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Numerical solution of box constrained separable convex quadratic programming problems

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Abstract

In this paper, minimization problems with a separable convex quadratic objective function subject to a linear equality constraint/linear equality constraints, and bounds on the variables (box constraints) are considered. A necessary and sufficient condition for a feasible solution to be an optimal solution to each of these problems has been established. A convergent polynomial algorithm for solving the problem with a linear equality constraint and bounded variables is proposed, and some numerical results, obtained by this algorithm, are presented.

Subject Classification: *(2020) 90C20, 90C25.*

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1. Introduction. Statement of the problem

Consider the following separable convex quadratic programming problem

(C_m)

$$\min\left\{c(\mathrm{x}) \equiv \sum_{j\in J} c_j(x_j) = \frac{1}{2}\sum_{j\in J} m_j x_j^2 + \sum_{j\in J} s_j x_j\right\} \tag{1}$$

subject to

$$\sum_{j\in J} d_{ij} x_j = \alpha_i, \quad i = 1,\ldots,m \tag{2}$$

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$$l_j \le x_j \le u_j, \quad j \in J, \tag{3}$$

where $m_j > 0, j \in J$, $d_{ij} > 0$ for each $i = 1,\dots,m$ and $j \in J$; $\alpha_i \in \mathbb{R}, i = 1,\dots,m$; $l_j, u_j \in \mathbb{R}, j \in J$; $\mathbf{x} = (x_j)_{j\in J}$, and $J \equiv \{1,...,n\}$.

Since $c_j''(x_j) = m_j > 0$ by assumption, then $c_j(x_j) = \frac{1}{2} m_j x_j^2 + s_j x_j, j \in J$, are strictly convex functions, with a minimum point at $h_j = -\frac{s_j}{m_j}$.

The feasible set X (2)-(3) is convex because it is an intersection of the convex set (2), which is defined by m hyperplanes, and the n – dimensional box (3), where $n = |J|$.

Problem (C_m) and problems, related to it, are considered in references [1-29], etc.

Rest of the paper is organized as follows. In Section 2, a necessary and sufficient condition (characterization theorem) for a feasible solution to be an optimal solution to problem (C_m) is stated and proved. In Section 3, a necessary and sufficient condition for the special case (C) of problem (C_m) is stated, a convergent algorithm of polynomial computational complexity for solving problem (C) is proposed, its convergence is proved, and results of some computational experiments are presented. In Section 4 we give conclusions where an open problem for future work is also formulated.

2. Characterization theorem for problem (C_m)

Assume that the following assumptions are satisfied.

(A.1) $l_j \le u_j, j \in J$. If $l_k = u_k$ for an index $k \in J$, then the value $x_k := l_k = u_k$ is determined *a priori*.

(A.2) $\sum_{j\in J} d_{ij} l_j \le \alpha_i \le \sum_{j\in J} d_{ij} u_j, i = 1,\dots,m$, otherwise constraints (2)-(3) are inconsistent and the feasible region X, defined by (2)-(3), is empty.

The Lagrangian function for problem (C_m) is defined as follows:

$$L(\mathrm{x},\mathrm{p},\mathrm{q},\lambda) = \frac{1}{2}\sum_{j\in J} m_j x_j^2 + \sum_{j\in J} s_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j\in J} d_{ij} x_j - \alpha_i \right) + \sum_{j\in J} p_j (l_j - x_j) + \sum_{j\in J} q_j (x_j - u_j), \tag{4}$$

where $\lambda \in \mathbb{R}^m$; $\mathrm{p}, \mathrm{q} \in \mathbb{R}_+^n$, and $\mathbb{R}_+^n$ is the totality of all n – vectors with real nonnegative components.

