

Theorem 2.7: *For any odd integer* $n \geq 3$, *for the cycle* C_l, *we have,*

$$Spec_{Ld}(C_l) = \begin{pmatrix} 2(2l-1) & 2(2l-1)\cos\frac{2\pi}{l} & \cdots & 2(2l-1)\cos\frac{2(l-1)\pi}{l} \\ 1 & 2 & \cdots & 2 \end{pmatrix}.$$

Further, $E_{Ld}(C_l) = (2l-1)E(C_l)$, where $E(C_l)$ is the energy of $E(C_l)$.

Proof: In the counterclockwise direction, we may identify the C_l cycle's vertices by labeling them as $\{w_1, w_2, ..., w_l\}$, obviously, for any vertex $\overrightarrow{w_i}$ we have ,

$$\overrightarrow{u_i} = \left(1, \underbrace{2,2,...,2,2,2}_{\left(\frac{l-1}{2}\right) times}\right)$$

So obviously, for any two different vertices w_i and w_j

$$\begin{aligned}\overrightarrow{w_i}.\overrightarrow{w_j} &= ((1)(1) + (2)(2) + ... + (2)(2) + (2)(2)) \\ &= 2l-1.\end{aligned}$$

Hence, $\mathbf{Ld}(C_l) = (2l-1)\mathbf{A}(C_l)$.
Therefore, by result 1.1, we have,

$$Spec_{Ld}(C_l) = \begin{pmatrix} 2(2l-1) & 2(2l-1)\cos\frac{2\pi}{l} & \cdots & 2(2l-1)\cos\frac{2(l-1)\pi}{l} \\ 1 & 2 & \cdots & 2 \end{pmatrix}.$$

Further, $E_{Ld}(C_l) = (2l-1)E(C_l)$.

Theorem 2.8: *For the complete bipartite* $K_{l,s}$, *where* $1 \leq l \leq s$. *we have*

$$Spec_{Ld}(K_{l,s}) = \begin{pmatrix} (2ls-l-s+2)\sqrt{ls} & -(2ls-l-s+2)\sqrt{ls} & 0 \\ 1 & 1 & l+s-2 \end{pmatrix}.$$

Further, $E_{Ld}(K_{l,s}) = (l(s-1) + s(l-1) + 2)\sqrt{ls}$.

Proof: For all *i* between 1 and *j* between 1 and *s*, by labeling the vertices of $K_{l,s}$ so that every pair of vertices, v_i and v_{l+j}, is adjacent. Suppose the two bipartite sets of the graph are V_1 and V_2 such that $|V_1| = l$ and $|V_2| = s$.

Then if $v_i \in V_1$ we have, $\overrightarrow{v_i}$ of v_i are given by: $\vec{v}_i = (1, s, l-1)$, and if $v_i \in V_2$ we have, $\overrightarrow{v_i}$ is given by: $\vec{v}_i = (1, l, s-1)$

For every pair of neighboring vertices v_i, v_j, in, $K_{l,s}$, it is clear that $\vec{v}_i . \vec{v}_j = 2ls - l - s + 2$. $Ld(K_{l,s}) = (2ls - l - s + 2)$, therefore

$$\mathbf{L}d(K_{l,s}) = (2ls - l - s + 2)\mathbf{A}(K_{l,s}).$$

Lemma 1.1 also allows us to obtain

$$Spec_{Ld}(K_{l,s}) = \begin{pmatrix} (2ls - l - s + 2)\sqrt{ls} & -(2ls - l - s + 2)\sqrt{ls} & 0 \\ 1 & 1 & l + s - 2 \end{pmatrix}.$$

Hence,

$$E_{Ld}(K_{l,s}) = (l(s-1) + s(l-1) + 2)\sqrt{ls}.$$

Theorem 2.8 has the immediate consequences listed below. :

1. So long as $m \geq 1$, let us assume that G is a complete bipartite graph $K_{l,l}$. If so, we get $E_L d(K_{l,l}) = 2l(l^2 - l + 1)$.
2. For any star graph $K_{1,l}$,

$$E_{Ld}(K_{1,l}) = \sqrt{l}(l+1)$$

3. Energy and Locating Degree Spectrum of the Regular and SRGS

Strongly regular graphs (or SRGÂ for short) are a family of regular graphs with several desirable characteristics. Combinatorics provides a foundation for the development of a wide variety of SRGS.

