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Similar *d*– even vertex odd mean labeling of diverse graphs

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Abstract

An injective function Φ where, $\Phi : V(\Gamma) \rightarrow \{0,2,4,\cdots, 2\beta+2d-2\}$ is said to be *d-even vertex odd mean labeling* (*d*-EVOML) of the graph $\Gamma(\alpha,\beta)$ when the induced mapping $\Phi^* : E(\Gamma) \rightarrow \{1,3,\cdots,2\beta-1\}$, given by: $\Phi^*(\lambda\omega) = \frac{\Phi(\lambda)+\Phi(\omega)}{2}$, is a bijective function. A d - even vertex odd mean graph is a graph which allows even vertex odd mean labelling. In this study, we identify the lowest value of d for which the graphs: Y-tree, star graph, $P_\xi \odot \overline{K_\eta}$, crown graph R_η, and rooted product $P_\eta \Diamond C_4$ have a d- even vertex odd mean labeling. Furthermore, we find the minimum number d for which the graphs: cycle graph C_η when $\eta \equiv 2 \bmod 4$, dragon graph $P_\xi(C_\eta)$ when $\eta \equiv 2 \bmod 4$, $\xi \geq 1$, prism graph Π_η, and Toeplitz graphs $T_\eta(1,3)$, $T_\eta(1,5)$, and $T_\eta(1,3,5)$ have a similar d- even vertex odd mean labeling. In the end, we establish that no odd cycle C_η is an even vertex odd mean graph for all d.

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1. Introduction

A graph $\Gamma = (V(\Gamma), E(\Gamma))$ throughout this paper is always a finite and a simple connected graph, where $\alpha = |V(\Gamma)|$ is the cardinality of $V(\Gamma)$ and $\beta = |E(\Gamma)|$ is that of $E(\Gamma)$.

For many mathematical issues employed in structural models, the field of graph theory plays a crucial role. Due to its widespread application in many fields, such as communication network addressing, circuit design, coding theory for radar, crystallography, x-ray, and graph decomposition addressing, graph labelling has grown to be a remarkable and wealthy area of research in graph theory. See [1–3] for more motivating applications of graph labeling.

A labelling of a graph is a mapping that, under certain restrictions, converts graph elements (edges, vertices, or both) to non-negative numbers. Rosa first proposed the concept of graph labelling at the beginning of the 1960s. Following this research, many graph labeling strategies have been investigated. Edge graceful labeling introduced by Lo [4] In 1985. An edge odd graceful labeling of a graph introduced by Solairaju and Chithra in 2009 [5]. An edge even graceful labeling of a graph introduced by Elsonbaty and Daoud in 2017 [6]. Zeen El Deen et al. [7–9] studied more graphs having an edge even graceful labeling. In 2020 Zeen El Deen [11] introduced the definition of an edge $\delta-$ graceful labeling of a graph G, for any positive integer δ. In [12]Zeen El Deen et al. introduced new classes of graphs with edge $\delta-$ graceful labeling. Furthermore, in [10], Basher studied odd-even graceful labeling of planar grid and prism graphs.

The concept of mean labeling was introduced and investigated by Somasundran and Ponraj [13, 14]. The concept of even vertex odd mean graphs were introduced by R. Vasuki, A Nagarajan, and S. Arockiaraj [15], they demonstrated that any graph G has K_3 as an induced subgraph, then G is not even vertex odd mean graph, in addition If G has an even vertex odd mean labeling, then G is a bipartite graph. They also looked at certain standard graphs having even vertex odd mean labeling. M. Basher [16–18] introduced more results on even vertex odd mean graphs. A summary of the outcomes on all known labels so far can be found in [19].

We will give a generalization of even vertex odd mean labeling to d-even vertex odd mean labeling in this research

Definition 1.1: A function Φ is known as d -even vertex odd mean labeling (d-EVOML) of the graph Γ if $\Phi : V(\Gamma) \to \{0, 2, 4, \cdots, 2\beta + 2d - 2\}$ is an injective function such that the induced mapping $\Phi^* : E(\Gamma) \to$

$\{1,3,\cdots,2\beta-1\}$, given by: $\Phi^*(\lambda\omega)=\frac{\Phi(\lambda)+\Phi(\omega)}{2}$, is a bijective function. The graph that is labeled with the d-even vertex odd mean is referred to as the d-even vertex odd mean graph. When $d=1$, this labeling is an even vertex odd mean labeling.

Definition 1.2: An injective function Φ where $\Phi: V(\Gamma) \to \{0,2,4,\cdots, 2\beta+2d-2\}$ is said to be similar d -even vertex odd mean labeling (similar d-EVOML) of the graph Γ if the induced mapping $\Phi^*: E(\Gamma) \to \{1,3,\cdots, 2\beta+2d-3\}$, given by: $\Phi^*(\lambda\omega)=\frac{\Phi(\lambda)+\Phi(\omega)}{2}$, is an injective function. The graph that accepts a similar d- even vertex odd mean labelling will be known as a similar d - even vertex odd mean graph.

The caterpillar graph shown in Figure 1 is bipartite, but it is not an even vertex odd mean graph, so we want to find the minimum number d to make this graph d-even vertex odd mean graph. Here the caterpillar has $\beta=10$, let $d=4$, the function $\Phi: V(\Gamma) \to \{0,2,4,\cdots, 2\beta+2d-2=26\}$ is an injective function such that the induced mapping $\Phi^*: E(\Gamma) \to \{1,3,\cdots,2\beta-1=19\}$, is a bijective function.

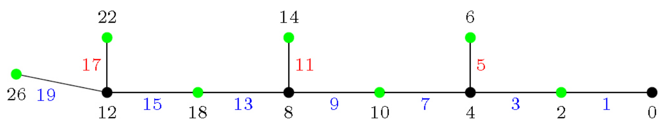

Figure 1

A 4-EVOML for caterpillar graph.

2. *d*-even vertex odd mean for path-related graphs

Lemma 2.1: *The Y-tree Y_η is $2-$ even vertex odd mean graph.*

Proof: From a path P_η, a Y-tree graph can be procured by supplementing an edge to a vertex of a path adjacent to an endpoint and it appears as Y_η where the cardinality of vertices of Y-tree is $\alpha=\eta$ and the cardinality of edges is $\beta=\eta-1$. Indicate the Y_η vertices and edges as shown in Figure 2

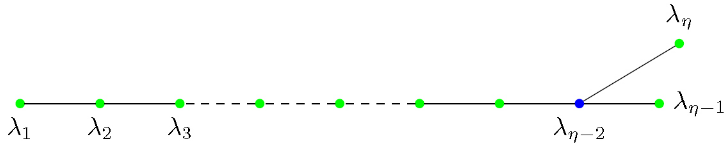

Figure 2

Y_η with normal labeling

It is easy to show that Y- tree is a bipartite graph but it is not even vertex odd mean graph.

Let $d = 2,$ we define the mapping $\Phi : V(G) \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 2\eta\}$ accordingly:

$$\Phi(\lambda_\kappa) = 2\kappa - 2 \text{ for } \kappa \in [1, \eta - 1], \text{ and } \Phi(\lambda_\eta) = 2\eta.$$

The resulting edge labels are then given by $\Phi^* : E(Y_\eta) \to \{1, 3, 5, \cdots, 2\beta - 1 = 2\eta - 3\}$

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = 2\kappa - 1, \quad \kappa \in [1, \eta - 2], \text{ and } \Phi^*(\lambda_{\eta-2}\lambda_\eta) = 2\eta - 3.$$

The proof is completed by the obvious oddness and distinctness of each edge label.

Illustration: A $2-$EVOML of the graph Y_{13} is depicted in Figure 3.

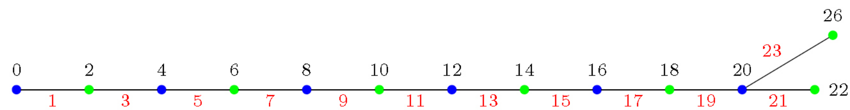

Figure 3

Y_{13} is labelled with a $2-$EVOML.

Theorem 2.2: *The star graph $K_{1,\eta}$ is $(\eta - 1)-$ even vertex odd mean graph.*

Proof: The star graph $K_{1,\eta}$ has $\alpha = \eta + 1$ vertices and $\beta = \eta$ edges, is produced by linking one vertex λ_0 to all the vertices λ_κ, where $1 \le \kappa \le \eta$. Suppose $d = \eta - 1,$ then $2\beta + 2d - 2 = 4\eta,$ we define the labeling

$$\Phi : V(K_{1,\eta}) \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 4\eta\} \text{ as follows:}$$

$$\Phi(\lambda_\kappa) = \begin{cases} 2 & \text{if } \kappa = 0; \\ 4\kappa - 4 & \text{if } \kappa \in [1, \eta + 1]. \end{cases}$$

Take into consideration the labels $4\kappa - 2,\ \kappa = 1, 2, 3, \cdots, \eta + 1$ are not included in the label of vertices.

The generated edge labels $\Phi^* : E(K_{1,\eta}) \to \{1, 3, 5, \cdots, 2\beta - 1 = 2\eta - 1\}$ are

$$\forall \ \kappa \in [1, \eta], \qquad \Phi^*(\lambda_0 \lambda_\kappa) = 2\kappa - 1.$$

Subsequently, the labels on the edges are all odd, and the proof is complete because there are no repeat numbers. ☐

Illustration: A 10-EVOML of the graph $K_{1,11}$ is offered in Figure 4.

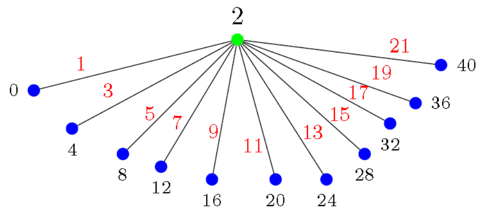

Figure 4

$K_{1,11}$ with 10-EVOML

Theorem 2.3:

(i) *For an odd positive integer* $\eta \geq 5$, *the graph* $\Delta_{\eta,\eta-2}$ *is* $2-$ *even vertex odd mean graph.*

(ii) *For an even positive integer* $\eta \geq 4$, *the graph* $\Delta_{\eta,\eta-2}$ *is an even vertex odd mean graph.*

Proof: From a path P_η, the graph $\Delta_{\eta,\eta-2}$ gotten by connecting precisely one pendant edge to every inner vertex of P_η. Its vertex set $V(\Delta_{\eta,\eta-2})$ = $\{\lambda_\kappa, \omega_\tau,\ \ \kappa \in [1,\eta],\ \ \tau \in [1,\eta-2]\}$ and $E(\Delta_{\eta,\eta-2})$ = $\{\lambda_\kappa \lambda_{\kappa+1},\ \ \omega_\tau \lambda_{\tau+1},\ \kappa \in [1,\eta-1],\ \ \tau \in [1,\eta-2]\}$, then $\alpha = 2\eta-2$ and $\beta = 2\eta-3$, see Figure 5.

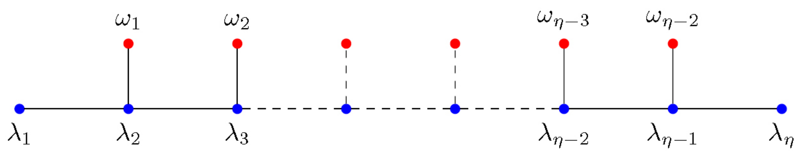

Figure 5

A standard labeling of the graph $\Delta_{\eta,\eta-2}$.