The Karush-Kuhn-Tucker (KKT) necessary and sufficient conditions for the minimum solution $\mathrm{x}^* = (x_j^*)_{j\in J}$ to problem (C_m) are as follows:

$$m_j x_j^* + s_j + \sum_{i=1}^{m} \lambda_i d_{ij} - p_j + q_j = 0, \quad j \in J, \tag{5}$$

$$p_j (l_j - x_j^*) = 0, \quad j \in J, \tag{6}$$

$$q_j (x_j^* - u_j) = 0, \quad j \in J, \tag{7}$$

$$\sum_{j \in J} d_{ij} x_j^* = \alpha_i, \quad i = 1, \ldots, m, \tag{8}$$

$$l_j \le x_j^* \le u_j, \quad j \in J, \tag{9}$$

$$\lambda_i \in \mathbb{R}^1, i = 1, \ldots, m, \quad p_j \in \mathbb{R}^1_+, \, q_j \in \mathbb{R}^1_+, \quad j \in J, \tag{10}$$

where $\lambda_i, i = 1, \ldots, m$ and $p_j, q_j, j \in J,$ denote the Lagrange multipliers (dual variables), which are associated with the constraints (2), $l_j \le x_j,$ and $x_j \le u_j, j \in J,$ respectively. In the case when $l_j = -\infty$ or $u_j = +\infty$ for some index $j \in J,$ the corresponding KKT condition (6) (KKT condition (7), respectively) and multiplier p_j (multiplier $q_j,$ respectively) are not considered.

The following Theorem 1 presents characterization (a necessary and sufficient optimality condition) of the optimal solution of the considered problem (C_m).

Theorem 1 : *Suppose that assumptions* $(A.1)-(A.2)$ *are satisfied. A feasible solution* $\mathrm{x}^* = (x_j^*)_{j \in J} \in X$ *is an optimal solution of problem* (C_m) *if and only if there exist scalars* $\lambda_i \in \mathbb{R}, i = 1, \ldots, m,$ *such that*

$$x_j^* = l_j, \quad j \in J_l^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -m_j l_j - s_j \le \sum_{i=1}^{m} \lambda_i d_{ij} \right\} \tag{11}$$

$$x_j^* = u_j, \quad j \in J_u^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -m_j u_j - s_j \ge \sum_{i=1}^{m} \lambda_i d_{ij} \right\} \tag{12}$$

$$x_j^* : -m_j x_j^* - s_j = \sum_{i=1}^{m} \lambda_i d_{ij},$$

$$j \in J^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -m_j u_j - s_j < \sum_{i=1}^{m} \lambda_i d_{ij} < -m_j l_j - s_j \right\}. \tag{13}$$

Proof: *Necessity*. Let $\mathrm{x}^* = (x_j^*)_{j\in J}$ be an optimal solution of problem (C_m). In accordance with the KKT theorem, there exist scalars $\lambda_i, i = 1,\ldots,m$ and $p_j, q_j, j \in J$, which satisfy the KKT conditions (5)-(10).

(a) If $x_j^* = l_j$, then $p_j \geq 0, q_j = 0$ in accordance with KKT conditions (10) and (7), respectively. Therefore from KKT conditions (5) and (10) it follows that

$$m_j x_j^* + s_j \equiv m_j l_j + s_j = p_j - \sum_{i=1}^{m} \lambda_i d_{ij} \geq -\sum_{i=1}^{m} \lambda_i d_{ij}. \tag{14}$$

If we multiply this inequality by (–1), we get

$$-m_j l_j - s_j \leq \sum_{i=1}^{m} \lambda_i d_{ij}.$$

(b) If $x_j^* = u_j$, then $p_j = 0, q_j \geq 0$ in accordance with KKT conditions (6) and (10), respectively. Therefore from KKT conditions (5) and (10) it follows that

$$m_j x_j^* + s_j \equiv m_j u_j + s_j = -q_j - \sum_{i=1}^{m} \lambda_i d_{ij} \leq -\sum_{i=1}^{m} \lambda_i d_{ij}. \tag{15}$$

