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Dominator coloring of Kneser graph

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Abstract

A dominator coloring (briefly DC) of a graph G is a proper coloring such that each vertex dominates every vertex of at least one color class (possibly its own class). The dominator chromatic number $\chi_d(G)$ of G is the smallest cardinality of color classes in a DC of G. . By using Steiner triple systems, in this paper, we establish that the dominator coloring of the Kneser graph $KG(n,2)$ is n for $n \geq 5$.

Subject Classification: *(2010) 05C05, 05C12, 05C76.*

Keywords: *Dominator coloring, Steiner triple systems, Kneser graph.*

1. Introduction

All graphs considered in this paper are connected, simple, undirected and finite. We refer the reader to [1] for the notation and terminology not defined in this paper. Let $G = (V, E)$ be a graph of order n. For any vertex $v \in V(G)$, the open neighborhood of v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A dominating set $S \subseteq V$ satisfies that the vertices in S dominate all the vertices of $V \setminus S$, that is $N(S) = V \setminus S$. The domination number of G is the minimum number of a dominating set in G, denoted by $\gamma(G)$. We abbreviate $\{1, 2, \ldots, n\}$ to $[n]$ where $n \in N$.

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A *proper vertex coloring* of a graph G is a function c from the set of vertices of G to a set of colors, satisfies that for any two vertices $u, v \in V(G)$, $c(u) \neq c(v)$ if $uv \in E(G)$. The chromatic number $\chi(G)$ is the smallest cardinality of colors in a proper coloring of G. A (*proper*) *r-coloring* of a graph G is a mapping $f: V(G) \to \{1, 2, \ldots, r\}$ satisfies that $f(u) \neq f(v)$ for any two adjacent vertices u, v. A color class of a proper coloring of G is a subset of vertices, the subset of vertices contains every vertex having the same color. When c is a *r-coloring* of G, the color classes $V_1, V_2, \ldots, V_r$ of c satisfies $c(u) = i$ for any vertex $u \in V_i$, $i \in [r]$, then we write simply $c = (V_1, V_2, \ldots, V_r)$. If $|V_i| = 1$, we call V_i is a *single color class*.

A *dominator coloring* (briefly DC) of a graph G is a proper coloring such that each vertex of G dominates every vertex of at least one color class (possibly its own class). The dominator chromatic number $\chi_d(G)$ of G is the smallest cardinality of color classes in a DC of G.

K. Kavitha, N. G. David researched that the relationship between the dominator chromatic number $\chi_d(G)$ and the domination number $\gamma(G)$, and gave the following proposition.

Proposition 1.1 : [4] Every graph G satisfies

$$max\{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G).$$

The concept of a dominator coloring of a graph was proposed by Gera et al.[7] in 2006. Recently, several researchers have investigated researched the dominator chromatic numbers of Fan graph F_n, Double fan graph $F_{2,n}$, Helm graph H_n, Gear graph G_r, Flower graph Fl_n, Sun flower graph Sf_n, Trees, Paths and Caterpillars [3-6]. Chellali et al. researched the χ_d of $P4$-free graphs [6]. Arumugam et al. proved that the problem of the dominator coloring on split graphs, planar and bipartite is NP-hard [8]. More results on the dominator coloring could be researched in [8-11] .

A *total dominator coloring* of a graph G is a proper coloring such that every vertex of G is adjacent to all of vertices of at least one color class. The total dominator chromatic number of G is the smallest cardinality of color classes in a total dominator coloring of G, denoted by $\chi_d^t(G)$. By the definition of the dominator coloring and the total dominator coloring, we know that $\chi_d(G) \leq \chi_d^t(G)$. The concept of a total dominator coloring of a graph was proposed by the author Adel P. Kazemi in [16]. He proved that every graph G of order n with no isolated vertices satisfies $max\{\chi_d(G), \gamma_t(G)\} \leq \chi_d^t(G) \leq n$. Recently, several researchers have investigated the total dominator coloring numbers of Mycieleskian graphs $M(G)$, Cental graphs $C(G)$, Kneser graph $KG(n,2)$ [16-18]. More results on the total dominator coloring could be found in [19-22]. Parvin Jalilolghadr etal. researched $\chi_d^t(KG(n,2))$ [19]. In this paper, we study the $\chi_d KG(n,2)$ and prove that $\chi_d KG(n,2) = \chi_d^t(KG(n,2))$ for $n \geq 6$.