Case (1) Assume *n* is an odd number, $\Delta_{\eta,\eta-2}$ is a bipartite graph but it is not even vertex odd mean graph. Let $d=2$, then $2\beta+2d-2 = 4\eta-4$. It is possible to define the labelling $\Phi : V(\Delta_{\eta,\eta-2}) \to \{0,2,4,\cdots,\ 2\beta+2 = 4\eta-4\}$ as listed below:

$$\Phi(\omega_\tau) = \begin{cases} 4\tau & \begin{array}{ll} \text{for } \tau \equiv 1 \bmod 2,\ ,\tau \in [1,\eta-3], & \text{and } \eta \text{ is even};\\ \text{or } \tau \equiv 1 \bmod 2,\ \ \tau \in [1,\eta-2], & \text{and } \eta \text{ is odd}; \end{array} \\ 4\tau-2 & \begin{array}{ll} \text{for } \tau \equiv 0 \bmod 2,\ \ \tau \in [2,\eta-2], & \text{and } \eta \text{ is even};\\ \text{or } \tau \equiv 0 \bmod 2,\ \ \tau \in [2,\eta-3], & \text{and } \eta \text{ is odd}, \end{array} \end{cases}$$

$$\Phi(\lambda_\kappa)=\begin{cases} 4\kappa-4 & \text{for } \kappa\equiv 1 \bmod 2,\ \kappa\in[1,\eta-1], \text{ and } \eta \text{ is even;} \\ & \text{or } \kappa\equiv 1 \bmod 2,\ \kappa\in[1,\eta-2], \text{ and } \eta \text{ is odd;} \\ 4\kappa-6 & \text{for } \kappa\equiv 0 \bmod 2,\ \kappa\in[2,\eta], \text{ and } \eta \text{ is even;} \\ & \text{or } \kappa\equiv 0 \bmod 2,\ \kappa\in[2,\eta-1], \text{ and } \eta \text{ is odd.} \end{cases}$$

So the induced edge labels $\Phi^*: E(\Delta_{\eta,\eta-2}) \to \{1,3,5,\cdots,2\beta-1 = 4\eta-7\}$ will be,

$$\forall \quad \kappa\in[1,\eta-1], \quad \Phi^*(\lambda_\kappa\lambda_{\kappa+1})=4\kappa-3,$$

$$\forall \quad \kappa\in[2,\eta-1], \quad \Phi^*(\lambda_\kappa\omega_{\kappa-1})=4\kappa-5.$$

Case (2) When n is an even number, the labeling $\Phi: V(\Delta_{\eta,\eta-2}) \to \{0,2,4,\cdots,2\beta=4\eta-6\}$ can be define as in the above case but with $d=1$

There is no repetition on the labels, and all of the labels on the edges are odd. Because of this, the graph $\Delta_{\eta,\eta-2}$ is $2-$ even vertex odd mean graph when η is an odd number and an even vertex odd mean graph when η is an even number. □

Illustration: The graphs $\Delta_{10,8}$ with an EVOML and $\Delta_{11,9}$ with $2-$EVOML are displayed in Figure 6.

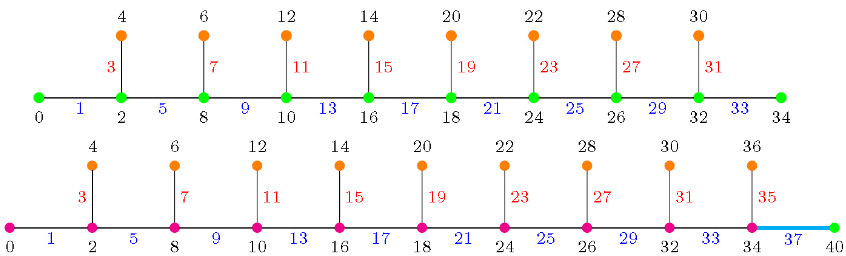

Figure 6

$\Delta_{10,8}$ with an EVOML and $\Delta_{11,9}$ with a $2-$EVOML.

Definition 2.4: [20] The corona $\Gamma \odot \Upsilon$ of two graphs Γ (with α vertices and β edges) and Υ (with ζ vertices and η edges) is characterized as the graph gotten by taking one copy of Γ and α copies of Υ, and then matching the $i-$th vertex of Γ with an edge to each vertex in the $i-$th copy of Υ. The corona's definition implies that $\Gamma \odot \Upsilon$ has $\alpha(1+\zeta)$ vertices and $\beta+\alpha(\eta+\zeta)$ edges.

Theorem 2.5:

(1) *When* ξ *is an even number,* $P_\xi \odot \overline{K_\eta}$ *is an even vertex odd mean graph.*

(2) *When* ξ *is n odd number,* $P_\xi \odot \overline{K_\eta}$ *is an* $\eta-$ *even vertex odd mean graph.*

Proof: In the graph $P_\xi \odot \overline{K_\eta}$ the cardinality of the vertices is $\alpha = \xi(\eta+1)$ and cardinality of the edges is $\beta = \xi(1+\eta)-1$. It is recommended to include vertex and edge indications exactly like the ones in Figure 7.

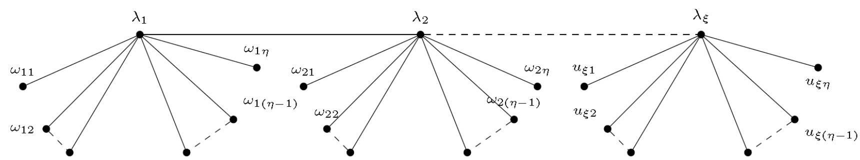

Figure 7

$P_\xi \odot \overline{K_\eta}$ **with ordinary labeling**

Case (1) When *m* is an even number.

The injective vertex labeling function $\Phi : V(P_\xi \odot \overline{K_\eta}) \to \{0, 2, 4, \cdots, 2\beta = 2\xi(1+\eta)-2\}$ is described as follows:

$$\Phi(\lambda_\kappa) = \begin{cases} 2\kappa\eta - 2\eta + 2\kappa - 2 & \text{if } \kappa \text{ is odd}, \quad \kappa \in [1, \xi-1]; \\ 2\kappa\eta + 2\kappa - 2 & \text{if } \kappa \text{ is even}, \quad \kappa \in [2, \xi], \end{cases}$$

$$\Phi(\omega_{\kappa\tau}) = \begin{cases} 4\tau - 2 & \text{if } \kappa = 1, \quad \tau \in [1, \eta]; \\ \Phi(\lambda_{\kappa-1}) + 4\tau & \text{if } \tau \in [1, \eta] \quad \kappa \in [2, \xi]. \end{cases}$$

Following that, the induced edge labels $\Phi^* : E(P_\xi \odot \overline{K_\eta}) \to \{1, 3, 5, \cdots, 2\beta - 1 = 2\xi(1+\eta)-3\}$ are

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = 2\kappa\,\eta + 2\,\kappa - 1\,, \quad \kappa \in [1, \xi - 1],$$

$$\Phi^*(\lambda_\kappa \omega_{\kappa\tau}) = 2\kappa\,\eta + 2\kappa - 2\eta - 3 + 2\tau\,, \quad \kappa \in [1, \xi\,], \quad \tau \in [1, \eta\,].$$

Case (2) If ξ **is an odd number**. Suppose that $d = \eta$ then $2\beta + 2d - 2 = 2\xi(1+\eta) + 2\eta - 4$.

The labeling function $\Phi : V(P_\xi \odot \overline{K_\eta}) \to \{0, 2, 4, \cdots, 2\xi(1+\eta) + 2\eta - 4\}$ should be defined as listed below:

$$\Phi(\lambda_\kappa) = \begin{cases} 2\kappa\eta + 2\kappa - 2 - 2\eta & \text{if } \kappa \in [1, \xi], \quad \kappa \text{ is odd}; \\ 2\kappa\eta + 2\kappa - 2 & \text{if } \kappa \in [2, \xi - 1] \quad \kappa \text{ is even}, \end{cases}$$

$$\Phi(\omega_{\kappa\tau}) = \begin{cases} 4\tau - 2 & \text{if} \quad \kappa = 1, \qquad \tau \in [1,\eta]; \\ \Phi(\lambda_{\kappa-1}) + 4\tau & \text{if} \quad \tau \in [1,\eta], \quad \kappa \in [2,\xi]. \end{cases}$$

Due to the stated labelling system, the created edge labels

$\Phi^* : E(P_\xi \odot \overline{K_\eta}) \to \{1,3,5,\cdots,2\beta - 1 = 2\xi(1+\eta) - 3\}$ are

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = 2\kappa\eta + 2\kappa - 1, \quad \kappa \in [1, \xi - 1],$$

$$\Phi^*(\lambda_\kappa \omega_{\kappa\tau}) = 2\kappa\eta + 2\kappa - 2\eta - 3 + 2\tau, \quad \kappa \in [1,\xi], \quad \tau \in [1,\eta].$$

It is obvious that all the edge label is different from others and odd. □

Illustration: The graphs $P_3 \odot \overline{K_6}$ with $6-$EVOML and $P_4 \odot \overline{K_6}$ with an EVOML are depicted in Figure 8.

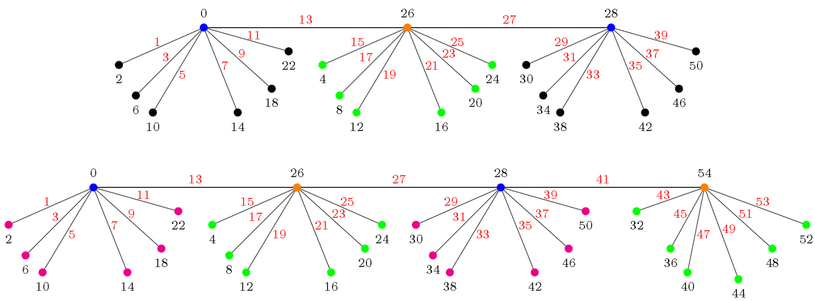

Figure 8

The graphs $P_3 \odot \overline{K_6}$ with $6-$EVOML and $P_4 \odot \overline{K_6}$ with an EVOML

3. d-even vertex odd mean labeling for various cycle-related graphs

Theorem 3.1: *Any odd cycle C_η is not $d-$ even vertex odd mean graph for all d.*

Proof: Let $\{\lambda_1, \lambda_2, \cdots, \lambda_\eta\}$ be the vertices of an odd cycle C_η, η is odd. To get odd edge labels, let the vertices $A = \{\lambda_1, \lambda_3, \lambda_5, \cdots, \lambda_{\eta-2}\}$ labeled by $0 \bmod 4$ and the vertices $B = \{\lambda_2, \lambda_4, \lambda_6, \cdots, \lambda_{\eta-1}\}$ labeled by $2 \bmod 4$. If the final vertex λ_η labeled by any even number $0 \bmod 4$, or $2 \bmod 4$, then the labeling of the edges $\lambda_{\eta-1}\lambda_\eta$ or $\lambda_\eta \lambda_0$ will be even, which complete the proof.