If we multiply this inequality by (–1), we obtain

$$-m_j u_j - s_j \geq \sum_{i=1}^{m} \lambda_i d_{ij}.$$

(c) If $l_j < x_j^* < u_j$, then $p_j = q_j = 0$ in accordance with KKT conditions (6) and (7), respectively. Therefore from KKT conditions (5) and (10) it follows that

$$-m_j x_j^* - s_j = \sum_{i=1}^{m} \lambda_i d_{ij}. \tag{16}$$

By using that $l_j < x_j^* < u_j$ in case (c), and $c_j(x_j) = \frac{1}{2} m_j x_j^2 + s_j x_j, j \in J$, are strictly convex functions, we obtain

$$c_j'(l_j) \equiv m_j l_j + s_j < c_j'(x_j^*) \equiv m_j x_j^* + s_j < c_j'(u_j) \equiv m_j u_j + s_j,$$

whence, after multiplying this chain of inequalities by (–1), in case (c) we have

$$-m_j u_j - s_j < \sum_{i=1}^{m} \lambda_i d_{ij} < -m_j l_j - s_j.$$

To describe cases (a), (b), (c), the index sets $J_l^\lambda, J_u^\lambda, J^\lambda$ are introduced, which are defined by (11), (12), and (13), respectively. Obviously, $J_l^\lambda \cup J_u^\lambda \cup J^\lambda = J$.

Sufficiency: Let $\mathrm{x}^* \in X$ and components of x^* satisfy conditions (11), (12), (13). Set:

$\sum_{i=1}^m \lambda_i d_{ij} = -m_j x_j^* - s_j, \quad p_j = q_j = 0 \quad$ for $j \in J^\lambda$;

$p_j = m_j l_j + s_j + \sum_{i=1}^m \lambda_i d_{ij}$ (≥ 0 in accordance with (11)), $q_j = 0$ for $j \in J_l^\lambda$;

$p_j = 0, \quad q_j = -m_j u_j - s_j - \sum_{i=1}^m \lambda_i d_{ij}$ (≥ 0 in accordance with (12)) for $j \in J_u^\lambda$.

Using these expressions, we can check that KKT conditions (5), (6), (7) and (10) are satisfied. Moreover, KKT conditions (8)-(9) are satisfied in accordance with the assumption that $\mathrm{x}^* \in X$, where the feasible set X is defined by (2)-(3).

Hence, in all possible cases, $\lambda_i, i = 1, \ldots, m$ and $x_j^*, p_j, q_j, \ j \in J$, satisfy the KKT conditions (5)-(10) for the problem (C_m), which are the necessary and sufficient conditions for a feasible solution to be an optimal solution to this problem. Therefore x^* is an optimal solution of problem (C_m).

Since the objective function of problem (C_m) is *strictly* convex, then optimal solution to this problem is unique. □

The importance of Theorem 1 is the fact that it characterizes components of the optimal solution of the considered problem (C_m) only via the Lagrange multipliers (dual variables) $\lambda_i, i = 1, \ldots, m,$ which are associated with the equality constraints (2).

3. The case of a linear equality constraint and box constraints

3.1 *Statement of the problem. Characterization theorem*

In this section, the special case (C) of problem (C_m) with the same separable convex quadratic objective function subject to a single linear equality constraint and box constraints is considered:

(C)

$$\min \left\{ c(\mathrm{x}) \equiv \sum_{j \in J} c_j(x_j) = \frac{1}{2} \sum_{j \in J} m_j x_j^2 + \sum_{j \in J} s_j x_j \right\} \qquad (17) \equiv (1)$$

subject to

$$\sum_{j\in J} d_j x_j = \alpha \tag{18}$$

$$l_j \le x_j \le u_j, \quad j \in J, \tag{19} \equiv (3)$$

where $m_j > 0, d_j > 0$ for every $j \in J$, and $\alpha \in \mathbb{R}$.

Assume that the following assumptions are satisfied.

(A.I) $l_j \le u_j, j \in J$. If $l_k = u_k$ for an index $k \in J$ then the value $x_k := l_k = u_k$ is determined *a priori*.

(A.II) $\sum_{j\in J} d_j l_j \le \alpha \le \sum_{j\in J} d_j u_j$. Otherwise constraints (18)-(19) are inconsistent and the feasible region X_1, defined by (18)-(19), is empty.

The Lagrangian function for problem (C) is defined as follows:

$$L(\mathbf{x}, \mathbf{p}, \mathbf{q}, \lambda) = \frac{1}{2}\sum_{j\in J} m_j x_j^2 + \sum_{j\in J} s_j x_j + \lambda\left(\sum_{j\in J} d_j x_j - \alpha\right) + \sum_{j\in J} p_j (l_j - x_j) + \sum_{j\in J} q_j (x_j - u_j), \tag{20}$$

where $\lambda \in \mathbb{R}$; $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n_+$, and $\mathbb{R}^n_+$ is the totality of all $n-$vectors with real nonnegative components.