A Steiner system $S(q,k,n)$ is a family Ω of k-point subsets of an n-point set such that every q-point subset of Ω lies in exactly one of the k-point sets. [2][19]. Perumalsamy et al. studied the bounds of the parameter forcing edge fixed steiner number of G and prove the existence theorem [3]. If there are $q=2, k=3$ with n points, we called that is a Steiner triple system of order n, denoted by $STS(n)$. A Steiner triple system of order n exists if and only if $n \equiv 1,3(mod\ 6)$ [2][19]. The idea of a Steiner triple system is used in the proof of the main result of the paper.

The Kneser graph $KG(n,k)$ with $k \leq \frac{n}{2}$ is a graph defined as:

- $V(KG(n,k)) = \{u \mid u = a_1a_2a_3\ldots a_k, a_i \in [n], i \in [k], a_1 < a_2 < \ldots < a_k,\ u \in \binom{[n]}{k}\}$,
- $E(KG(n,k)) = \{uv \mid u = a_1a_2a_3\ldots a_k, v = b_1b_2b_3\ldots b_k, u \cap v = \phi$ for any $u,v \in V(KG(n,k))\}$.

In fact, when $k=2$, $KG(n,2)$ has the vertex set $\{u \mid u = a_1a_2,\ a_1, a_2 \in [n],\ a_1 \neq a_2,\ u \in \binom{[n]}{2}\}$, and the edge set $\{uv \mid u = a_1a_2,\ v = b_1b_2,\ u \cap v = \{a_1,a_2\} \cap \{b_1,b_2\} = \phi$ for any $u,v \in V(KG(n,2))\}$ (see Figure 1).

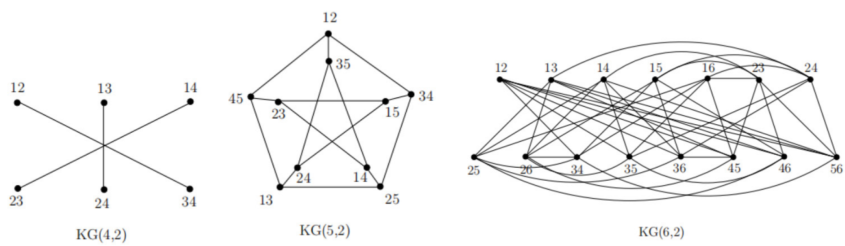
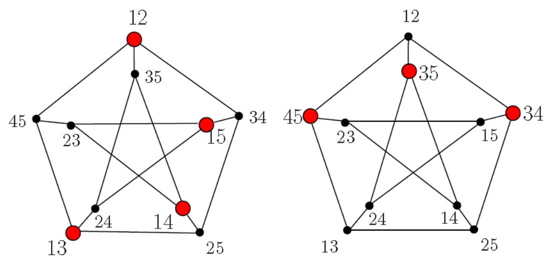


Figure 1

***KG*(4, 2), *KG*(5, 2) and *KG*(6, 2).**

About the domination number of $KG(n,2)$, J. Ivanco, B. Zelinka. etablished the following proposition.

Proposition 1.2 : For $n \geq 3$, $\gamma(KG(n,2)) = 3$.



(a) A starlike color class (b) A triangular color class

Figure 2

(a) A starlike color class and (b) A triangular color class.