□

Lemma 3.2: *If any graph* Γ *induces an odd cycle* C_η, *as a subgraph, then* Γ *is not a* $d-$*even vertex odd mean graph for all d.*

Proof: Directly from Theorem 3.1. □

Theorem 3.3: *The cycle* C_η *is a similar* $2-$ *even vertex odd mean graph if and only if* $\eta \equiv 2 \bmod 4$.

Proof: Let $V(C_\eta)=\{\lambda_1,\lambda_2,\cdots,\lambda_\eta\}$ and the edges of C_η are $e_\kappa=(\lambda_\kappa\lambda_{\kappa+1})$, $1\le\kappa\le\eta$, where κ is cyclicly module η. In [15] they proved that when $n\equiv 0 \bmod 4$, C_η is an even vertex odd mean graph.

For $\eta\equiv 2\bmod 4$, suppose $d=2$, the mapping $\Phi: V(C_\eta)\to\{0,2,4,\cdots,2\beta+2d-2 = 2\eta+2\}$ can be defined as follows:

$$\Phi(\lambda_\kappa)=\begin{cases}2\kappa-2 & \text{if} \quad \kappa\in\left[1,\frac{\eta}{2}+1\right];\\ 2\kappa+2 & \text{if} \quad k\in\left[\frac{\eta}{2}+2,\eta\right].\end{cases}$$

There are two labels that are not included in the label of vertices: $\eta+2$ and $\eta+4$.

The generated edge labels are as follows $\Phi^*: E(C_\eta)\to\{1,3,5,\cdots,2\beta+2d-3 = 2\eta+1\}$

$$\Phi^*(\lambda_1\lambda_\eta)=\eta+1,$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1})=\begin{cases}2\kappa-1 & \text{if} \quad k\in\left[1,\frac{\eta}{2}\right];\\ \eta+3 & \text{if} \quad k=\frac{\eta}{2}+1;\\ 2\kappa+3 & \text{if} \quad k\in\left[\frac{\eta}{2}+2,\ \eta-1\right].\end{cases}$$

Once more, the label $\eta+5$ don't include in the label of edges. Each edge label are all odd and distinctive. As a result, the cycle graph C_η, when $\eta\equiv 2\bmod 4$, is a similar $2-$ even vertex odd mean graph. □

Illustration: In Figure 9, we present a similar $2-$EVOML of C_{10}.

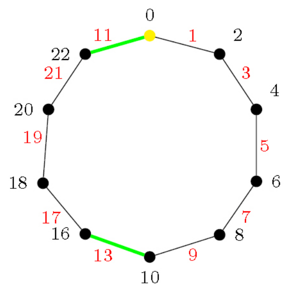

Figure 9

A similar $2-$EVOML of C_{10}.

Theorem 3.4: *When $\eta \equiv 2 \bmod 4$, and $\xi \geq 1$, the dragon graph $P_\xi(C_\eta)$ is a similar $2-$ even vertex odd mean graph.*

Proof: Let $V(C_\eta)=\{\lambda_1,\lambda_2,\cdots,\lambda_\eta\}$ and $E(C_\eta)$ = $\{e_\kappa=(\lambda_\kappa\lambda_{\kappa+1})\}$ for $1\leq\kappa\leq\eta$ of the cycle C_η where κ is cyclicly module η. Also, let $\{\omega_1,\omega_2,\cdots,\omega_\xi\}$ be the vertices of the path P_ξ and the edges are e_κ = $(\omega_\kappa\omega_{\kappa+1})$ for $1\leq\kappa\leq\xi-1$. The dragon $P_\xi(C_\eta)$ can be derived by linking one vertex λ_1 of C_n to an end vertex ω_ξ of the path. In $P_\xi(C_\eta)$, $\alpha=|V[P_\xi(C_\eta)]|$ = β = $|E[P_\xi(C_\eta)]|$ = $\eta+\xi-1$, see Figure 10.

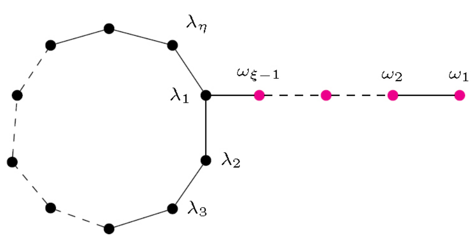

Figure 10

The dragon graph $P_\xi(C_\eta)$ with ordinary labeling

Let $d=2$, we define the mapping $\Phi: V[P_\xi(C_\eta)] \rightarrow \{0,2,4,\cdots,$ $2\beta+2d-2$ = $2\xi+2\eta\}$ as follows:

$$\Phi(\omega_\kappa)=2\kappa-2, \qquad \kappa\in[1,\xi-1],$$

$$\Phi(\lambda_\kappa)=\begin{cases} 2\xi+2\kappa-4 & \text{if} \quad \kappa\in\left[1,\frac{\eta}{2}+1\right], \\ 2(\xi+\kappa) & \text{if} \quad \kappa\in\left[\frac{\eta}{2}+2,\eta\right]. \end{cases}$$

Be aware that we aren't including the labels $\eta+2\xi-2$ and $\eta+2\xi$ in the label of vertices.

Based on the labelling pattern above, the produced edge labels $\Phi^*: E[P_\xi(C_\eta)] \to \{1,3,5,\cdots,2\beta+2d-3=2\eta+2\xi-1\}$ are

$$\Phi^*(\omega_\kappa\omega_{\kappa+1}) = 2\kappa-1, \quad \kappa\in[1,\xi-1],$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1}) = \begin{cases} 2\kappa+2\xi-3 & \text{if} \quad \kappa\in\left[1,\frac{\eta}{2}\right], \\ 2\xi-1+2\kappa & \text{if} \quad \kappa\in\left[\frac{\eta}{2}+1,\ \eta-1\right], \\ 2\xi-1+\eta & \text{if} \quad \kappa=\eta. \end{cases}$$

We leave out the labels $\eta+2\xi+3$ in the label of edges in this labelling. Once more, the proof is completed by the odd and distinct nature of each edge label. $\square$

Illustration: In Figure 11, we depict the dragon graph $P_5(C_{10})$ with $2-$ EVOML.

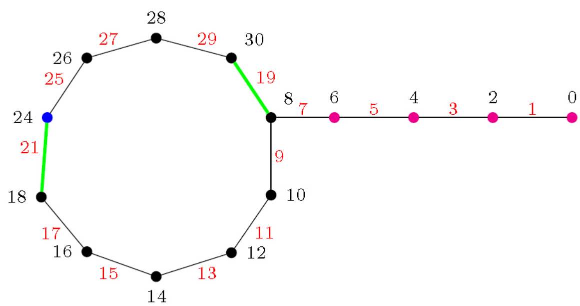

Figure 11

A similar $2-$EVOML of $P_5(C_{10})$.

Proposition 3.5: When $\eta\equiv 2 \bmod 4$, the crown graph R_η is $2-$ even vertex odd mean graph.

Proof: The cycle $C_\eta=\{\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_\eta\}$ transformed into a crown graph R_η by including η more pendent vertices $\omega_1,\omega_2,\omega_3,\cdots,\omega_\eta$ and η additional edges $(\lambda_\kappa\omega_\kappa)$, $\kappa=1,2,\cdots,\eta$ for $\eta\geq 3$ are present. The crown graph R_η has $\alpha=|V(R_\eta)|=2\eta$ and $\beta=|E(R_\eta)|=2\eta$ edges. Take a look at Figure 12, which depicts the crown graph R_η.

Let $d = 2$, we define the mapping $\Phi : V(R_\eta) \rightarrow \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 4\eta + 2\}$ as follows:

$$\Phi\ (\lambda_\kappa) = \begin{cases} 4\kappa - 2 & \text{if} \quad \kappa \equiv 0 \bmod 2, \quad \kappa \in [2, \eta], \\ 4\kappa - 4 & \text{if} \quad \kappa \equiv 1 \bmod 2, \quad \kappa \neq \frac{\eta}{2} + 1; \\ 2\eta + 6 & \text{if} \quad \kappa = \frac{\eta}{2} + 1. \end{cases}$$

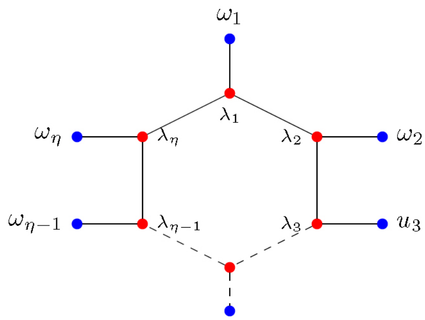

Figure 12

Standard labeling of the crown graph R_η.

$$\Phi(\omega_\kappa) = \begin{cases} 4\kappa - 2 & \text{if} \quad \kappa \equiv 1 \bmod 2, \quad \kappa \in [1, \frac{\eta}{2} - 1]; \\ 4\kappa + 6 & \text{if} \quad \kappa \equiv 1 \bmod 2, \quad \kappa \in [\frac{\eta}{2} + 2, \eta - 1]; \\ 4\kappa - 4 & \text{if} \quad \kappa \equiv 0 \bmod 2, \quad \kappa \in [2, \frac{\eta}{2} + 1]; \\ 4\kappa + 4 & \text{if} \quad \kappa \equiv 0 \bmod 2, \quad \kappa \in [\frac{\eta}{2} + 3, \eta]. \end{cases}$$

Through the previously mentioned labelling pattern, the generated edge labels $\Phi^* : E(R_\eta) \rightarrow \{1, 3, 5, \cdots, 2\beta - 1 = 4\eta - 1\}$ are as comes next.

$$\Phi^*(\lambda_1 \lambda_\eta) = 2\eta - 1,$$

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = \begin{cases} 4\kappa - 1 & \text{if} \quad \kappa \in [1, \frac{\eta}{2} - 2]; \\ 2\eta + 1 & \text{if} \quad \kappa = \frac{\eta}{2} - 1; \\ 2\kappa + 3 & \text{if} \quad \kappa \in [\frac{\eta}{2} + 2, \eta - 1], \end{cases}$$

$$\Phi^*(\lambda_\kappa \omega_\kappa) = \begin{cases} 4\kappa - 3 & \text{if} \quad \kappa \in [1, \frac{\eta}{2}]; \\ 2\eta + 3 & \text{if} \quad \kappa = \frac{\eta}{2} + 1; \\ 4\kappa + 1 & \text{if} \quad \kappa \in [\frac{\eta}{2} + 2, \eta - 1]; \\ 4\eta - 1 & \text{if} \quad \kappa = \eta. \end{cases}$$

Each edge labels is obviously distinct and odd, therefore the proof has been conclusive.

Illustration: In Figure 13, we display the crown R_{14} with 2– EVOML.

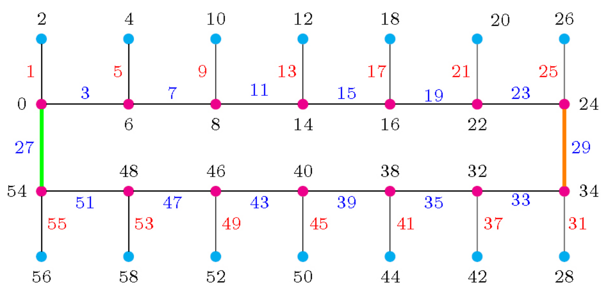

Figure 13

A 2– EVOML for R_{14}.