A graph with m vertices that is regular with valency k and meets the following characteristics is called SRG with parameters (m, k, λ, μ) in [12]:

- there are precisely λ neighbors between any pair of adjacent vertices;
- exactly μ neighbors connect any two non-adjacent vertices.

Theorem 3.1: *Consider the SRG G to have parameters* (l, t, λ, μ) *and its spectral is* $\begin{pmatrix} t & \theta & \tau \\ 1 & f & g \end{pmatrix}$. *Then*

$$Spec_{Ld}(G) = \begin{pmatrix} k\zeta & \theta\zeta & \tau\zeta \\ 1 & f & g \end{pmatrix},$$

even more, $E_{Ld}(G) = \zeta E(G)$, where $\zeta = 1 + t^2 + (l - t - 1)^2$ and $E(G)$ stands for G energy.

Proof: Take the SRG G with parameters (l, t, λ, μ). Any pair of locating degree vectors has a product of $1 + t^2 + (l - t - 1)^2$, and this holds true for any vertex v where $\vec{v} = (1, t, l - t - 1)$. Hence

$$Ld(G) = (1 + t^2 + (l - t - 1)^2)A(G).$$

If we put $\zeta = (1 + t^2 + (l - t - 1)^2)$, then

$$Spec_{Ld}(G) = \begin{pmatrix} t\zeta & \theta\zeta & \tau\zeta \\ 1 & f & g \end{pmatrix},$$

and $E_{Ld}(G) = \zeta E(G)$

Notation:

If $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_l \\ m_1 & m_2 & \cdots & m_l \end{pmatrix}$ is a graph G's spectrum, then we have

$$\zeta\ spec(G) = \zeta \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_l \\ m_1 & m_2 & \cdots & m_l \end{pmatrix} = \begin{pmatrix} \zeta\lambda_1 & \zeta\lambda_2 & \cdots & \zeta\lambda_l \\ m_1 & m_2 & \cdots & m_l \end{pmatrix},$$

for any ζ that is a real value.

This next theorem generalizes the previous one 3.1:

Theorem 3.2: *Consider the t-regular graph of diameter 2. If* $Spec(G) = \begin{pmatrix} t & \lambda_2 & \dots & \lambda_l \\ 1 & t_2 & \dots & t_l \end{pmatrix}$, *then*

$$Spec_{Ld}(G) = \zeta Spec(G).$$

Further, $E_{Ld}(G) = \zeta E(G)$, such that $\zeta = 1 + t^2 + (l - t - 1)^2$ where $E(G)$ stands for standard G energy.

4. Constraints on the Locating Energy

Theorem 4.1: *For any SRG with locating vectors* $\overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3}, \ldots, \overrightarrow{w_l}$,

$$\sqrt{\alpha} \le E_{Ld}(G) \le \sqrt{l\alpha},$$

namely, that $\alpha = 2\sum_{1 \le i < j \le l} (\overrightarrow{w_i} . \overrightarrow{w_j})^2$.