The KKT necessary and sufficient conditions for the minimum solution $\mathbf{x}^* = (x_j^*)_{j\in J}$ of problem (C) are

$$m_j x_j^* + s_j + \lambda d_j - p_j + q_j = 0, \quad j \in J, \tag{21}$$

$$p_j (l_j - x_j^*) = 0, \quad j \in J, \tag{22}$$

$$q_j (x_j^* - u_j) = 0, \quad j \in J, \tag{23}$$

$$\sum_{j\in J} d_j x_j^* = \alpha, \tag{24}$$

$$l_j \le x_j^* \le u_j, \quad j \in J, \tag{25}$$

$$\lambda \in \mathbb{R}^1, \quad p_j \in \mathbb{R}^1_+, q_j \in \mathbb{R}^1_+, \quad j \in J, \tag{26}$$

where λ and $p_j, q_j, j \in J$, are the Lagrange multipliers (dual variables), associated with the constraints (18), $l_j \le x_j$, and $x_j \le u_j, j \in J$, respectively. Again, in the case when $l_j = -\infty$ or $u_j = +\infty$ for some index $j \in J$, the

corresponding KKT condition (22) (KKT condition (23), respectively) and multiplier p_j (multiplier q_j, respectively) are not considered.

The following Theorem 2 gives a necessary and sufficient condition (characterization) of the optimal solution of problem (C).

Theorem 2 : *Suppose that assumptions* (A.I)-(A.II) *are satisfied. A feasible solution* $\mathrm{x}^* = (x_j^*)_{j\in J} \in X_1$ *is an optimal solution of problem* (C) *if and only if there exists a scalar* $\lambda \in \mathbb{R}$ *such that*

$$x_j^* = l_j, \quad j \in J_l^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -\frac{m_j l_j + s_j}{d_j} \le \lambda \right\} \tag{27}$$

$$x_j^* = u_j, \quad j \in J_u^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -\frac{m_j u_j + s_j}{d_j} \ge \lambda \right\} \tag{28}$$

$$x_j^* = -\frac{s_j + \lambda d_j}{m_j}, \quad j \in J^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -\frac{m_j u_j + s_j}{d_j} < \lambda < -\frac{m_j l_j + s_j}{d_j} \right\}. \tag{29}$$

The proof of Theorem 2 is omitted because it is similar to the proof of Theorem 1.

3.2 *The Algorithm*

The following Algorithm 1 is proposed for solving problem (C).

Algorithm 1

1. (Initialization) $J := \{1, \ldots, n\}$, $k := 0$, $\alpha^{(0)} := \alpha$, $n^{(0)} := n$, $J^{(0)} := J$, $J_l^\lambda := \varnothing$, $J_u^\lambda := \varnothing$.
   If $\sum_{j\in J} d_j l_j \le \alpha \le \sum_{j\in J} d_j u_j$ go to 2, else go to 9.

2. $J^{\lambda(k)} := J^{(k)}$. Calculate $\lambda^{(k)}$ by using the explicit expression
$$\lambda^{(k)} = \left(\sum_{j\in J^{\lambda(k)}} \frac{d_j^2}{m_j} \right)^{-1} \left(-\sum_{j\in J^{\lambda(k)}} \frac{d_j s_j}{m_j} - \alpha^{(k)} \right).$$
   Go to 3.

3. Construct the index sets $J_l^{\lambda(k)}$, $J_u^{\lambda(k)}$, $J^{\lambda(k)}$, using expressions (27), (28), (29), respectively (with $j \in J^{(k)}$ instead of $j \in J$) and determine their cardinalities $|J_l^{\lambda(k)}|$, $|J_u^{\lambda(k)}|$, $|J^{\lambda(k)}|$, respectively. Go to 4.