Definition 3.6: Let's say that C_η resembles a cycle of length η, the corona of all vertices of C_η with the exception of vertex $\{\lambda_1\}$, with the complement graph $\overline{K_{2\xi-1}}$ is indicate by $\{C_\eta - \{\lambda_1\}\} \odot \overline{K_{2\xi-1}}$, in this graph $\alpha = |V[\{C_\eta - \{\lambda_1\}\} \odot \overline{K_{2\xi-1}}]| = \beta = |E[\{C_\eta - \{\lambda_1\}\} \odot \overline{K_{2\xi-1}}]| = 2\xi(\eta - 1) + 1$.

Theorem 3.7: *On the assumption that η is an even number, the graph $\{C_\eta - \{\lambda_1\}\} \odot \overline{K_3}$ is a similar $(4\eta - 11)-$ even vertex odd mean graph*

Proof: Within the graph $\{C_\eta - \{\lambda_1\}\} \odot \overline{K_3}$, $\alpha = |V[\{C_\eta - \{\lambda_1\}\} \odot \overline{K_3}]| = \beta = |E[\{C_\eta - \{\lambda_1\} \odot \overline{K_3}]| = 4\eta - 3$. Assume that Figure 14 includes the vertex and edge symbols.

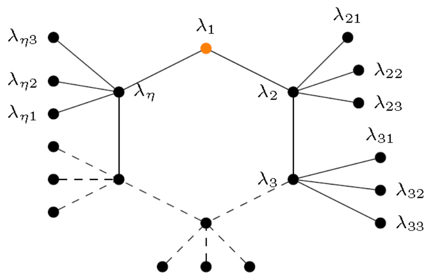

Figure 14

$\{C_\eta - \{\lambda_1\}\} \odot \overline{K_3}$

Let $d = (4n-11)$ the labeling $\Phi : V[\{C_\eta - \{\lambda_1\}\} \odot \overline{K_3}] \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 16\eta - 30\}$ can be defined as follows:

$$\Phi\ (\lambda_\kappa) = \begin{cases} 8\kappa - 8 & \text{if} \quad \kappa \equiv 1 \bmod 2, \quad \kappa \in [1, \eta - 1]; \\ 8\kappa - 14 & \text{if} \quad \kappa \equiv 0 \bmod 2, \quad \kappa \in [2, \eta - 2], \end{cases}$$

$$\Phi\ (\lambda_n) = 16\eta - 30,$$

$$\Phi\ (\lambda_{\kappa\tau}) = \begin{cases} 4(2\kappa + \tau - 4) & \text{if} \quad \kappa \equiv 0 \bmod 2, \quad \kappa \in [2, \eta], \quad \tau \in [1, 3]; \\ 2(4\kappa + 2\tau - 11) & \text{if} \quad \kappa \equiv 1 \bmod 2 \quad \text{and } \kappa \in [3, \eta - 1], \quad \tau \in [1, 3]. \end{cases}$$

Therefore, $\Phi^* : E[\{C_\eta - \{\lambda_1\}\} \odot \overline{K_3}] \to \{1, 3, 5, \cdots, 2\beta + 2d - 3 = 16\eta - 31\}$ produces the induced vertex labels as follows:

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = \begin{cases} 8\kappa - 7 & \text{if} \quad \kappa \in [1, \eta - 2]; \\ 8\eta + d - 4 & \text{if} \quad \kappa = \eta - 1; \\ 8\eta - 15 & \text{if} \quad \kappa = \eta, \end{cases}$$

$$\Phi^*(\lambda_\kappa \lambda_{\kappa\tau}) = 8\kappa + 2\tau - 15 \quad \text{if} \quad \kappa \in [1, \eta\ ], \ \tau \in [1, 3].$$

The proof has been done since every pendant edge and every label on the cycle edges take a different odd number.

Illustration: In Figure 15, we display a similar 13– even vertex odd mean labeling for $\{C_6 - \{\lambda_1\}\} \odot \overline{K_3}$.

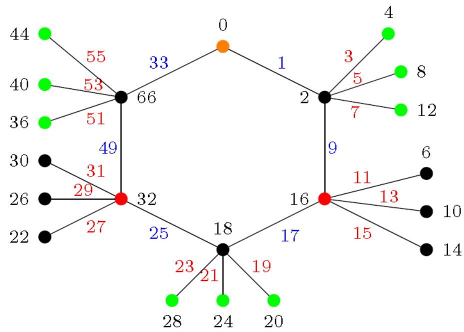

Figure 15

A similar 13– EVOML for $\{C_6 - \{\lambda_1\}\} \odot \overline{K_3}$

Theorem 3.8: *When* $\eta \equiv 2 \bmod 4$, *the prism graph* Π_η *is a similar 4– even vertex odd mean graph.*

Proof: The prism graph Π_η can be formed by linking each vertex λ_κ of a cycle $C_\eta = \{\lambda_1, \lambda_2, \cdots, \lambda_\eta\}$ to the corresponding vertex ω_κ of $C'_\eta = \{\omega_1, \omega_2, \cdots, \omega_\eta\}$, a copy of C_η, for all $\kappa \in \{1,2,3,\cdots,\eta\}$. Thus, the prism Π_η possesses $\alpha(\Pi_\eta) = 2\eta$ and its edge set is $E(\Pi_\eta) = \{\lambda_\kappa \lambda_{\kappa+1}, \lambda_\kappa \omega_\kappa, \omega_\kappa \omega_{\kappa+1}, \kappa = 1,2,\cdots,\eta\}$, then $\beta(\Pi_\eta) = 3\eta$.

Let $d = 4$, we define the mapping $\Phi : V(\Pi_\eta) \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 6\eta + 6\}$ as follows:

$$\Phi\ (\omega_\kappa) = \begin{cases} 2\kappa - 2 & \text{if} \quad \kappa \in [1, \frac{\eta}{2} + 1]; \\ 2\kappa + 2 & \text{if} \quad \kappa \in [\frac{\eta}{2} + 2, \eta], \end{cases}$$

$$\Phi\ (\lambda_\kappa) = \begin{cases} 4\eta + 8 + 2\kappa & \text{if} \quad \kappa \in [1, \eta - 1]; \\ 2\eta + 4 & \text{if} \quad \kappa = \eta. \end{cases}$$

With the labelling pattern described above, the generated edge labels are generated by

$$\Phi^* : E(\Pi_\eta) \to \{1, 3, 5, \cdots, 2\beta + 2d - 3 = 6\eta + 5\}$$

$$\Phi^*(\omega_\kappa \omega_{\kappa+1}) = \begin{cases} 2\kappa - 1 & \text{if} \quad \kappa \in [1, \frac{\eta}{2}]; \\ \eta + 3 & \text{if} \quad \kappa = \frac{\eta}{2} + 1; \\ 2\kappa + 3 & \text{if} \quad \kappa \in [\frac{\eta}{2} + 2, \eta - 1], \end{cases}$$

$$\Phi^*(\omega_1\omega_\eta) = \eta+1,$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1}) = \begin{cases} 4\eta+2\kappa+9 & \text{if} \quad \kappa\in[1,\eta-2]; \\ 4\eta-5 & \text{if} \quad \kappa=\eta-1, \end{cases}$$

$$\Phi^*(\lambda_1\lambda_\eta) = 3\eta+7,$$

$$\Phi^*(\lambda_\kappa\omega_\kappa) = \begin{cases} 2\kappa+2\eta+3 & \text{if} \quad \kappa\in[1,\frac{\eta}{2}+1]; \\ 2\eta+2\kappa+5 & \text{if} \quad \kappa\in[\frac{\eta}{2}+2,\eta-1]; \\ 2\eta+3 & \text{if} \quad \kappa=\eta. \end{cases}$$

It's important to keep in mind that the labels $\eta+5, 4\eta+7$, and $4\eta+9$ are not included in the label of edges. All of edge labels are evidently odd and distinctive. The proof is finished as a result.

Illustration: In Figure 16, we present a similar 4– even vertex odd mean labeling of the prism Π_{10}.

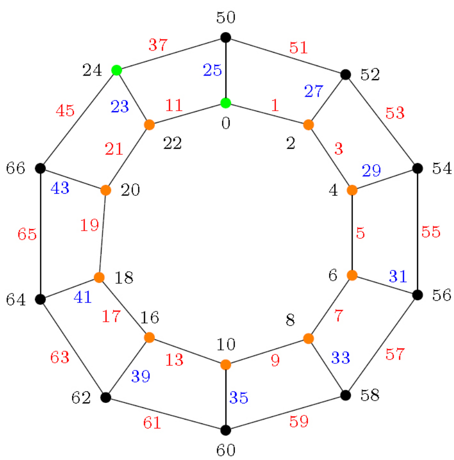

Figure 16

A similar 4– EVOML for Π_{10}

Definition 3.9: For the two graphs Γ and Υ the rooted product, denoted by $(\Gamma\Diamond\Upsilon)$, is gotten by taking $|V(\Gamma)|$ copies of Υ and then, for each vertex λ_κ of Γ, matching λ_κ with the root vertex of the κth copy of Υ.

Theorem 3.10:

(i) *When η is an odd natural, the rooted product graph $P_\eta \lozenge C_4$ is 2– even vertex odd mean graph.*

(ii) *When η is even natural , the rooted product graph $P_\eta \lozenge C_4$ is an even vertex odd mean graph.*

Proof: In the graph $P_\eta \lozenge C_4$, let $\{\omega_1, \omega_2, \cdots, \omega_\eta\}$ be the vertices of P_n, then the vertex set of the graph $P_\eta \lozenge C_4$ will be $\{\omega_\kappa,\ \lambda_{\kappa\tau},\ 1 \le \kappa \le \eta,\ \tau = 1,2,3\}$ and the edge set of the graph $P_\eta \lozenge C_4$ will be $\{\omega_\kappa \omega_{\kappa+1},\ 1 \le \kappa \le \eta - 1\} \cup \{\omega_\kappa \lambda_{\kappa 1},\ \omega_\kappa \lambda_{\kappa 3},\ \lambda_{\kappa\tau} \lambda_{\kappa(\tau\tau+1)}\ 1 \le \kappa \le \eta, \tau = 1,2\}$, then $\alpha = 4\eta$ and $\beta = 5\eta - 1$, see Figure 17.