Proof: Let $Ld-$ eigenvalues of G be denoted by $\gamma_1,\gamma_2,\ldots,\gamma_l$. According to the Cauchy-Schwarz inequality, if $(a_1,\ldots,a_l)$ and $(b_1,\ldots,b_l)$ are l-vectors, where $a_1,\ldots,a_l,b_1,\ldots,b_l$ are real numbers, then

$$\left(\sum_{i=1}^{l} a_i b_i\right)^2 \le \left(\sum_{i=1}^{l} a_i^2\right)\left(\sum_{i=1}^{l} b_i^2\right).$$

If putting $a_i=1$ and $b_i=|\gamma_i|$, for $i=1,2,\ldots,l$, we obtain

$$\left(\sum_{i=1}^{l} |\gamma_i|\right)^2 \le \left(\sum_{i=1}^{l} 1^2\right)\left(\sum_{i=1}^{l} |\gamma_i|^2\right).$$

So, using the Lemma 2.2, we get

$$\sum_{i=1}^{l} |\gamma_i| \le \sqrt{l\sum_{i=1}^{l} |\gamma_i|^2} = \sqrt{2l\sum_{1\le i<j\le l}(\overrightarrow{v_i}.\overrightarrow{v_j})^2}$$

and so

$$E_{Ld}(G) \le \sqrt{l\alpha}.$$

For the other bound,

$$\left(E_{Ld}(G)\right)^2 = \left(\sum_{i=1}^{l} |\gamma_i|\right)^2 \ge \sum_{i=1}^{l}(|\gamma_i|)^2 = 2\sum_{1\le i<j\le l}(\overrightarrow{v_i}.\overrightarrow{v_j})^2.$$

Due to this,

$$E_{Ld}(G) \ge \sqrt{\alpha}.$$

Theorem 4.2: *If a graph's diameter is d, and there are l vertices in it, where* $l\ge 2$, *then*

$$\sqrt{8(l-1)} \le E_{Ld}(G) \le l(1+(\lceil (l-1)/d \rceil)^2 d)\sqrt{l-1}\,.$$

Proof: With G assumed to be a graph consisting of vertices $w_1,w_2,\ldots,w_l$, with a diameter d, we have that for any pair of locating vectors $\overrightarrow{w_i}$, $\overrightarrow{w_j}$ in G, $\overrightarrow{w_i}.\overrightarrow{w_j}=2$ in the minimum case, this implies,

$$\overrightarrow{w_i}.\overrightarrow{w_j} \ge 2,$$

and hence

$$(\overrightarrow{w_i}.\overrightarrow{w_j})^2 \geq 4.$$

Because G is connected, there are at least $l-1$ values of $(\overrightarrow{w_i}.\overrightarrow{w_j})^2$ such that $1 \leq i < j \leq l$ and $\overrightarrow{w_i}.\overrightarrow{w_j}$ doesn't equal to zero. Therefore,

$$\sum_{1\leq i<j\leq l}^{l} (\overrightarrow{w_i}.\overrightarrow{w_j})^2 \geq 4(l-1).$$

By Theorem 4.1 we get,

$$E_{Ld}(G) \geq \sqrt{2\sum_{1\leq i<j\leq l} (\overrightarrow{w_i}.\overrightarrow{w_j})^2} \geq \sqrt{8(l-1)}.$$

The same holds true for the upper bound; here, we have

$$\overrightarrow{w_i}.\overrightarrow{w_j} \leq 1 + (\lceil (l-1)/d \rceil)^2 d,$$

and we get,

$$(\overrightarrow{w_i}.\overrightarrow{w_j})^2 \leq (1 + (\lceil (l-1)/d \rceil)^2 d)^2.$$

Any graph G has a maximum of $l(l-1)/2$ non-zero values for the locating product $\overrightarrow{w_i}.\overrightarrow{w_j}$. Therefore,

$$\sum_{1\leq i<j\leq l}^{l} (\overrightarrow{w_i}.\overrightarrow{w_j})^2 \leq \frac{1}{2} l(l-1)(1 + (\lceil (l-1)/d \rceil)^2 d)^2.$$

The upper bound established in 4.1 is used here, and the result is,

$$E_{Ld}(G) \leq \sqrt{2l\sum_{1\leq i<j\leq l} (\overrightarrow{w_i}.\overrightarrow{w_j})^2} \leq l(1 + (\lceil (l-1)/d \rceil)^2 d)\sqrt{l-1}.$$

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