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# On magic total labelings in combinatorial designs

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**Abstract**

Let $H$ be a graph and $\mathcal{B}$ be an $H$-decomposition of $G$. The $H$-magic total labeling of a graph $G$ is an assignment of integers set $[1,|V(G)\bigcup E(G)|]$ to $V(G)\bigcup E(G)$ such that: any vertex and any edge of $G$ receive distinct integers, and the sum of all vertices receive integers and all edges receive integers on $B$ is a constant for any $B \in \mathcal{B}$. In this paper we study the $H$-super magic problems of balanced incomplete block designs and resolvable balanced incomplete block designs.

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***Keywords:*** *H-(super) magic decomposition, Magic total labeling, k-factorable, Partition of set, Balanced incomplete block design.*

## 1. Introduction

Graph labelings provide useful mathematical models for a wide range of applications, such as communication networks, various coding theory problems, x-ray crystallography, cryptography, data security, astronomy and mobile telecommunication systems. About applications of graph labelings can be found in Bloom and Golomb's papers [1] and [2]. Kalantari et al. in [3] tried to find applications of magic labeling in optimization theory. Baskoro et al. in [4] proposed a secret sharing scheme construction using edge-magic labeling. Wallis in [5] proposed edge-magic total labeling for assigning addresses of communication networks

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and radar pulse codes. Hartnell and Rall in [6] proposed a game based on vertex-magic edge labeling.

Let $v$ and $k$ be integers with $v \geq k \geq 2$. A balanced incomplete block design (denoted by $BIBD(v,k,1)$) on $v$ points with block-size $k$ is an incidence structure $D=(V,\mathcal{B},I)$ with: $|V|=v$, $|B|=k$ for all $B \in \mathcal{B}$, and for any set $T$ of 2 points there are exactly one block incident with all points in $T$. So all blocks have the same size and every 2-subset of the point set is contained in the same number of blocks. An $H$-decomposition of a graph $G$ is a set of subgraphs of $G$, say $\{H_1, H_2, ..., H_n\}$, such that each edge of $G$ belongs to exactly one of the subgraphs $H_i, i \in [1,n]$ and each $H_i$ is isomorphic to $H$ for any $i \in [1,n]$. We also shall write $H$-decomposition of $G$ for $GD(G,H)$. Let $K_v$ denote the complete graph with $v$ vertices. It is easy to see that a $BIBD(v,k,1)$ is a $GD(K_v,K_k)$.

Let $Z$ be the ring of integers, and let $[a,b]=\{x \mid x \in Z, a \leq x \leq b\}$, $[a,b]_k = \{x \mid x \in Z, a \leq x \leq b, x \equiv a (\bmod k)\}$. We stipulate $[a,b]=\phi$ if $a>b$. We denote $\{f(x) \mid x \in A\}$ by $f(A)$ if $f: C \rightarrow D$ is a mapping and $A \subseteq C$ and denote $\sum_{i \in S} i$ by $\sum S$ if $S$ is a number set.

We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph $G$, respectively. Let the simple graph $G$ admits an $H$-decomposition $\mathcal{B}$. The $H$-decomposition of $G$ is said to be $H$-magic decomposition if there exists a bijection $f: V(G) \bigcup E(G) \rightarrow [1, |V(G) \bigcup E(G)|]$ such that for all $B \in \mathcal{B}$, $\sum_{v \in V(B)} f(v) + \sum_{e \in E(B)} f(e)$ is a constant (this constant is called weight of $B$ or the $H$-magic decomposition). An $H$-magic decomposition of $G$ is said to be $H$-super magic decomposition (simply, $H$-SMD) if $f(V(G))=[1,|V(G)|]$. The $f$ is said to be an $H$-(super) magic total labeling of $G$ (simply, $H$-(S) MTL).

Kotzig and Rosa [7] studied $H$-MTLs of some $G$ if $H \cong K_2$. Inayah et al. in [8] discussed magic and antimagic $H$-decompositions of some graphs. Zhu and Liang in [9] discussed the super $(a,d)$-$H$-antimagic total labelings of path chain graph $PCG(H,U,n)$. So far there is an enormous literature built up on several kinds of labelings of graphs. For a survey of various graph labeling problems one may refer to Gallian [10].

In this paper we mainly investigate the existence of $K_k$-super magic total labeling of $K_v$ and factor-super magic total labeling of $K_v$.