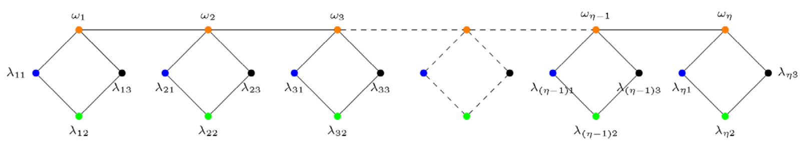

Figure 17

$P_\eta \lozenge C_4$

Case (i) if η is an odd number. Let $d = 2$ the labeling $\Phi : V[P_\eta \lozenge C_4] \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 10\eta\}$ can be defined as follows:

$$\Phi(\omega_\kappa) = \begin{cases} 10\kappa - 10 & \text{if} \quad \kappa \equiv 1 \bmod 2; \\ 10\kappa - 2 & \text{if} \quad \kappa \equiv 0 \bmod 2, \end{cases}$$

$$\Phi(\lambda_{\kappa 1}) = \begin{cases} 10\kappa - 8 & \text{if} \quad \kappa \equiv 1 \bmod 2; \\ 10\kappa - 12 & \text{if} \quad \kappa \equiv 0 \bmod 2, \end{cases}$$

$$\Phi\ (\lambda_{\kappa 2}) = 10\kappa - 6, \quad \kappa \in [1, \eta],$$

$$\Phi(\lambda_{\kappa 3}) = \begin{cases} 10\kappa & \text{if} \quad \kappa \equiv 1 \bmod 2; \\ 10\kappa - 4 & \text{if} \quad \kappa \equiv 0 \bmod 2. \end{cases}$$

Then the induced edge labels $\Phi^* : E(P_n \lozenge C_4) \to \{1, 3, 5, \cdots, 2\beta - 1 = 10\eta - 3\}$ are

$$\Phi^*(\omega_\kappa \omega_{\kappa+1}) = 10\kappa - 1, \quad \kappa \in [1, \eta - 1],$$

$$\Phi^*(\omega_\kappa \lambda_{\kappa 1}) = \begin{cases} 10\kappa - 9 & \text{if} \quad \kappa \equiv 1 \bmod 2; \\ 10\kappa - 7 & \text{if} \quad \kappa \equiv 0 \bmod 2, \end{cases}$$

$$\Phi^*(\omega_\kappa \lambda_{\kappa 3}) = \begin{cases} 10\kappa - 5 & \text{if} \quad \kappa \equiv 1 \bmod 2; \\ 10\kappa - 3 & \text{if} \quad \kappa \equiv 0 \bmod 2, \end{cases}$$

$$\Phi^*(\lambda_{\kappa 1} \lambda_{\kappa 2}) = \begin{cases} 10\kappa - 7 & \text{if} \quad \kappa \equiv 1 \bmod 2; \\ 10\kappa - 9 & \text{if} \quad \kappa \equiv 0 \bmod 2, \end{cases}$$

$$\Phi^*(\lambda_{\kappa 2} \lambda_{\kappa 3}) = \begin{cases} 10\kappa - 3 & \text{if} \quad \kappa \equiv 1 \bmod 2; \\ 10\kappa - 5 & \text{if} \quad \kappa \equiv 0 \bmod 2. \end{cases}$$

Case (ii) When η is an even number, the one to one vertex labeling function

$\Phi : V(P_\eta \Diamond C_4) \to \{0, 2, 4, \cdots, 2\beta = 10\eta - 2\}$ can be define as in the above case but with $d = 1$.

It is worth mentioning that the proof is completed by the odd and different edge labels on each of them.

Illustration: In Figure 18, we present a 2– even vertex odd mean labeling of the graph $P_5 \Diamond C_4$ and an even vertex odd mean labeling of the graph $P_6 \Diamond C_4$

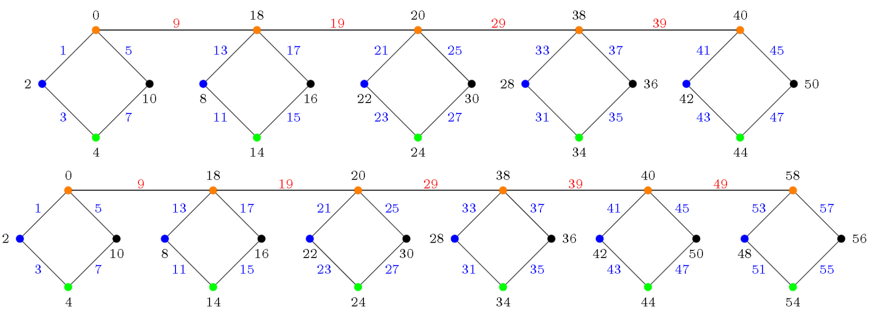

Figure 18

A 2– EVOML for $P_5 \Diamond C_4$ and an EVOML for $P_6 \Diamond C_4$

4. *d*-even vertex odd mean labeling for Toeplitz graphs

Van Dal et al. [21] explored the undirected Toeplitz graphs with respect to hamiltonicity.

Definition 4.1: If the adjacency matrix $A(\Gamma)$ of a simply undirected graph Γ of order α is Toeplitz, the graph is said to be a Toeplitz graph. A Toeplitz matrix is a $(\alpha\times\alpha)$ symmetric matrix with constant values along all diagonals that are parallel to the main diagonal, i.e. $\theta_{\kappa,\tau}=\theta_{\kappa+1,\tau+1}$ for each $\kappa,\tau=1,2,\cdots,\alpha-1.$ The labels $0,1,2,\cdots,\alpha-1$ denote the α distinct diagonals of a $(\alpha\times\alpha)$ symmetric Toeplitz adjacency matrix. In order to prevent loops in the Toeplitz graph, diagonal 0 is the main diagonal and it only contains zeros, i.e., $\theta_{\kappa\kappa}=0$ for every $\kappa=1,2,\cdots,\alpha.$ The initial row of $A(\Gamma),$ a $(0-1)-$ sequence, serves as the only identifier of a Toeplitz graph $\Gamma.$

Let $x_1,x_2,\cdots,x_k,$ where $1\le x_1<x_2<\cdots<x_k<p,$ be the diagonals containing ones. Then, $T_p\langle x_1,\cdots,x_{ki}\rangle.$ will be used to represent the relevant Toeplitz graph. That is, $T_n\langle x_1,\cdots,x_{k_i}\rangle$ is the graph with the vertex set $V(T)=\{\lambda_\kappa:\kappa=1,2,\cdots,\alpha\}$ in which two vertices λ,ω of T being connected by an edge if and only if $|\lambda-\omega|\in\{x_1,x_2,\cdots,x_k\}.$ If $x_j, j=1,2,\cdots,k,$ is a diagonal containing ones then the diagonal elements $\theta_{\kappa,x_{\tau+\kappa}},\kappa=1,2,\cdots,\alpha-x_j,$ determining the edges $\lambda_i\lambda_{x_{j+i}}$ in the Toeplitz graph.

For results regarding different properties of Toeplitz graphs such as connectivity, bipartiteness, planarity, and colourability, see {22, 23}.

Theorem 4.2

(1) *When* $\eta\equiv 0 \bmod 4,$ *Toeplitz graph* $T_\eta(1,3)$ *is 2– even vertex odd mean graph.*

(2) *When* $\eta\equiv 2 \bmod 4,$ *Toeplitz graph* $T_\eta(1,3)$ *is a similar 2– even vertex odd mean graph.*

(3) *When* $\eta\equiv 3 \bmod 4,$ *Toeplitz graph* $T_\eta(1,3)$ *is an even vertex odd mean graph.*

(4) *When* $\eta\equiv 1 \bmod 4,$ *Toeplitz graph* $T_\eta(1,3)$ *is 3– even vertex odd mean graph.*

Proof: Toeplitz graph $T_\eta(1,3)$ has $V[T_\eta(1,3)]=\{\lambda_\kappa:1\le\kappa\le\eta\}$ and edge set

$$E[T_n(1,3)]=\{\lambda_\kappa\lambda_{\kappa+1}:1\le\kappa\le\eta-1\}\cup\{\lambda_\kappa\lambda_{\kappa+3}:1\le\kappa\le\eta-3\}.$$

Case (1): In Toeplitz graph $T_\eta(1,3)$, and $\eta \equiv 0 \bmod 4$. Let $\eta = 2l+4$, $l = 0, 2, \cdots, \frac{\eta-4}{2}$, then $\alpha = \eta = 2l+4$ and $\beta = 4l+4$. Assume that $d = 2$, then $2\beta + 2d - 2 = 4\eta - 6$.

The labeling $\Phi : V[T_\eta(1,3)] \to \{0, 2, 4, \cdots, 2\beta+2 = 4\eta-6 = 8l+10\}$ can be defined as follows:

$$\Phi(\lambda_\kappa) = \begin{cases} 2\kappa-2 & \text{if} \quad 1 \le \kappa \le 3; \\ 4\kappa-6 & \text{if} \quad \kappa = 4, 6, 8, \cdots, 2l+2; \\ 4\kappa-4 & \text{if} \quad \kappa = 5, 9, 13, \cdots, 2l+1; \\ 4\kappa-8 & \text{if} \quad \kappa = 7, 11, 15, \cdots, 2l+3, \end{cases}$$

$$\Phi(\lambda_\eta) = 8l+10.$$

Using the labelling pattern mentioned above, the generated edge labels $\Phi^* : E[T_\eta(1,3)] \to \{1, 3, 5, \cdots, 2\beta-1 = 4\eta-9 = 8l+7\}$ are:

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = \begin{cases} 4\kappa-3 & \text{for} \quad \kappa = 1, 5, 9, 13, \cdots, 2l+1; \quad \text{or } \kappa = 4, 8, 12, \cdots, 2l; \\ 4\kappa-5 & \text{for} \quad \kappa = 2, 6, 10, \cdots, 2l+2; \quad \text{or } \kappa = 3, 7, 11, \cdots 2l+3; \end{cases}$$

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+3}) = \begin{cases} 4\kappa+1 & \text{for} \quad \kappa = 1, 5, 9, 13, \cdots, 2l+1; \quad \text{or } 2, 6, 10, \cdots, 2l-2; \\ 4\kappa-1 & \text{for} \quad \kappa = 4, 8, 12, \cdots, 2l; \quad \text{or } \kappa = 3, 7, 11, \cdots 2l-1, \end{cases}$$

$$\Phi^*(\lambda_{2l+3} \lambda_{2l+4}) = 8l+7.$$

Case (2) In $T_\eta(1,3)$ when $\eta \equiv 2 \bmod 4$. Let $\eta = 2l+4$ and $l = 1, 3, \cdots, \frac{\eta}{2} - 2$, then $\beta = 4k+4$. Assume that $d = 2$, then $2\beta + 2d - 2 = 4\eta - 6$.

The labeling $\Phi : V[T_\eta(1,3)] \to \{0, 2, 4, \cdots, 2\beta+2 = 4\eta-6 = 8l+10\}$ can be defined as follows:

$$\Phi(\lambda_\kappa) = \begin{cases} 2\kappa-2 & \text{if} \quad 1 \le \kappa \le 3; \\ 4\kappa-6 & \text{if} \quad 4 \le \kappa = 4, 6, \cdots, 2l+2; \\ 4\kappa-4 & \text{if} \quad \kappa = 5, 9, 13, \cdots, 2l+3; \\ 4\kappa-8 & \text{if} \quad \kappa = 7, 11, 15, \cdots, 2l+1, \end{cases}$$

$$\Phi(\lambda_n) = 8l+10.$$

Then the induced vertex labels $\Phi^* : E[T_\eta(1,3)] \to \{1, 3, 5, \cdots, 2\beta+1 = 4\eta-7 = 8l+9\}$ are:

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+4}) = \begin{cases} 4\kappa-3 & \text{for} \quad \kappa = 1,5,9,13,\cdots,2l+3; \text{ or } \kappa = 4,8,12,\cdots,2l+2; \\ 4\kappa-5 & \text{for} \quad \kappa = 2,6,10,\cdots,2l; \text{ or } \kappa = 3,7,11,\cdots 2l+1, \end{cases}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+3}) = \begin{cases} 4\kappa+1 & \text{for} \quad \kappa = 1,5,9,13,\cdots,2l+1; \text{ or } \kappa = 2,6,10,\cdots,2l; \\ 4\kappa-1 & \text{for} \quad \kappa = 4,8,12,\cdots,2l-2; \text{ or } \kappa = 3,7,11,\cdots 2l-1. \end{cases}$$

In this labeling, we do not take the label $4\eta-9$ in the label of edges.