Let "i" is a sign, for any $i \in [n]$. For any proper coloring of $KG(n,2)$, the color classes of $KG(n,2)$ have only two special structures. One is starlike color class and another is triangular color class. A starlike color class satisfies that all vertices in this color class have a common sign, such as $\{ab, ac, ad, ae\}$ ("a" is called to be a central sign). A triangular color class has three vertices which pairwise vertices have a common sign and these signs are different, such as $\{ab, ac, bc\}$ (see Figure 2).

L. Lovász obtained the following result of the chromatic number of $KG(n,k)$.

Proposition 1.3 : [14] For $n \geq 3$, $\chi(KG(n,k)) = n - 2k + 2$.

Proposition 1.4 : [15][19] Let c be a $(n-2)$ - *coloring* of $KG(n,2)$ with $n \geq 5$, then there only has one triangular color class. Moreover, if it is necessary, by renaming the signs $1,2,\ldots,n,$ the color classes $V_1, V_2, \ldots, V_{n-2}$ satisfy the following properties:

(i) $V_{n-2} = \{(n-1)(n-2), n(n-1), n(n-2)\}$, and V_{n-2} is a triangular color class;

(ii) $\{x(n-1), x(n-2), xn\} \subseteq V_x$, and the sign x is the central sign of V_x, for $x \in [n-3]$,;

(iii) If the vertex xy satisfies $\{x, y\} \subseteq [n-3]$, the vertex xy is either in V_x or V_y.

Proposition 1.5 : For $n \geq 5$,

$$\chi_d^t(KG(n,2)) = \begin{cases} 6, n=5, \\ n, n \geq 6. \end{cases}$$

Proposition 1.6 : $\chi_d(KG(5,2)) \leq 5$.

Proof: We take the following coloring for $KG(5,2)$:

$$V_1 = \{21\},\ V_2 = \{15\},\ V_3 = \{52\},\ V_4 = \{31,32,34,35\},\ V_5 = \{41,42,45\},$$

then it is a dominator coloring of $KG(5,2)$, so $\chi_d(KG(5,2)) \leq 5$.

□

By proposition 1.1, 1.2, 1.3, 1.5 and 1.6, the following result holds.

Corollary 1.7 : *For* $n \geq 5$, $n-2 \leq \chi_d(KG(n,2)) \leq n$.

In this paper, we investigate the dominator chromatic number of $KG(n,2)$ and establish $\chi_d(KG(n,2)) = n$ for $n \geq 5$.

2. Main results

For $i \in [n]$, if a color class V_i has a vertex containing the sign "i", we say that the color class V_i has the sign "i". For any $i, j \in [n]$, we have $|\{ij \mid i \neq j\}| = n-1$, that is, $n-1$ vertices of $V(KG(n,2))$ contains the sign "i". Therefore, for any $i \in [n]$, there are at most $n-1$ color classes having the sign "i".

Lemma 2.1 : *For any positive integers* $n \geq 5$ *and r with* $n-2 \leq r \leq n-1$. *Let* $c = (V_1, V_2, \ldots, V_r)$ *be a r-coloring of* $KG(n,2)$. *For some* $i \in [n]$, *if each color class has the sign "i", then c is not a DC.*

Proof: Let p be the cardinality of single color classes. The following research is divided into three cases:

Case 1 : $p = 0$.

When $p = 0$, each color class has at least 2 vertices. Since there exists a sign "i" satisfies that each color class has the sign "i", each vertex which contains the sign "i" can not dominate one color class, so c is not a DC.

Case 2 : $1 \leq p < r$.

Suppose that $p = 1$ and $|V_1| = 1$, $|V_j| \geq 2$ for $2 \leq j < r$. Let $ij \in V_j$, $j \in [n] \setminus \{i\}$, so ij can not dominate its own color class. Since each color class has the sign "i" for some $i \in [n]$, ij can not dominate at least one color class. So c is not a DC.