## 2. The preliminary results

A subset family $\mathcal{P}=\{P_i \mid i \in [1,m]\}$ of set $[1,mk]$ is said to be an $(m,k)$-partition of $[1,mk]$ if it satisfies the following conditions: for any

$1 \le i, j \le m$, $|P_i| = k$, $P_i \cap P_j = \emptyset (i \ne j)$ and $\bigcup_{i=1}^{m} P_i = [1, mk]$. If $\mathcal{P} = \{P_i \mid i \in [1, m]\}$ is $(m,k)$-partition of $[1, mk]$ and $\sum P_1 = \sum P_2 = \cdots = \sum P_m$, then $\mathcal{P}$ is called $(m,k)$-magic partition of $[1, mk]$.

Let the simple graph $G$ admits $H$-decomposition $\mathcal{B}$. The graph $G$ is said to be $H$-vertex (edge) magic if there exists a bijection $f_v : V(G) \to [1, |V(G)|]$ $(f_e : E(G) \to [1, |E(G)|])$ such that for all $B \in \mathcal{B}$, $\sum_{v \in V(B)} f_v(v)$ $(\sum_{e \in E(B)} f_e(e))$ is a constant. The sum of all vertex (edge) labels on graph $H$ is denoted by $\sum f_v(H)$ $(\sum f_e(H))$.

In what follows, we are going to list some useful results.

**Theorem 2.1 :** *(see [11])* $(m,k)$*-magic partition of* $[1, mk]$ *exists if and only if k is even and m is any, or both k and m are odd.*

**Theorem 2.2 :** *(see [12])*

(1) *If a* $GD(G,F)$ *and a* $GD(H,G)$ *exist, then a* $GD(H,F)$ *exists.*

(2) *If the set* $[1,t]$ *can be decomposed into m subsets such that the sum of all the elements in each subset is a constant, then so does the set* $[a+1, a+t]$ *for any positive integer a.*

**Theorem 2.3 :** *(see [12])*

(1) *An H-edge magic decomposition of G exists is equivalent to there exists a* $(m, |E(H)|)$*-magic partition* $\{A_i \mid i \in [1, m]\}$ *of* $[1, |E(G)|]$.

(2) *If both an H-vertex magic decomposition and an H-edge magic decomposition of G exist, then H-SMD of G exists.*

(3) *Suppose that the graph H is a connected spanning subgraph with even size of G. If a* $GD(G,H)$ *exists, then H-SMD of G exists.*

## 3. The $K_3$-SMD of $K_v$

The following Theorem 3.1 is easy to prove, we simply state it.

**Theorem 3.1 :** *A* $K_3$*-decomposition of* $K_v$ *exists is equivalent to there exists a* $BIBD(v,3,1)$. *And if a* $BIBD(v,3,1)$ *exists, then there is* $v \equiv 1$ *(mod 2).*

**Theorem 3.2 :** *(see [13]) A* $BIBD(v,3,1)$ *exists if and only if* $v \equiv 1, 3$ *(mod 6).*

**Theorem 3.3 :** *A* $K_3$*-vertex magic decomposition of* $K_v$ *exists if and only if* $v = 3$.

***Proof:*** A $K_3$-decomposition of $K_v$ contains $\frac{v(v-1)}{6}$ $K_3$'s. Every vertex of $K_v$ occurs in $\frac{v-1}{2}$ $K_3$'s. Let $f_v$ be the $K_3$-vertex magic labeling of $K_v$ and $a$ the weight of $K_3$. Since $f_v(V(K_v)) = [1, |V(K_v)|] = [1, v]$, we have

$$\frac{av(v-1)}{6} = \frac{v-1}{2}\sum_{i=1}^{v} i = \frac{v-1}{2}\frac{v(v+1)}{2}.$$

Thus $a = \frac{3(v+1)}{2}$. Again by Theorem 3.2 we have $v \equiv 1, 3 \pmod 6$.

Because the vertex labeled $v$ and the vertex labeled $v-1$ must occur in one $K_3$, then $a > v + (v-1) = 2v-1$, i.e. $\frac{5-v}{2} > 0$. Thus $v = 3$. It is obvious that there exists a $K_3$-vertex magic decomposition of $K_v$ if $v = 3$.

The converse of this assertion is also valid. □

It follows immediately from Theorem 2.3 that the following result is true.