Case (3) When $\eta \equiv 3 \bmod 4$, in $T_\eta(1,3)$, let $\eta = \alpha = 2l+3$, $k = 0,2,\cdots,\frac{\eta-3}{2}$, then $\beta = 4l+2$.

The labeling $\Phi : V[T_\eta(1,3)] \to \{0,2,4,\cdots,2\beta = 8l+4\}$ can be defined as follows:

$$\Phi\ (\lambda_\kappa) = \begin{cases} 2\kappa-2 & \text{if} \quad 1 \le \kappa \le 3; \\ 4\kappa-6 & \text{if} \quad 4 \le \kappa = 4,6,\cdots,2l+2; \\ 4\kappa-4 & \text{if} \quad \kappa = 5,9,13,\cdots,2l+1; \\ 4\kappa-8 & \text{if} \quad \kappa = 7,11,15,\cdots,2l+3. \end{cases}$$

Then the induced vertex labels $\Phi^* : E[T_\eta(1,3) = T_{2k+3}(1,3)] \to \{1,3,5,\cdots,\ 2\beta-1 = 4\eta-9 = 8l+3\}$ are:

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1}) = \begin{cases} 4\kappa-3 & \text{for} \quad \kappa = 1,5,9,13,\cdots,2l+1; \text{ or } \kappa = 4,8,12,\cdots,2l; \\ 4\kappa-5 & \text{for} \quad \kappa = 2,6,10,\cdots,2l+2; \text{ or } \kappa = 3,7,11,\cdots 2l-1; \end{cases}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+3}) = \begin{cases} 4\kappa+1 & \text{for} \quad \kappa = 1,5,9,13,\cdots,2l-3; \text{ or } \kappa = 2,6,10,\cdots,2l-2; \\ 4\kappa-1 & \text{for} \quad \kappa = 4,8,12,\cdots,2l; \text{ or } \kappa = 3,7,11,\cdots 2l-1. \end{cases}$$

Case (4) When $\eta \equiv 1 \bmod 4$, in $T_\eta(1,3)$, let $\eta = \alpha = 2l+3$, $l = 1,3,5,\cdots,\frac{\eta-3}{2}$, then $\beta = 4l+2$.

Let $d = 3$, the labeling $\Phi : V[T_\eta(1,3)] \to \{0,2,4,\ \cdots,\ 2\beta+2d-2 = 4\eta-4 = 8l+8\}$ can be defined as follows:

$$\Phi\ (\lambda_\kappa) = \begin{cases} 2\kappa-2 & \text{if} \quad 1 \le \kappa \le 3; \\ 4\kappa-6 & \text{if} \quad \kappa = 4,6,8\cdots,2l+2; \\ 4\kappa-4 & \text{if} \quad \kappa = 5,9,13,\cdots,2l+3; \\ 4\kappa-8 & \text{if} \quad \kappa = 7,11,15,\cdots,2l+1. \end{cases}$$

In this labelling, the labels $4\eta-8$ and $4\eta-6$ are absent in the label of vertices.

Given the labelling pattern mentioned previously, the induced vertex labels $\Phi^* : E[T_\eta(1,3)] \to \{1,3,5,\ \cdots,\ 2\beta+1 = 4\eta-7 = 8l+5\}$ are:

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = \begin{cases} 4\kappa - 3 & \text{for} \quad \kappa = 1,5,9,13,\cdots,2l-1; \text{ or } \kappa = 4,8,12,\cdots,2l+2; \\ 4\kappa - 5 & \text{for} \quad \kappa = 2,6,10,\cdots,2l; \text{ or } \kappa = 3,7,11,\cdots 2l+1; \end{cases}$$

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+3}) = \begin{cases} 4\kappa + 1 & \text{for} \quad \kappa = 1,5,9,13,\cdots,2l-1; \text{ or } \kappa = 2,6,10,\cdots,2l; \\ 4\kappa - 1 & \text{for} \quad \kappa = 4,8,12,\cdots,2l-2; \text{ or } \kappa = 3,7,11,\cdots 2l-3. \end{cases}$$

In this labeling, we do not take the label $4\eta - 9$ in the label of edges.

There is no repetition on the labels, and all of the edges' labels are obviously odd. The graph $T_\eta(1,3)$ accepts an even vertex odd mean labeling as a result.

Illustration: The graphs $T_{12}(1,3)$ with a 2– EVOML, $T_{13}(1,3)$ with a 3– EVOML, $T_{14}(1,3)$ with a similar 2– EVOML, and $T_{15}(1,3)$ with an EVOML are each depicted in Figure 19, respectively.

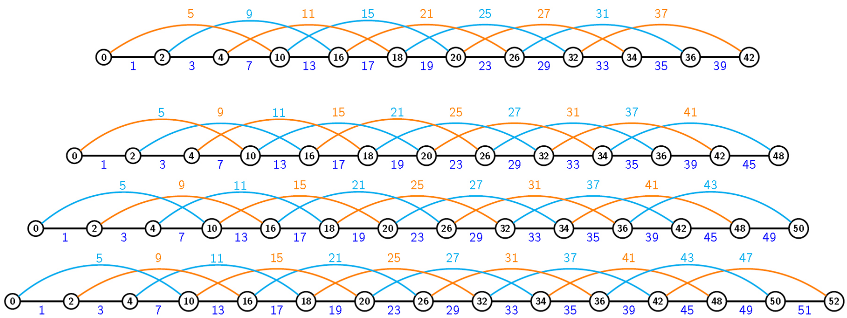

Figure 19

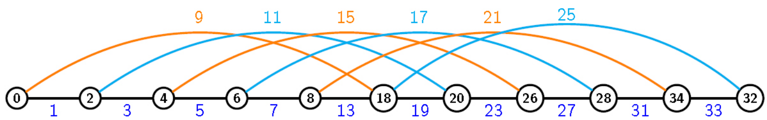

Figure 20

Theorem 4.3: *In Toeplitz graph* $T_\eta(1,5), \eta > 10,$ *we have the following cases*

(1) *Toeplitz graph* $T_{11}(1,5)$ *is a similar 2– even vertex odd mean graph while Toeplitz graph* $T_{12}(1,5)$ *is 2– even vertex odd mean graph.*

(2) *Toeplitz graph* $T_\eta(1,5)$ *is a similar 5– even vertex odd mean graph, if* η *is an odd integer number.*

(3) *Toeplitz graph* $T_\eta(1,5)$ *is a similar 4– even vertex odd mean graph, when* $\eta \equiv 2 \bmod 4$.

(4) *Toeplitz graph* $T_\eta(1,5)$ *is a similar 6– even vertex odd mean graph, when* $\eta \equiv 0 \bmod 4$.

Proof: Toeplitz graph $T_\eta(1,5)$ has vertex set $V[T_\eta(1,5)] = \{\lambda_\kappa : 1 \le \kappa \le \eta\}$ and edge set $E[T_\eta(1,5)] = \{\lambda_\kappa \lambda_{\kappa+1} : 1 \le \kappa \le \eta - 1\} \cup \{\lambda_\kappa \lambda_{\kappa+5} : 1 \le \kappa \le \eta - 5\}$, so the numbers of vertices and edges are $\alpha = \eta$ and $\beta = 2\eta - 6$.

Case (1) When $\eta = 11$, let $d = 2$, the labeling $\Phi : V[T_{11}(1,5)] \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 34\}$ and the induced vertex labels $\Phi^* : E[T_{11}(1,5)] \to \{1, 3, 5, \cdots, 2\beta + 2d - 3 = 33\}$ is shown in Figure 20.

Also, when $\eta = 12$, let $d = 2$, the labeling $\Phi : V[T_{12}(1,5)] \to \{0, 2, 4, \cdots, 2q + 2d - 2 = 38\}$ and the induced vertex labels $\Phi^* : E[T_{12}(1,5)] \to \{1, 3, 5, \cdots, 2q - 1 = 35\}$: is shown in Figure 21.

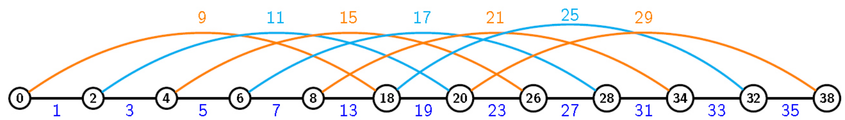

Figure 21

Case (2) **If** η **is odd number**. Let $d = 5$, the one to one vertex labeling function $\Phi : V[T_\eta(1,5)] \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 4\eta - 4\}$ can be defined as follows:

$$\Phi(\lambda_\kappa) = \begin{cases} 2\kappa - 2 & \text{if} \quad 1 \le \kappa \le 5; \\ 4\kappa - 6 & \text{if} \quad \kappa \equiv 2 \bmod 4,\ 6 \le \kappa; \\ 4\kappa - 2 & \text{if} \quad \kappa \equiv 0 \bmod 4, 16 \le \kappa; \\ 4\kappa - 8 & \text{if} \quad \kappa = 7, 9; \\ 26 & \text{if} \quad \kappa = 8; \\ 32 & \text{if} \quad \kappa = 11; \\ 38 & \text{if} \quad \kappa = 12; \\ 4\kappa - 4 & \text{if} \quad \kappa \equiv 1 \bmod 2,\ 13 \le \kappa \le \eta. \end{cases}$$

Assuming the above labelling scheme, the induced vertex labels $\Phi^* : E[T_\eta(1,5)] \to \{1, 3, 5, \cdots, 2\beta + 2d - 3 = 4\eta - 5\}$ are:

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1})=\begin{cases}2\kappa-1 & \text{if} \quad 1\le\kappa\le 4;\\ 4\kappa-7 & \text{if} \quad \kappa=5,10;\\ 4\kappa-5 & \text{if} \quad \kappa=6,7,8,9;\\ 35 & \text{if} \quad \kappa=11;\\ 43 & \text{if} \quad \kappa=12;\\ 4\kappa-3 & \begin{matrix}\text{for} & 13\le\kappa, & \kappa\equiv 1 \bmod 4;\\ \text{or} & 14\le\kappa, & \kappa\equiv 2 \bmod 4;\end{matrix}\\ 4\kappa-1 & \begin{matrix}\text{for} & 15\le\kappa, & \kappa\equiv 3 \bmod 4;\\ \text{or} & 16\le\kappa, & \kappa\equiv 0 \bmod 4.\end{matrix}\end{cases}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+5})=\begin{cases}9 & \text{if} \quad \kappa=1;\\ 3+4\kappa & \begin{matrix}\text{for} & \kappa=2,3;\\ \text{or} & \kappa=9,11,12;\end{matrix}\\ 1+4\kappa & \text{if} \quad 4\le\kappa\le 7;\\ 5+4\kappa & \text{if} \quad \kappa=8,10;\\ 5+4\kappa & \begin{matrix}\text{for} & 13\le\kappa, & \kappa\equiv 1 \bmod 4;\\ \text{or} & 14\le\kappa, & \kappa\equiv 2 \bmod 4;\end{matrix}\\ 7+4\kappa & \begin{matrix}\text{for} & 15\le\kappa, & \kappa\equiv 3 \bmod 4;\\ \text{or} & 16\le\kappa, & \kappa\equiv 0 \bmod 4.\end{matrix}\end{cases}$$