Suppose that there exist p single color classes $(1 \le p \le r-1)$, say $V_1, V_2, \ldots, V_p$, and $|V_{p+1}| \ge 2$, $i(p+1) \in V_{p+1}$. So $i(p+1)$ can not dominate its own color class. Since each color class has the sign "i" for some $i \in [n]$, each color class contains a vertex with sign "i", and so $i(p+1)$ can not dominate at least one color class. So c is not a DC.

Case 3 : $p = r$.

Since $|V(KG(n,2))| = \binom{n}{2} = \frac{n(n-1)}{2} > n-1 \ge r$ for $n \ge 5$, it is impossible that all color classes are single color classes. So c is not a DC. □

Lemma 2.2: *For any positive integers $n \ge 5$ and r with $n-2 \le r \le n-1$. Let c be a r-coloring of $KG(n,2)$. If there is no single color class, and there exists a sign "i" satisfies that $r-1$ color class has the sign "i", then c is not a DC.*

Proof: Since there is no single color class in c, each vertex can not dominate its own color class. Since at most $n-1$ color classes have the sign "i" for any $i \in [n]$. Now, suppose that the are $r-1$ color classes having the sign "i" with $n-2 \le r \le n-1$, then there exists a color class V_j without the sign "i". Let $i'j$ be a vertex of V_j, there exists a vertex ii' can not dominate at least one color class, so c is not a DC. □

Proposition 2.3 : For $n \ge 5$, $n-1 \le \chi_d(KG(n,2)) \le n$.

Proof: For any $(n-2)$-coloring of $KG(n,2)$, by Proposition 1.4, there exist 3 signs such that every color class of c has these 3 signs. Thus, by Corollary 1.7, Lemma 2.1 and Proposition 1.6,

$$n-1 \le \chi_d(KG(n,2)) \le n$$

holds. □

Lemma 2.4: *Let c be a $(n-1)$-coloring of $KG(n,2)$ with every color class of c is a starlike color class, then c is not a DC.*

Proof: Since every color class of c is a starlike color class and c is a $(n-1)$-coloring of $KG(n,2)$, we have at most $n-1$ different central signs. And there has a sign "i", the sign "i" is not central sign. Hence every color class contains at most one vertex having the sign "i". Since $|\{ij \mid i \ne j\}| = n-1$ for any $i, j \in [n]$, every color class only has one vertex with the sign "i". By Lemma 2.1, c is not a DC.

Lemma 2.5: *Let c be a $n-1$ dominator coloring of $KG(n,2)$ for $n\geq 5$, then the dominator coloring of $KG(n,2)$ has s triangular color classes with $1\leq s\leq 5$.*

Proof: By Lemma 2.4, we know that $s\geq 1$. Let t be the cardinality of starlike color classes, then

$$t=(n-1)-s\leq n-2.$$

Assume that $V_n, V_{n-1},\ldots,V_{n-t+1}$ are t starlike color classes, $B=\{i\,|\,i$ is the central sign of V_i, for any $i\in[n]\setminus[n-t]$, $B\subseteq[n]\setminus[n-t]$. Then $n-t$ signs $1,2,\ldots,n-t$ are not central signs. $\binom{n-t}{2}$ vertices of $V(KG(n,2))$ whose all signs in $[n-t]$ are not in t starlike color classes. We know that every triangular color class has 3 vertices, then these $\binom{n-t}{2}$ vertices are distributed to at least $\frac{1}{3}\binom{n-t}{2}$ triangular color classes. Hence,

$$s\geq\frac{1}{3}\binom{n-t}{2}=\frac{(n-t)(n-t-1)}{6},$$

that is,

$$s\geq\frac{(s+1)s}{6}.$$

Hence $1\leq s\leq 5$. □

Theorem 2.6: *For $n\geq 5$, $\chi_d(KG(n,2))=n$.*

Proof: We know that $n-1\leq\chi_d(KG(n,2))\leq n$ for $n\geq 5$, by Proposition 2.3. Let c be a $n-1$ dominator coloring of $KG(n,2)$, then we will lead to a contradiction.