4. Calculate

$$\delta(\lambda^{(k)}) \stackrel{\text{def}}{=} \sum_{j\in J_l^{\lambda(k)}} d_j l_j + \sum_{j\in J_u^{\lambda(k)}} d_j u_j - \sum_{j\in J^{\lambda(k)}} \frac{d_j s_j}{m_j} - \lambda^{(k)} \sum_{j\in J^{\lambda(k)}} \frac{d_j^2}{m_j} - \alpha^{(k)}.$$

Go to 5.

5. If $\delta(\lambda^{(k)}) = 0$ or $J^{\lambda(k)} = \varnothing$ then $\lambda := \lambda^{(k)}$, $J_l^{\lambda} := J_l^{\lambda} \cup J_l^{\lambda(k)}$, $J_u^{\lambda} := J_u^{\lambda} \cup J_u^{\lambda(k)}$, $J^{\lambda} := J^{\lambda(k)}$,
go to 8
else if $\delta(\lambda^{(k)}) > 0$ go to 6
else if $\delta(\lambda^{(k)}) < 0$ go to 7.

6. $x_j^* := l_j$ for $j \in J_l^{\lambda(k)}$, $\alpha^{(k+1)} := \alpha^{(k)} - \sum_{j\in J_l^{\lambda(k)}} d_j l_j$, $J^{(k+1)} := J^{(k)} \setminus J_l^{\lambda(k)}$, $n^{(k+1)} := n^{(k)} - |J_l^{\lambda(k)}|$, $J_l^{\lambda} := J_l^{\lambda} \cup J_l^{\lambda(k)}$, $k := k+1$. Go to 2.

7. $x_j^* := u_j$ for $j \in J_u^{\lambda(k)}$, $\alpha^{(k+1)} := \alpha^{(k)} - \sum_{j\in J_u^{\lambda(k)}} d_j u_j$, $J^{(k+1)} := J^{(k)} \setminus J_u^{\lambda(k)}$, $n^{(k+1)} := n^{(k)} - |J_u^{\lambda(k)}|$, $J_u^{\lambda} := J_u^{\lambda} \cup J_u^{\lambda(k)}$, $k := k+1$. Go to 2.

8. $x_j^* := l_j$ for $j \in J_l^{\lambda}$, $x_j^* := u_j$ for $j \in J_u^{\lambda}$, $x_j^* := -\frac{s_j + \lambda d_j}{m_j}$ for $j \in J^{\lambda}$. Go to 10.

9. Problem (C) has no optimal solution because its feasible set X_1, defined by (18)-(19), is empty.

10. End.

3.3 *Convergence and computational complexity of Algorithm 1*

The following Theorem 3 states convergence of Algorithm 1.

Theorem 3 : *Let* $\{\lambda^{(k)}\}$ *be the numerical sequence, generated by Algorithm 1. Then the following relations hold true:*

(i) *if* $\delta(\lambda^{(k)}) > 0$ *then* $\lambda^{(k)} \le \lambda^{(k+1)}$;

(ii) *if* $\delta(\lambda^{(k)}) < 0$ *then* $\lambda^{(k)} \ge \lambda^{(k+1)}$.

Proof: Denote by $x_j^{(k)}$ the components of $\mathrm{x}^{(\mathrm{k})} = (x_j)_{j\in J^{(k)}}$ at iteration k of the implementation of Algorithm 1.

(i) Let $\delta(\lambda^{(k)}) > 0$. From Step 6 of Algorithm 1, which is performed when $\delta(\lambda^{(k)}) > 0$, we get

$$\sum_{j\in J^{\lambda(k+1)}} d_j x_j^{(k)} \equiv \sum_{j\in J^{(k+1)}} d_j x_j^{(k)} = \sum_{j\in J^{(k)}\setminus J_l^{\lambda(k)}} d_j x_j^{(k)} = \alpha^{(k)} - \sum_{j\in J_l^{\lambda(k)}} d_j x_j^{(k)}. \tag{30}$$

Let $j \in J_l^{\lambda(k)}$. According to definition (27) of index set $J_l^{\lambda(k)}$ we have

$$-\frac{m_j l_j + s_j}{d_j} \le \lambda^{(k)} = -\frac{m_j x_j^{(k)} + s_j}{d_j}.$$

Since $m_j > 0, d_j > 0$ by assumption, then $l_j \ge x_j^{(k)}, j \in J_a^{\lambda(k)}$.