Case (3) When $\eta\equiv 2 \bmod 4$. Let $d=4$, the labeling $\Phi: V[T_\eta(1,5)]\to \{0,2,4,\cdots,2\beta+2d-2=4\eta-6\}$ can be defined as follows:

$$\Phi\ (\lambda_\kappa)=\begin{cases}2\kappa-2 & \text{if} \quad 1\le\kappa\le 5;\\ 4\kappa-8 & \text{if} \quad \kappa=7,9;\\ 26 & \text{if} \quad \kappa=8;\\ 32 & \text{if} \quad \kappa=11;\\ 38 & \text{if} \quad \kappa=12;\\ 4\kappa-6 & \text{if} \quad \kappa\equiv 2 \bmod 4,\ 6\le\kappa\le\eta;\\ 4\kappa-2 & \text{if} \quad \kappa\equiv 0 \bmod 4, 16\le\kappa\le\eta-2;\\ 4\kappa-4 & \text{if} \quad 13\le\kappa\le\eta-1.\end{cases}$$

The labelling pattern discussed earlier allows the induced edge labels $\Phi^*: E[T_\eta(1,5)]\to \{1,3,5,\cdots,2\beta+2d-3=4\eta-7\}$ to be:

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1})=\begin{cases}2\kappa-1 & \text{if} \quad 1\le\kappa\le 4;\\ 4\kappa-7 & \text{if} \quad \kappa=5,10;\\ 4\kappa-5 & \text{if} \quad \kappa=6,7,8,9;\\ 35 & \text{if} \quad \kappa=11;\\ 43 & \text{if} \quad \kappa=12;\\ 4\kappa-3 & \begin{array}{l}\text{for} \quad 3\le\kappa\le\eta-1,\ \ \kappa\equiv 1 \bmod 4;\\ \text{or} \quad 14\le\kappa\le\eta-4,\ \ \kappa\equiv 2 \bmod 4;\end{array}\\ 4\kappa-1 & \begin{array}{l}\text{for} \quad 15\le\kappa\le\eta-3,\ \ \kappa\equiv 3 \bmod 4;\\ \text{or} \quad 16\le\kappa\le\eta-2,\ \ \kappa\equiv 0 \bmod 4,\end{array}\end{cases}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+5})=\begin{cases}9 & \text{if} \quad \kappa=1;\\ 3+4\kappa & \begin{array}{l}\text{for} \quad \kappa=2,3;\\ \text{or} \quad \kappa=9,11,12;\end{array}\\ 1+4\kappa & \text{if} \quad 4\le\kappa\le 7;\\ 5+4\kappa & \text{if} \quad \kappa=8,10;\\ 5+4\kappa & \begin{array}{l}\text{for} \quad 13\le\kappa\le\eta-5,\ \ \kappa\equiv 1 \bmod 4;\\ \text{or} \quad 14\le\kappa\le\eta-8,\ \ \kappa\equiv 2 \bmod 4;\end{array}\\ 7+4\kappa & \begin{array}{l}\text{for} \quad 15\le\kappa\le\eta-7,\ \ \kappa\equiv 3 \bmod 4;\\ \text{or} \quad 16\le\kappa\le\eta-6,\ \ \kappa\equiv 0 \bmod 4.\end{array}\end{cases}$$

Case (4) When $\eta\equiv 0 \bmod 4$. Let d = 6, the labeling $\Phi:V[T_n(1,5)]\to \{0,2,4,\cdots,2\beta+2d-2=4\eta-2\}$ can be defined as follows:

$$\Phi(\lambda_\kappa)=\begin{cases}2\kappa-2 & \text{if} \quad 1\le\kappa\le 5;\\ 4\kappa-8 & \text{if} \quad \kappa=7,9;\\ 26 & \text{if} \quad \kappa=8;\\ 32 & \text{if} \quad \kappa=11;\\ 38 & \text{if} \quad \kappa=12;\\ 4\kappa-6 & \text{if} \quad \kappa\equiv 2 \bmod 4,\ 6\le\kappa\le\eta-2;\\ 4\kappa-2 & \text{if} \quad \kappa\equiv 0 \bmod 4, 16\le\kappa\le\eta;\\ 4\kappa-4 & \text{if} \quad \kappa\equiv 1 \bmod 2,\ 13\le\kappa\le\eta-1.\end{cases}$$

The edge labels $\Phi^*:E[T_\eta(1,5)]\to\{1,3,5,\cdots,2\beta+2d-3=4\eta-5\}$ that are induced based on the labelling pattern mentioned above are

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1})=\begin{cases}2\kappa-1 & \text{if} \quad 1\le\kappa\le 4;\\ 4\kappa-7 & \text{if} \quad \kappa=5,10;\\ 4\kappa-5 & \text{if} \quad \kappa=6,7,8,9;\\ 35 & \text{if} \quad \kappa=11;\\ 43 & \text{if} \quad \kappa=12;\\ 4\kappa-3 & \text{for} \quad 13\le\kappa\le\eta-3,\ \ \kappa\equiv 1 \bmod 4;\\ & \text{or} \quad 14\le\kappa\le\eta-2,\ \ \kappa\equiv 2 \bmod 4;\\ 4\kappa-1 & \text{for} \quad 15\le\kappa\le\eta-1,\ \ \kappa\equiv 3 \bmod 4;\\ & \text{or} \quad 16\le\kappa\le\eta-4,\ \ \kappa\equiv 0 \bmod 4,\end{cases}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+5})=\begin{cases}9 & \text{if} \quad \kappa=1;\\ 3+4\kappa & \text{for} \quad \kappa=2,3;\\ & \text{or} \quad \kappa=9,11,12;\\ 1+4\kappa & \text{if} \quad 4\le\kappa\le 7;\\ 5+4\kappa & \text{if} \quad \kappa=8,10;\\ 5+4\kappa & \text{for} \quad 13\le\kappa\le\eta-7,\ \ \kappa\equiv 1 \bmod 4;\\ & \text{or} \quad 14\le\kappa\le\eta-6,\ \ \kappa\equiv 2 \bmod 4;\\ 7+4\kappa & \text{for} \quad 15\le\kappa\le\eta-5,\ \ \kappa\equiv 3 \bmod 4;\\ & \text{or} \quad 16\le\kappa\le\eta-8,\ \ \kappa\equiv 0 \bmod 4.\end{cases}$$

The proof has been verified because it is obvious that the labels of the edges are all odd and there is no duplication in the labels of the graph $T_\eta(1,5)$, $\eta>10$.

Illustration: A similar 5– EVOML for $T_{17}(1,5)$, a similar 4– EVOML, for $T_{18}(1,5)$, and a similar 6– EVOML, for $T_{20}(1,5)$, are demonstrated in Figure 22.

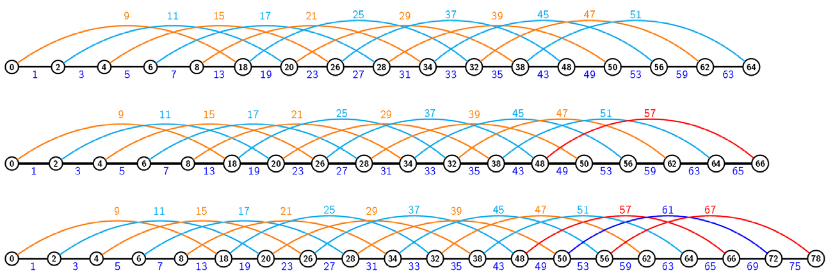

Figure 22

Theorem 4.4: *The following cases arise in the Toeplitz graph* $T_\eta(1,3,5)$, $\eta > 10$

(1) *When* $\eta \equiv 0 \mod 6$, *Toeplitz graph* $T_\eta(1,3,5)$ *is a similar 7– even vertex odd mean graph.*

(2) *When* $\eta \equiv 2 \mod 6$, *Toeplitz graph* $T_\eta(1,3,5)$ *is a similar 11– even vertex odd mean graph.*

(3) *When* $\eta \equiv 4 \mod 6$, *or* η *is an odd number, Toeplitz graph* $T_\eta(1,3,5)$ *is a similar 9– even vertex odd mean graph.*

Proof: The vertex set of Toeplitz graph $T_\eta(1,3,5)$ is $V[T_\eta(1,3,5)] = \{\lambda_\kappa : 1 \le \kappa \le \eta\}$ while the edge set is $E[T_\eta(1,3,5)] = \{\lambda_\kappa \lambda_{\kappa+1} : 1 \le \kappa \le \eta - 1\} \cup \{\lambda_\kappa \lambda_{\kappa+3} : 1 \le \kappa \le \eta - 3\} \cup \{\lambda_\kappa \lambda_{\kappa+5} : 1 \le \kappa \le \eta - 5\}$, so $\alpha = \eta$ and $\beta = 3\eta - 9$.

Case (1) When $\eta \equiv 0 \mod 6$. Let $d = 7$, the labeling $\Phi : V[T_\eta(1,3,5)] \to \{0, 2, 4, \cdots, 2\beta + 2d - 2 = 6\eta - 6\}$ can be defined as follows:

$$\Phi(\lambda_\kappa) = \begin{cases} 2\kappa - 2 & \text{if} \quad 1 \le \kappa \le 3; \\ 6\kappa - 14 & \text{if} \quad \kappa = 4, 5; \\ 6\kappa - 6 & \text{if} \quad \kappa \equiv 0 \mod 6, 6 \le \kappa \le \eta; \\ 6\kappa + 2 & \text{if} \quad \kappa \equiv 2 \mod 6, 8 \le \kappa \le \eta - 4; \\ 6\kappa - 2 & \text{for} \quad 10 \le \kappa \le \eta - 2, \text{ if } \kappa \equiv 4 \mod 6; \\ & \text{or} \quad 7 \le \kappa \le \eta - 1, \text{ if } \kappa \text{ is odd}, \end{cases}$$