Let $A\subseteq[n]$ be the set of central signs, $\overline{A}=[n]\setminus A$, t be the cardinality of starlike color classes and s be the cardinality of triangular color classes. By Lemma 2.5, we have $1\leq s\leq 5$.

Case 1: s = 5.

Assume that the triangular color classes are V_1,V_2,V_3,V_4,V_5, then $t=n-1-s=n-6$, $|A|\leq n-6$ and $|\overline{A}|\geq 6$, that is at least 6 signs which are not central signs. Without loss the generality, assume that $1,2,3,4,5,6$ are not central signs, then $\{1,2,3,4,5,6\}\subseteq\overline{A}$. Note that when $x\neq y$ and $\{x,y\}\subseteq\overline{A}$, every starlike color class dose not contains the vertex xy.

$$\{xy\,|\,x\neq y,\{x,y\}\subseteq\overline{A}\}\subseteq\bigcup_{i=1}^{5}V_i,$$

and

$$|\{xy \mid x \neq y, \{x, y\} \subseteq \{1,2,3,4,5,6\}\}| = \binom{6}{2} = 15 = \sum_{i=1}^{5} |V_i|$$

holds.

By Steiner triple system ($n = 6,\ q = 2,\ k = 3$), for any $s \in [5]$, let K_i be the set of all signs in V_i. Since three signs construct a triangular color class V_i, we have $|K_i| = 3$ for $1 \leq i \leq 5$. For any two signs x, y with $\{x, y\} \subseteq [6]$, there exists exactly one set K_i, $\{1 \leq i \leq 5\}$, such that $\{x, y\} \subseteq K_i$. This is a contradiction with the existence of a $STS(6)$.

Case 2: $s = 4$.

Assume that the triangular color classes are V_1, V_2, V_3, V_4, then $t = n - 1 - s = n - 5$, $|A| \leq n - 5$ and $k = |\overline{A}| \geq 5$, that is at least 5 signs which are not central signs. When $k \geq 6$, $\binom{k}{2}$ vertices $\{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\}$ are assigned to these 4 triangular color classes. Hence,

$$\binom{k}{2} \geq \binom{6}{2} 15 > 12 = \bigcup_{i=1}^{4} V_i,$$

which is a contradiction. Thus, there are exactly 5 signs which are not central signs, say $1,2,3,4,5$, that is $\overline{A} = \{1,2,3,4,5\}$. Since each V_i is an independent set, $1 \leq i \leq 4$ and

$$\{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\} \subseteq \bigcup_{i=1}^{4} V_i, |\{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\}| = 10, |\bigcup_{i=1}^{4} V_i| = 12,$$

there have 2 vertices xy and $x'y'$ in $\bigcup_{i=1}^{4} V_i$ satisfy that

$$\{x, x'\} \subseteq \overline{A},\ \{y, y'\} \subseteq A.$$

The following research is divided into two subcases:

Subcase 2.1: xy and $x'y'$ are in the same triangular color class.

Since xy and $x'y'$ are in the same triangular color class, we have $\{xy, x'y'\} \subseteq V_i$, $1 \leq i \leq 4$.

When $x = x'$, $V_i = \{xy, xy', yy'\}$ is a triangular color class. $\{y, y'\}$ is the 2-set of $\overline{A}$, but $\{y, y'\} \subseteq A$, which is a contradiction.

When $y = y'$, $V_i = \{xy, x'y, xx'\}$ is a triangular color class. Without loss the generality, assume that $V_i = \{1y, 2y, 12\}$. Then three vertices $13, 14, 15$ are assigned to three remaining triangular color classes. If the vertices $13, 14, 15$ are respectively in different triangular color classes, then every triangular color class exactly contains two vertices with the sign "1", which is a contradiction. If two of the vertices $13, 14, 15$ in the same triangular color class, without loss the generality, assume that $V_i = \{13, 14, 34\}$, then the triangular color class which contains vertex 15 has a new vertex with the sign "1", which is a contradiction. If the vertices $13, 14, 15$ are in the same triangular color class, the color class is a starlike color class, which is a contradiction.