**Theorem 3.4 :** *If a* $(\frac{v(v-1)}{2|E(H)|}, |E(H)|)$ *-magic partition of the set* $[1, v(v-1)/2]$ *exists, then a H-edge magic decomposition of* $K_v$ *exists.*

**Theorem 3.5 :** *A* $K_3$*-edge magic decomposition of* $K_v$ *exists if and only if* $v \equiv 3, 7 \pmod{12}$*.*

***Proof :*** A $K_3$-decomposition of $K_v$, say $\mathcal{B}$, contains $\frac{v(v-1)}{6}$ $K_3$'s. Let $f_v$ be the $K_3$-vertex magic labeling of $K_v$ and $a$ the weight of $K_3$. If a $K_3$-edge magic decomposition of $K_v$ exists, then

$$\frac{av(v-1)}{6} = \sum_{i=1}^{v(v-1)/2} i = \frac{v(v-1)(v^2-v+2)}{8}.$$

Thus $a = \frac{3(v^2-v+2)}{4}$, i.e. $v \equiv 2, 3 \pmod 4$. Again by Theorem 3.2 we have $v \equiv 3, 7 \pmod{12}$.

On the other hand, when $v = 12t+3$ (or $12t+7$), $\frac{v(v-1)}{6} = 24t^2 + 10t + 1$ (or $24t^2 + 26t + 7$) is odd. From Theorem 2.1 we obtain that a $(v(v-1)/6, 3)$ -magic partition of the set $[1, v(v-1)/2]$ exists. Again from Theorem 3.4 we obtain a $K_3$-edge magic decomposition of $K_v$ exists. □

**Theorem 3.6 :** *If a* $K_3$*-SMD of* $K_v$ *exists, then* $v \equiv 3, 7 \pmod{12}$*.*

***Proof :*** Let $f:V(K_v)\bigcup E(K_v)\to[1,|V(K_v)\bigcup E(K_v)|]$ be a $K_3$-super magic labeling of $K_v$, and $\mathcal{B}$ a $K_3$-SMD of $K_v$ with weight $a$. Then

$$\frac{av(v-1)}{6}=\sum_{B\in\mathcal{B}}[\sum f(V(B))+\sum f(E(B))]=\frac{v(v-1)(v+1)}{4}+\frac{v^2(v-1)}{2}+\frac{v(v-1)(v^2-v+2)}{8}.$$

Thus $a=\frac{3(v^2+5v+4)}{4}$, i.e. $v\equiv 0,3 \pmod 4$. Again by Theorem 3.2 we have $v\equiv 3,7 \pmod{12}$.

**Example 1 :** A $K_3$-SMD of $K_7$ with weight 66 is as follows.

Let $V(K_7)=[1,7]$ and $f:V(K_7)\bigcup E(K_7)\to[1,28]$. Define $f(i)=i$, if $i\in V(K_7)$. If $V(K_3)=\{a,b,c\}$, then we denote $K_3$ by $\{a,b,c\}$.

It is easy to verify that $\mathcal{B}=\{\{1,2,4\},\{2,3,5\},\{1,3,7\},\{1,5,6\},\{3,4,6\},\{2,6,7\},\{4,5,7\}\}$ is a $K_3$-decomposition of $K_7$. Define $\{28,8,23\}$, $\{27,9,20\}$, $\{26,10,19\}$, $\{25,11,18\}$, $\{24,12,17\}$, $\{22,15,14\}$, $\{21,16,13\}$ are the sets of edge labeling of the cycles $\{1,2,4\}$, $\{2,3,5\}$, $\{1,3,7\}$, $\{1,5,6\}$, $\{3,4,6\}$, $\{2,6,7\}$, $\{4,5,7\}$, respectively. Then $\mathcal{B}$ is a $K_3$-SMD of $K_7$ with weight 66.