The induced vertex labels $\Phi^* : E[T_\eta(1,3,5)] \to \{1, 3, 5, \cdots, 2\beta + 2d - 3 = 6\eta - 7\}$ are given by

$$\Phi^*(\lambda_1\lambda_2) = 1, \ \Phi^*(\lambda_2\lambda_3) = 3, \Phi^*(\lambda_3\lambda_4) = 7, \Phi^*(\lambda_4\lambda_5) = 13, \ \Phi^*(\lambda_5\lambda_6) = 23, \tag{4.1}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1}) = \begin{cases} 6\kappa - 1 & \text{for } 6 \le \kappa \le \eta - 6, \ \kappa \equiv 0 \mod 6; \\ & \text{or } 11 \le \kappa \le \eta - 1, \ \kappa \equiv 5 \mod 6; \\ 6\kappa + 1 & \text{for } 9 \le \kappa \le \eta - 3, \ \kappa \equiv 3 \mod 6; \\ & \text{or } 10 \le \kappa \le \eta - 2, \ \kappa \equiv 4 \mod 6; \\ 6\kappa + 3 & \text{for } 7 \le \kappa \le \eta - 5, \ \kappa \equiv 1 \mod 6; \\ & \text{or } 8 \le \kappa \le \eta - 4, \ \kappa \equiv 2 \mod 6, \end{cases}$$

$$\Phi^*(\lambda_1\lambda_4) = 5, \ \Phi^*(\lambda_2\lambda_5) = 9, \Phi^*(\lambda_3\lambda_6) = 17, \Phi^*(\lambda_4\lambda_7) = 25, \ \Phi^*(\lambda_5\lambda_8) = 33, \tag{4.2}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+3})=\begin{cases}6\kappa+5 & \text{for} \quad 6\le\kappa\le\eta-3, \quad \kappa\equiv 0 \bmod 3;\\ 6\kappa+7 & \text{for} \quad 7\le\kappa\le\eta-5, \quad \kappa\equiv 1 \bmod 3;\\ 6\kappa+9 & \text{for} \quad 8\le\kappa\le\eta-4, \quad \kappa\equiv 2 \bmod 3,\end{cases}$$

$$\Phi^*(\lambda_1\lambda_6)=15,\ \Phi^*(\lambda_2\lambda_7)=21, \Phi^*(\lambda_3\lambda_8)=27, \Phi^*(\lambda_4\lambda_9)=31,\ \Phi^*(\lambda_5\lambda_{10})=37, \tag{4.3}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+5})=\begin{cases}6\kappa+11 & \text{if} \quad \kappa=6l \quad \text{or} \quad \kappa=6l+1,\ 1\le l\le\eta-1;\\ 6\kappa+15 & \text{if} \quad \kappa=6l+2 \quad \text{or} \quad \kappa=6l+3,\ 1\le l\le\eta-9;\\ 6\kappa+13 & \text{if} \quad \kappa=6l+4 \quad \text{or} \quad \kappa=6l+5,\ 1\le l\le\eta-7.\end{cases}$$

Case (2) When $\eta\equiv 2 \bmod 6$. Let d = 11, It can be possible to define the vertex labelling $\Phi: V[T_\eta(1,3,5)]\to\{0,2,4,\cdots,2\beta+2d-2=6\eta+2\}$ in the following manner:

$$\Phi(\lambda_\kappa)=\begin{cases}2\kappa-2 & \text{if} \quad 1\le\kappa\le 3;\\ 6\kappa-14 & \text{if} \quad \kappa=4,5;\\ 6\kappa-6 & \text{if} \quad \kappa\equiv 0 \bmod 6, 6\le\kappa\le\eta-2;\\ 6\kappa+2 & \text{if} \quad \kappa\equiv 2 \bmod 6, 8\le\kappa\le\eta;\\ 6\kappa-2 & \text{for} \quad 10\le\kappa\le\eta-4, \quad \kappa\equiv 4 \bmod 6;\\ & \text{or} \quad 7\le\kappa\le\eta-1, \quad \kappa \text{ is odd.}\end{cases}$$

From the labelling structure previously described, we can deduce the induced vertex labels $\Phi^*: E[T_\eta(1,3,5)]\to\{1,3,5,\cdots,6\eta-3\}$ are:

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+1})=\begin{cases}6\kappa-1 & \text{for} \quad 6\le\kappa\le\eta-2, \quad \kappa\equiv 0 \bmod 6;\\ & \text{or} \quad 11\le\kappa\le\eta-3, \quad \kappa\equiv 5 \bmod 6;\\ 6\kappa+1 & \text{for} \quad 9\le\kappa\le\eta-5, \quad \kappa\equiv 3 \bmod 6;\\ & \text{or} \quad 10\le\kappa\le\eta-4, \quad \kappa\equiv 4 \bmod 6;\\ 6\kappa+3 & \text{for} \quad 7\le\kappa\le\eta-1, \quad \kappa\equiv 1 \bmod 6;\\ & \text{or} \quad 8\le\kappa\le\eta-6, \quad \kappa\equiv 2 \bmod 6.\end{cases}$$

$$\Phi^*(\lambda_\kappa\lambda_{\kappa+3})=\begin{cases}6\kappa+5 & \text{for} \quad 6\le\kappa\le\eta-5, \kappa\equiv 0 \bmod 3;\\ 6\kappa+7 & \text{for} \quad 7\le\kappa\le\eta-4, \kappa\equiv 1 \bmod 3;\\ 6\kappa+9 & \text{for} \quad 8\le\kappa\le\eta-3, \kappa\equiv 2 \bmod 3,\end{cases}$$

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+5}) = \begin{cases} 6(\kappa+2)-1 & \text{if } \kappa = 6l \text{ or } \kappa = 6l+1,\ 1 \le l \le \eta-7; \\ 6(\kappa+2)+3 & \text{if } \kappa = 6l+2 \text{ or } \kappa = 6l+3,\ 1 \le l \le \eta-5; \\ 6(\kappa+2)+1 & \text{if } \kappa = 6l+4 \text{ or } \kappa = 6l+5,\ 1 \le l \le \eta-9. \end{cases}$$

The labeling of the unmentioned edges will match as within Eqs.(4.1-4.3).

Case (3) **When η is an odd number** or $\eta \equiv 4 \bmod 6$. Let $d = 9$, the labeling $\Phi : V[T_\eta(1,3,5)] \to \{0,2,4,\cdots,2\beta+2d-2 = 6\eta-2\}$ can be defined as follows:

$$\Phi(\lambda_\kappa) = \begin{cases} 6\kappa-6 & \text{if } \kappa \equiv 0 \bmod 6,\ 6 \le \kappa; \\ 6\kappa+2 & \text{if } \kappa \equiv 2 \bmod 6, 8 \le \kappa; \\ 6\kappa-2 & \text{for } 10 \le \kappa,\ \kappa \equiv 4 \bmod 6; \\ & \text{or } 7 \le \kappa,\ \kappa \text{ is odd.} \end{cases}$$

From the labelling pattern mentioned earlier, we can deduce the induced vertex labels Φ^*

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+1}) = \begin{cases} 6\kappa-1 & \text{for } 6 \le \kappa \le \eta-2,\ \kappa \equiv 0 \bmod 6; \\ & \text{or } 11 \le \kappa \le \eta-3,\ \kappa \equiv 5 \bmod 6; \\ 6\kappa+1 & \text{for } 9 \le \kappa \le \eta-5,\ \kappa \equiv 3 \bmod 6; \\ & \text{or } 10 \le \kappa \le \eta-4,\ \kappa \equiv 4 \bmod 6; \\ 6\kappa+3 & \text{for } 7 \le \kappa \le \eta-1,\ \kappa \equiv 1 \bmod 6; \\ & \text{or } 8 \le \kappa \le \eta-6,\ \kappa \equiv 2 \bmod 6. \end{cases}$$

$$\Phi^*(\lambda_\kappa \lambda_{\kappa+3}) = \begin{cases} 6\kappa+5 & \text{if } \kappa \equiv 0 \bmod 3, 6 \le \kappa; \\ 6\kappa+7 & \text{if } \kappa \equiv 1 \bmod 3, 7 \le \kappa; \\ 6\kappa+9 & \text{if } \kappa \equiv 2 \bmod 3, 8 \le \kappa, \end{cases}$$

$$\Phi^*(v_\kappa v_{\kappa+5}) = \begin{cases} 6\kappa+11 & \text{if } \kappa = 6l \text{ or } \kappa = 6l+1,\ 1 \le l \le \eta-9; \\ 6\kappa+15 & \text{if } \kappa = 6l+2 \text{ or } \kappa = 6l+3,\ 1 \le l \le \eta-7; \\ 6\kappa+13 & \text{if } \kappa = 6l+4 \text{ or } \kappa = 6l+5,\ 1 \le l \le \eta-5. \end{cases}$$

The labeling of the unmentioned edges will match as within Eqs.(4.1-4.3).

It should be noted that we have three cases

(i) If $\eta \equiv 4 \bmod 6$ or $\eta \equiv 5 \bmod 6$, then $\Phi^* : E[T_\eta(1,3,5)] \to \{1,3,5,\cdots,6\eta-5\}$.

(ii) If $\eta \equiv 3 \bmod 6$, $\Phi^* : E[T_\eta(1,3,5)] \to \{1,3,5,\cdots,6\eta-3\}$.

(iii) If $\eta \equiv 1 \bmod 6$, $\Phi^* : E[T_\eta(1,3,5)] \to \{1,3,5,\cdots,6\eta-7\}$.

There are no repeating labels on the edges, and they are all obviously odd. In light of this, the graph $T_\eta(1,3,5)$ is d– even vertex odd mean graph.

Illustration: A similar 7– EVOML, for $T_{18}(1,3,5)$, a similar 9– EVOML, for $T_{19}(1,3,5)$, and a similar 11– EVOML, for $T_{20}(1,3,5)$, are displayed in Figure 23.

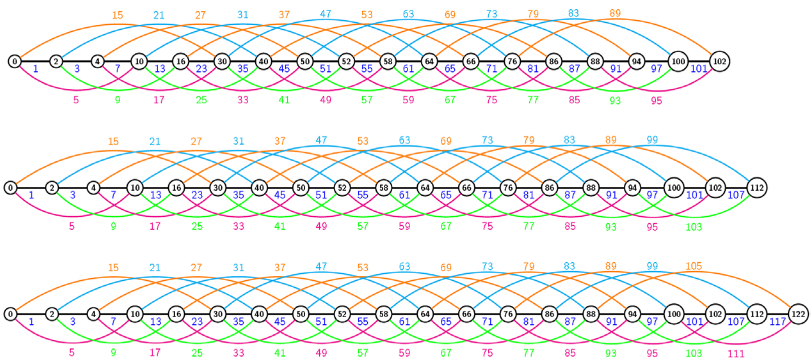

Figure 23

Conclusions

Graph labeling is one of the key topics under investigation in modern graph theory research. It has broad applications in a variety of fields, including astronomy, communication networks, radar, X-rays, and coding theory. We have provided an even vertex odd mean labelling generalisation to d-even vertex odd mean labelling. A function Φ is called d-even vertex odd mean labeling} of the graph Γ if $\Phi : V(\Gamma) \to \{0,2,4,\cdots,2\beta+2d-2\}$ is an injective function such that the induced edge function $\Phi^* : E(\Gamma) \to \{1,3,\cdots,2\beta-1\}$, given by: $\Phi^*(\lambda\omega) = \frac{\Phi(\lambda)+\Phi(\omega)}{2}$, is a bijective function. A d - even vertex odd mean graph is a graph which allows even vertex odd mean labelling. We have discovered the necessary conditions that various path and cycle-related graphs must satisfy in order to exhibit a d- even vertex odd mean labelling or a similar d- even vertex odd mean labeling. We determine the smallest integer d for which the graphs : Y-tree, star graph, $P_\xi \odot \overline{K_\eta}$, crown graph R_η, and rooted product $P_n \Diamond C_4$ have a d– even vertex odd mean labeling. Furthermore we find the least number

d for which the graphs: Cycle graph C_η when $\eta \equiv 2 \bmod 4$, Dragon graph $P_\xi(C_\eta)$ when $\eta \equiv 2 \bmod 4$, $\xi \geq 1$, prism graph, and Toeplitz graphs $T_\eta(1,3), T_\eta(1,5)$, and $T_\eta(1,3,5)$ have a similar d- even vertex odd mean labeling. In the end, we establish that no odd cycle C_η is an even vertex odd mean graph for all d.

Competing interests

The authors affirm that they do not have any competing interests.

Availability of data and material

The article contains the data that were utilised to support the results of the research.

Author's contributions

The final manuscript was read by all authors, who gave their approval.

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