Subcase 2.2: xy and $x'y'$ are in different triangular color classes

Since xy and $x'y'$ are in different triangular color classes, we have $xy \in V_i$ and $x'y' \in V_{i'}$ with $i \neq i'$, and $1 \leq i, i' \leq 4$. However, V_i has 2 vertices with the sign y, $V_{i'}$ has 2 vertices with the sign y'. That is,

$$\left| \bigcup_{i=1}^{4} V_i \setminus \{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\} \right| = 4 > 2,$$

which is a contradiction.

Case 3: $s = 3$.

Assume that the triangular color classes are V_1, V_2, V_3, then $t = n - 1 - s = n - 4$, $|A| \leq n - 4$ and $k = |\overline{A}| \geq 4$, that is at least 4 signs which are not central signs. When $k \geq 5$, $\binom{k}{2} \geq \binom{5}{2} = 10$ and the 3 triangular color classes needs 9 vertices. When $k = 4$, there are 4 signs which are not central signs, say $\overline{A} = \{1, 2, 3, 4\}$. Since

$$\{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\} \subseteq \bigcup_{i=1}^{3} V_i, \ |\{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\}| = \binom{4}{2} = 6, \ |\bigcup_{i=1}^{3} V_i| = 9,$$

there have 3 vertices xy, $x'y'$ and $x''y''$ in $\bigcup_{i=1}^{3} V_i$ satisfy that

$$\{x, x', x''\} \subseteq \overline{A}, \ \{y, y', y''\} \subseteq A.$$

The following research is divided into three subcases:

Subcase 3.1: $xy, x'y', x''y''$ are in the same triangular color class

Since $xy, x'y', x''y''$ are in the same triangular color class, say $\{xy, x'y', x''y''\} \subseteq V_1$. V_2, V_3 need 6 vertices which all signs in $\{1, 2, 3, 4\}$, it is impossible for $STS(4)$.

Subcase 3.2: Two of $xy, x'y', x''y''$ are in the same triangular color class

Since two of $xy, x'y', x''y''$ are in the same triangular color class, say $\{xy, x'y'\} \subseteq V_1$ and $\{x''y''\} \subseteq V_2$. V_2 contains 2 vertices with the sign y''. That is,

$$\left| \bigcup_{i=1}^{3} V_i \setminus \{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\} \right| \geq 4,$$

which is a contradiction.

Subcase 3.3: $xy, x'y', x''y''$ are respectively in different triangular color classes

Since $xy, x'y', x''y''$ are respectively in different triangular color classes, we have $xy \in V_1$, $x'y' \in V_2$, $x''y'' \in V_3$. V_1 has 2 vertices having the sign y, V_2 has 2 vertices having the sign y', V_3 has 2 vertices having the sign y''. That is,

$$\left| \bigcup_{i=1}^{3} V_i \setminus \{xy \mid x \neq y, \{x, y\} \subseteq \overline{A}\} \right| \geq 6,$$

which is contradiction.

Case 4: $s = 2$.

Assume that the triangular color classes are V_1, V_2, then $t = n-1-s = n-3$ and $|A| \leq n-3$, then there are $k = |\overline{A}| \geq 3$ signs which are not central signs. When $k = 4$, 6 vertices is needed in 2 triangular color classes and $\binom{4}{2} = 6$, contradicting with $STS(4)$. When $k \geq 5$, 2 triangular color classes need 6 vertices and $\binom{k}{2} \geq \binom{5}{2} = 10 > 6$, which is a contradiction. When $k = 3$, there are 3 signs which are not central signs, say $1, 2, 3$. Hence, three vertices $23, 12, 13$ are assigned to 2 triangular color classes. By the Pigenhole principle, we have $|V_i \cap \{23, 12, 13\}| \geq 2$ for some $i \in \{1, 2\}$, and $V_i = \{23, 12, 13\}$. Assume that $V_1 = \{23, 12, 13\}$ and $V_2 = \{xy, xz, yz\}$. We know $V_1 \cap V_2 = \phi$, then $|\{1, 2, 3\} \cap \{x, y, z\}| \leq 1$.