From (30), using that $d_j > 0, l_j \ge x_j^{(k)}, j \in J_a^{\lambda(k)}$, and Step 6 of Algorithm 1, we get

$$\sum_{j\in J^{\lambda(k+1)}} d_j x_j^{(k)} = \alpha^{(k)} - \sum_{j\in J_l^{\lambda(k)}} d_j x_j^{(k)} \ge \alpha^{(k)} - \sum_{j\in J_l^{\lambda(k)}} d_j l_j = \alpha^{(k+1)} = \sum_{j\in J^{\lambda(k+1)}} d_j x_j^{(k+1)}.$$

Since $d_j > 0, j \in J$, then there exists at least one $j_0 \in J^{\lambda(k+1)}$ such that $x_{j_0}^{(k)} \ge x_{j_0}^{(k+1)}$. Then

$$\lambda^{(k)} = -\frac{m_{j_0} x_{j_0}^{(k)} + s_{j_0}}{d_{j_0}} \le -\frac{m_{j_0} x_{j_0}^{(k+1)} + s_{j_0}}{d_{j_0}} = \lambda^{(k+1)}.$$

We have used that the relationship between $\lambda^{(k)}$ and $x_j^{(k)}$ is given by (29) for $j \in J^{\lambda(k)}$ (Step 2 of Algorithm 1), that $m_j > 0$ and that $-(m_j x_j^{(k)} + s_j) > 0, j \in J^{\lambda(k)}$, according to (29) with $\lambda^{(k)} \ge 0$, $d_j > 0$.

The proof of part (ii) of Theorem 3 is omitted because it is similar to the proof of part (i). □

Feasibility of $\mathrm{x}^* = (x_j^*)_{j\in J}$, generated by Algorithm 1.

Components $x_j^* = l_j, j \in J_l^\lambda$, and $x_j^* = u_j, j \in J_u^\lambda$, obviously satisfy constraint (19). From

$$\frac{-m_j u_j + s_j}{d_j} < \lambda \equiv -\frac{m_j x_j^* + s_j}{d_j} < -\frac{m_j l_j + s_j}{d_j}, \quad j \in J^\lambda,$$

it follows that

$$-\frac{m_j u_j}{d_j} < -\frac{m_j x_j^*}{d_j} < -\frac{m_j l_j}{d_j}, \quad j \in J^\lambda,$$

and since $d_j > 0, m_j > 0$, then $l_j < x_j^* < u_j$ for $j \in J^\lambda$. Hence all $x_j^*, j \in J$, satisfy (19).

Since at each iteration, $\lambda^{(k)}$ is calculated using the current equality constraint (18) (Step 2 of Algorithm 1) and since $x_j, j \in J$, are calculated according to $\lambda^{(k)}$ at each iteration (Steps 5, 6, 7, 8 of Algorithm 1), then x^* satisfies (18).

Therefore Algorithm 1 determines $\mathbf{x}^*$ that is feasible for problem (C), which is an assumption of Theorem 2.

Theorem 2, definitions of index sets J_l^λ (27), J_u^λ (28), J^λ (29), and Steps 6, 7 and 8 of Algorithm 1 show that $J_l^{\lambda(k)} \subseteq J_l^{\lambda(k+1)}, J_u^{\lambda(k)} \subseteq J_u^{\lambda(k+1)}$, and $J^{\lambda(k)} \supseteq J^{\lambda(k+1)}$. Hence, if j belongs to current set $J_l^{\lambda(k)}$ then j belongs to the next index set $J_l^{\lambda(k+1)}$ and, continuing in the same manner, it follows that this index j belongs to the optimal index set J_l^λ; the same holds true about the index sets $J_u^{\lambda(k)}$ and J_u^λ. Hence, $\lambda^{(k)}$ converges to the optimal value λ of Theorem 2 and $J_l^{\lambda(k)}, J_u^{\lambda(k)}, J^{\lambda(k)}$ "converge" to the optimal index sets $J_l^\lambda, J_u^\lambda, J^\lambda$, respectively. This means that calculation of λ, operations $x_j^* := l_j, j \in J_l^{\lambda(k)}$ (Step 6), $x_j^* := u_j, j \in J_u^{\lambda(k)}$ (Step 7), and the construction of final optimal index sets $J_l^\lambda, J_u^\lambda, J^\lambda$ and assignment of final values of $x_j^*, j \in J$ (Step 8) are in accordance with Theorem 2.