**Example 2 :** A $K_3$-SMD of $K_{15}$ with weight 228 is as follows.

Let $V(K_{15})=[1,15]$ and $f:V(K_{15})\bigcup E(K_{15})\to[1,120]$. Define $f(i)=i$, if $i\in V(K_{15})$. If $V(K_3)=\{a,b,c\}$, then we denote $K_3$ and $f(E(K_3))$ by $a,b,c$ and $f(ab),f(bc),f(ca)$, respectively. For any $B\in\mathcal{B}$, $f(E(B))$ is defined as follows:

| $B$ | 1,2,3 | 4,8,12 | 5,10,14 | 6,11,13 | 7,9,15 |
|---|---|---|---|---|---|
| $f(E(B))$ | 120,16,86 | 107,27,70 | 101,35,63 | 92,52,54 | 81,39,77 |
| $\sum f(V(B))$ | 6 | 24 | 29 | 30 | 31 |
| $\sum f(E(B))$ | 222 | 204 | 199 | 198 | 197 |

| $B$ | 1,4,5 | 2,8,10 | 3,13,15 | 6,9,14 | 7,11,12 |
|---|---|---|---|---|---|
| $f(E(B))$ | 119,17,82 | 112,22,74 | 91,21,85 | 100,20,79 | 98,25,75 |
| $\sum f(V(B))$ | 10 | 20 | 31 | 29 | 30 |
| $\sum f(E(B))$ | 218 | 208 | 197 | 199 | 198 |

| $B$ | 1,6,7 | 2,9,11 | 3,12,14 | 4,10,15 | 5,8,13 |
|---|---|---|---|---|---|
| $f(E(B))$ | 116,45,53 | 111,26,69 | 99,29,71 | 84,32,83 | 106,34,62 |
| $\sum f(V(B))$ | 14 | 22 | 29 | 29 | 26 |
| $\sum f(E(B))$ | 214 | 206 | 199 | 199 | 202 |

| $B$ | 1,8,9 | 2,12,15 | 3,5,6 | 4,11,14 | 7,10,13 |
|---|---|---|---|---|---|
| $f(E(B))$ | 113,24,73 | 97,44,58 | 117,19,78 | 96,31,72 | 88,46,64 |
| $\sum f(V(B))$ | 18 | 29 | 14 | 29 | 30 |
| $\sum f(E(B))$ | 210 | 199 | 214 | 199 | 198 |

| $B$ | 1,10,11 | 2,13,14 | 3,4,7 | 5,9,12 | 6,8,15 |
|---|---|---|---|---|---|
| $f(E(B))$ | 110,28,68 | 95,47,57 | 115,23,76 | 105,36,61 | 94,49,56 |
| $\sum f(V(B))$ | 22 | 29 | 14 | 26 | 29 |
| $\sum f(E(B))$ | 206 | 199 | 214 | 202 | 199 |

| $B$ | 1,12,13 | 2,4,6 | 3,9,10 | 5,11,15 | 7,8,14 |
|---|---|---|---|---|---|
| $f(E(B))$ | 104,38,60 | 118,18,80 | 109,30,67 | 114,41,42 | 93,51,55 |
| $\sum f(V(B))$ | 26 | 12 | 22 | 31 | 29 |
| $\sum f(E(B))$ | 202 | 216 | 206 | 197 | 199 |

| $B$ | 1,14,15 | 2,5,7 | 3,8,11 | 4,9,13 | 6,10,12 |
|---|---|---|---|---|---|
| $f(E(B))$ | 89,43,66 | 90,37,87 | 108,48,50 | 103,40,59 | 102,33,65 |
| $\sum f(V(B))$ | 30 | 14 | 22 | 26 | 28 |
| $\sum f(E(B))$ | 198 | 214 | 206 | 202 | 200 |

Then $f(V(K_{15}))=[1,15]$, $f(E(K_{15}))=[16,120]$ and $f(B)=228$ for any $B\in\mathcal{B}$. Thus $\mathcal{B}$ is a $K_3$-SMD of $K_{15}$ with weight 228.

For further investigation we propose the following open problem:

**Open problem 1.** If $v\equiv 3,7 \pmod{12}$, whether a $K_3$-SMD of $K_v$ exists.

## 4. The factor-SMD of $K_v$

A factor of a graph $G$ is a spanning subgraph of $G$. A $k$-factor of $G$ is a factor of $G$ that is $k$-regular. Thus, a 1-factor of $G$ is a matching that saturates all the vertices of $G$. A 2-factor of $G$ is a factor of $G$ that is a disjoint union of cycles of $G$. A graph $G$ is $k$-factorable if $G$ is an edge-disjoint union of $k$-factors of $G$. A parallel class in a $BIBD(v,k,1)$ is a set of blocks that partition the point set. A resolvable balanced incomplete block design (simply $RBIBD(v,k,1)$) is a $BIBD(v,k,1)$ whose blocks can be partitioned into parallel classes. In this case each parallel class is called a $K_k$-factor of $K_v$. A decomposition of graph $K_v$ into a $K_k$-factor is called a $K_k$-factorization of $K_v$. A graph $K_v$ with $K_k$-factorization is $K_k$-factorable.