The following research is divided into two subcases:

Subcase 4.1: $|\{1, 2, 3\} \cap \{x, y, z\}| = 0$.

Then $\{x, y, z\} \subseteq \{4, 5, \ldots, n\}$. For any $i, j \in [n]$, we have $|ij \mid i \neq j| = n-1$. "1", "2", "3" are not central signs and all triangular color classes has 2 vertices having "1", "2" or "3". Since "1", "2", "3" appear at most once in every star color class for each star color class which has at least 2 vertices, by the pigeonhole principle, each color class is not single color class, and $n-2$ color classes contains one vertex having the sign "1". By Lemma 2.2, we know that c is not a DC.

Subcase 4.2: $|\{1,2,3\}\cap\{x,y,z\}|=1$.

Without loss the generality, we assume that $\{1,2,3\}\cap\{x,y,z\}=\{1\}$. For any $i,j\in[n]$, we have $|ij\,|\,i\neq j|=n-1$. "1", "2", "3" are not central signs and "2", "3" appear twice in all triangular color classes. By the pigeonhole principle, "2", "3" appear at most once in every star color class for each star color class which has at least 2 vertices, and each color class is not single color class. Then "2", "3" appear in $n-2$ color classes. By Lemma 2.2, we know that c is not a DC.

Case 5: $s = 1$.

Assume that the triangular color class is V_1, $t=n-1-s=n-2$, $|A|\leq n-2$ and $k=|\overline{A}|\geq 2$, that is at least 2 signs which are not central signs. Suppose that $\{1,2\}\subseteq\overline{A}$, and $V_1=\{12,1x,2x\}$. For any $i,j\in[n]$, we have $|ij\,|\,i\neq j|=n-1$. "1" is a sign which are not central signs and the triangular color class V_1 contains two vertices with "1". Every star color class contains at most one vertex with the sign "1" which has at least two vertices. By the pigeonhole principle, we have $n-3$ star color classes containing the sign "1" at least two vertices, there exists a star color class V_j not containing the sign "1".

The following research is divided into two subcases:

Subcase 5.1: $|V_j|\geq 2$.

By the pigeonhole principle, "1" appears in $n-2$ color classes. And by Lemma 2.2, we know that c is not a DC.

Subcase 5.2: $|V_j|=1$.

Since "1", "2" are not central signs, V_j is not containing sign "1", "2". Assume $V_j=\{xy\}$, $|V_j|=1$, and $x,y\in A$, $x\neq y$. By the pigeonhole principle, sign "1", "2" appear in remaining star color classes. There exists a vertex $1x$ $(1x\in V_i$ and $|V_i|\geq 2)$, it can not dominate every vertex of at least one color class and we know that c is not a DC. □

By Proposition 1.5 and Theorem 2.6, the following corollary holds.

Corollary 2.7: For $n\geq 6$, $\chi_d(KG(n,2))=\chi_d^t(KG(n,2))=n$.

3. Problem

The research can be further extend to the dominated coloring, the domination coloring, the dominator edge coloring of $KG(n,k)$, we list the following open problems.

Problem 1 : When $k=2$, determine the value of the dominated coloring, the dominator edge coloring and the domination coloring of $KG(n,2)$.

Problem 2 : For $k \leq \frac{n}{2}$, determine the bound of $\chi_d(KG(n,k))$ and $\chi_d^t(KG(n,k))$.

Problem 3 : Analyze the value of k such that $\chi_d(KG(n,k)) = \chi_d^t(KG(n,k))$.

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