Computational complexity of Algorithm 1.

At each iteration, Algorithm 1 determines the value of at least one variable (Steps 6, 7, 8) and at each iteration, a problem of the form (C) of *less* dimension is solved (Steps 2-7). Therefore Algorithm 1 is finite and it converges with at most $n = |J|$ iterations, that is, the iteration complexity of Algorithm 1 is $O(n)$. Since each step takes time which is bounded by $O(n)$, then Algorithm 1 has $O(n^2)$ running time and therefore it is a polynomial algorithm.

3.4 *Computational results*

In this subsection, computational results, obtained by applying Algorithm 1 to problem (C), are presented.

The computations have been performed on an Intel Pentium(R) Dual-Core CPU E5800 3.20GHz/2.00GB. Each problem was run 30 times. Parameters have been randomly generated in the appropriate intervals.

Number of variables	n = 1200	n = 1500
(Average) Number of iterations	2.07	2.10
Average run time (in seconds)	0.00009	0.0001

Numerical results, obtained by Algorithm 1, applied to some illustrative problems of the form (*C*), are presented below.

Example 1:

$$\min\left\{c(\mathrm{x}) = \sum_{j=1}^{10} x_j^2 + 4x_1 + 2x_2 + 6x_3 + 5x_4 - 12x_5 - x_6 - 9x_7 - x_8 + 2x_9 + 5x_{10}\right\}$$

subject to

$$\begin{aligned}
\sum_{j=1}^{10} x_j &= 57.5 \\
7 \le x_1 &\le 10 \\
-1 \le x_2 &\le 1.5 \\
-2 \le x_3 &\le 2.4 \\
-2.5 \le x_4 &\le 0.5 \\
18 \le x_5 &\le 21 \\
1.5 \le x_6 &\le 3.5 \\
14.5 \le x_7 &\le 19.5 \\
3.5 \le x_8 &\le 6.1 \\
1 \le x_9 &\le 2.5 \\
-1.5 \le x_{10} &\le 5.5
\end{aligned}$$

The optimal solution, obtained by Algorithm 1, is

$$\mathrm{x}^* = (7, 1.5, 2, 0.5, 18, 3.5, 14.5, 5.5, 2.5, 2.5),$$

$$c_{\min} = c(\mathrm{x}^*) = 352.25$$

Example 2: $\min\{c(\mathbf{x}) = 4x_1^2 + 10x_2^2 + 4x_3^2 + 3x_4^2 + 7x_5^2 + 3x_6^2 + x_7^2\}$
subject to

$$\begin{aligned}
\sum_{j=1}^{7} x_j &= 72 \\
4 \le x_1 &\le 7
\end{aligned}$$

$$4.5 \le x_2 \le 10$$
$$8 \le x_3 \le 13$$
$$5 \le x_4 \le 8$$
$$4 \le x_5 \le 7$$
$$30 \le x_6 \le 40$$
$$4 \le x_7 \le 7$$

The optimal solution, obtained by Algorithm 1, is

$$\mathrm{x}^* = (7, 4.5, 9.8636, 8, 5.6363, 30, 7),$$

$$c_{\min} = c(\mathrm{x}^*) = 3951.0454$$

4. Conclusions

In this paper, minimization problems with a separable convex quadratic objective function subject to a linear equality constraint/linear equality constraints, and bounds on the variables (box constraints) are considered. A necessary and sufficient condition for a feasible solution to be an optimal solution to each of the considered problems is established. A convergent algorithm of polynomial computational complexity for solving the problem with a linear equality constraint and bounded variables is proposed, and some numerical results obtained by this algorithm are presented.

An open question for future research is to develop a convergent polynomial algorithm for solving the considered problem (C_m) with several linear equality constraints.

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