**Theorem 4.1 :** (see [14])

1. *$K_{n,n}$ and $K_{2n}$ are 1-factorable.*
2. *The number of 1-factors of $K_{n,n}$ is $n!$ and $K_{2n}$ is $\frac{(2n)!}{2^n n!}$.*
3. *Let k be a positive integer. A k-regular bipartite graph is 1-factorable.*

**Theorem 4.2 :**

1. *(see [15]) $K_{2n+1}$ is 2-factorable where every 2-factor is a $C_{2n+1}$.*
2. *(see [16]) $K_{6n+3}$ is 2-factorable where every 2-factor is a disjoint union of 3-cycles of $K_{6n+3}$.*

**Theorem 4.3 :** Let *H* be a *k*-factor of *G*, and *G k*-factorable. There exists an *H*-SMD of *G* with weight *a* if and only if there exists an *H*-edge magic decomposition of *G* with weight $a'$.

***Proof:*** Let $a = a' + \frac{|V(G)|(|V(G)|+1)}{2}$.

We first show necessity. Let $f: V(G) \cup E(G) \to [1, |V(G) \cup E(G)|]$ be *H*-SMD of *G* with weight *a*, and $\mathcal{B}$ a *k*-factorization of *G*. Then for each $H \in \mathcal{B}$ has $\sum f(V(H)) = \frac{|V(G)|(|V(G)|+1)}{2}$. Thus $\sum f(E(H)) = a - \sum f(V(H))$ is a constant for each $H \in \mathcal{B}$. We define $f_e(e) = f(e) - |V(G)|$ if $e \in E(G)$. Then $f_e : E(G) \to [1, |E(G)|]$ is *H*-edge magic decomposition of *G* with weight $a'$.

We now show sufficiency. Let $f_e$ be an *H*-edge magic decomposition of *G* with weight $a'$. We define a mapping $f: V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$ by $f(v_i) = i$ if $v_i \in V(G)$ and $f(e) = f_e(e) + |V(G)|$ if $e \in E(G)$.

We have

$$\begin{aligned}\sum f(H_i) &= \sum f(V(H_i)) + \sum f(E(H_i)) \\ &= \sum f(V(H_i)) + \sum f_e(E(H_i)) + |E(H)| \cdot |V(G)| \\ &= \frac{|V(G)|(|V(G)|+1)}{2} + a' + |E(H)| \cdot |V(G)|.\end{aligned}$$

Then $\sum f(H_i)$ is a constant for each $H_i \in \mathcal{B}$. Since *f* is a bijection, the mapping $f: V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$ is an *H*-SMD labeling of *G* with weight *a*.

**Theorem 4.4 :**

1. *If H is a 1-factor of* $K_{2n}$ $(K_{n,n})$, *then there exists an H-SMD of* $K_{2n}$ $(K_{n,n})$.
2. *Let n be odd. There exists a* $C_{2n+1}$*-SMD of* $K_{2n+1}$.

***Proof:***

1. Since there exists a $(2n-1,n)$-magic partition of $[1,n(2n-1)]$ (a $(n,n)$-magic partition of $[1,n^2]$) by Theorem 2.1, there exists an $H$-SMD of $K_{2n}$ $(K_{n,n})$ by Theorem 4.3.
2. Since there exists a $(n,2n+1)$-magic partition of $[1,n(2n+1)]$ by Theorem 2.1, there exists a $C_{2n+1}$-SMD of $K_{2n+1}$ by Theorem 4.3. □

**Theorem 4.5 :** (see [17])

1. *Necessary conditions for the existence of an* $RBIBD(v,k,1)$ *are* $v-1\equiv 0$ *(mod* $(k-1)$*) and* $v\equiv 0$ *(mod k).*
2. *An* $RBIBD(v,3,1)$ *is a Kirkman triple system* $KTS(v)$. *A* $KTS(v)$ *exists if and only if* $v\equiv 3$ *(mod 6).*
3. *An* $RBIBD(v,4,1)$ *exists if and only if* $v\equiv 4,16,28$ *(mod 36).*

**Theorem 4.6 :**

1. *Kirkman triple system* $KTS(6n+3)$ *is* $(K_3$*-factor)-super magic.*
2. *An* $RBIBD(v,4,1)$ *is* $(K_4$*-factor)-super magic.*

***Proof:***

1. Let $H$ be a $K_3$-factor of $K_{6n+3}$. Then there exists an $H$-decomposition of $K_{6n+3}$ by Theorem 4.5. Since there exists a $(3n+1,6n+3)$-magic partition of $[1,(3n+1)(6n+3)]$ by Theorem 2.1, there exists an $H$-SMD of $K_{6n+3}$ by Theorem 4.3. This implies that $KTS(6n+3)$ is $(K_3$-factor)-super magic.
2. An $RBIBD(v,4,1)$ exists if and only if $v\equiv 4,16,28$ (mod 36) (i.e. a $K_4$-factorization of $K_v$ exists if and only if $v\equiv 4,16,28$ (mod 36)). A $K_4$-factor of $K_v$ has $3v/2$ edges because $K_4$ has 6 edges and a $K_4$-factor of $K_v$ contains $v/4$ $K_4$'s. A $K_v$ has $(v-1)/3$ $K_4$-factor's. When $v\equiv 4,16,28$ (mod 36), we have $3v/2$ even. Since there exists a $((v-1)/3,3v/2)$-magic partition of $[1,v(v-1)/2]$ by Theorem 2.1, there exists an $(K_4$-factor)-SMD of $K_v$ by Theorem 4.3. This implies that an $RBIBD(v,4,1)$ is $(K_4$-factor)-super magic. □

We propose the following open problem:

**Open problem 2.** If $k \geq 5$, whether an $RBIBD(v,k,1)$ is ($K_k$-factor)-super magic.

## References


[1] G.S. Bloom and S.W. Golomb, Applications of numbered undirected graphs, Proc. IEEE, 65(1977), 562-570.

[2] G.S. Bloom and S.W. Golomb, Numbered complete graphs, unusual rules, and assorted applications, In: Theory and Applications of Graphs, *Lecture Notes in Math.*, 642(1978), 53-65.

[3] B. Kalantari and G.B. Khosrovshahi, Magic labeling in graphs: Bounds, complexity, and an application to a variant of TSP, Networks, 28(1996), 211-219.

[4] E.T. Baskoro, R. Simanjuntak and M.T. Adithia, Two level secret sharing schemes based on magic labelings, Proceedings INA-CISC 2005, *Indonesia Cryptology and Information Security*, March 30-31 (2005), 151-154.

[5] W.D. Wallis, Magic Graphs, Birkhäuser, Boston-Basel-Berlin, 2001.

[6] B. Hartnell and D. Rall, A vertex-magic edge labeling game, *Congr. Numer.* 161(2003), 163-167.

[7] A. Kotzig, A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.,* 13(1970), 451-461.

[8] ]N. Inayah, A. Lladó and J. Moragas, Magic and antimagic H-decompositions, *Discrete Math.,* 312 (2012), 1367-1371.

[9] Dongxu Zhu, Zhihe Liang, On the super (a,d)-H-antimagic total labelings of path chain graphs, *Journal of Information and Optimization Sciences,* 42(3) (2021), 689-700.

[10] J.A. Gallian, A dynamic survey of graph labelings, Electron. J. Combin., (2017) # DS6, http://www.combinatorics.org/surveys/ds6.pdf.

[11] Zhihe Liang, On the *G*-supermagic coverings of graphs, *Acta Mathematicae Applicatae Sinica* (Chinese Series), 37(5) (2014), 857-864.

[12] Zhihe Liang, Cycle-supermagic decompositions of complete multipartite graphs, *Discrete Mathematics*, 312 (2012), 3342-3348.

[13] T. P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. Journal*, 2(1847), 191-204.

[14] R. Balakrishnan, K. Ranganathan, A Textbook of Graph Theory (Second Edition), Springer Science Business Media, New York, 2012.

[15] B. Alspach and H. Gavlas, Cycle decompositions of $K_n$ and $K_n - I$, *J. Combin. Theory* (B), 81(2001), 77-99.

[16] C. C. Lindner, C. A. Rodger, Design theory, CRC Press, New York, 1997, 78.

[17] C. J. Colbourn, J. H. Dinitz, Handbook of Combinatorial Designs, 2nd Ed., Chapman & Hall/CRC, 2007, 124-125.

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