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Fair dominating sets of paths

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Abstract

Let $G=(V,E)$ be a simple graph. A dominating set of G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The cardinality of the smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. For $i \geq 1$, a i-fair dominating set (iFD-set) in G, is a dominating set S such that $|N(v) \cap D| = i$ for every vertex $v \in V \setminus D$. A fair dominating set, in G is a iFD-set for some integer $i \geq 1$. In this paper, we present the structure of fair dominating sets of a path and also we count the number of these sets.

Subject Classification: *05C25.*

Keywords: *Composition of a number, Fair dominating set, Path.*

1. Introduction

Let $G=(V,E)$ be a simple graph with n vertices. The distance between two vertices u and v denoted by $d(u,v)$ is the number of edges in a shortest path (also called a graph geodesic) connecting them. Let $S \subseteq V$ be any subset of vertices of G. The induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S. A set $D \subseteq V(G)$ is a dominating set, if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G ([9]).

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For $i \geq 1$, a i-fair dominating set (iFD-set) in G, is a dominating set D such that $|N(v) \cap D| = i$ for every vertex $v \in V \setminus D$. The i-fair domination number of G, denoted by $fd_i(G)$, is the minimum cardinality of a iFD-set. A iFD-set of G of cardinality $fd_i(G)$ is called a $fd_i(G)$-set. A fair dominating set, abbreviated FD-set, in G is a iFD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality $fd(G)$ is called a $fd(G)$-set. By convention, if $G = \overline{K_n}$, we define $fd(G) = n$. By the definition, it is easy to see that for any graph G of order n, $\gamma(G) \leq fd(G) \leq n$ and $fd(G) = n$ if and only if $G = \overline{K_n}$. Caro, Hansberg and Henning in [8] showed that for a disconnected graph G (without isolated vertices) of order $n \geq 3$, $fd(G) \leq n - 2$, and they constructed an infinite family of graphs achieving equality in this bound.

Caro, Hansberg, and Henning in [8] proved that if T is a tree of order $n \geq 2$, then $fd(T) \leq \frac{n}{2}$ with equality if and only if $T = T' \circ K_1$ for some tree T'.

Regarding the enumerative side of dominating sets, Alikhani and Peng in [4], have introduced the domination polynomial of a graph. The domination polynomial of graph G is the generating function for the number of dominating sets of G, i.e., $D(G, x) = \sum_{k=1}^{|V(G)|} d(G, k)x^k$, where $d(G, k)$ is the number of dominating sets of G with cardinality k (see [1, 4]). This polynomial and its roots has been actively studied in recent years (see for example [7, 12]). It is natural to count the number of another kind of dominating sets ([3]).

Let $D_f(G, k)$ be the family of the fair dominating sets of a graph G with cardinality k and let $d_f(G, k) = |D_f(G, k)|$. It is natural to study the structure of $D_f(G, k)$ and the number of fair-dominating sets of graphs. Recently, the number of fair dominating sets of graphs has considered in [5]. In this paper, we consider the path graph P_n and study the structures of fair-dominating sets of P_n and investigate $d_f(P_n, i)$. We denote the set $\{1, 2, \ldots, n\}$ simply by $[n]$.

2. Main results

In this section, we present the structure of fair dominating sets of path. As a consequence, we obtain the number of fair dominating sets of paths. Let P_n, be the path with n vertices $V(P_n) = [n]$ and $E(P_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$. Let $f\mathcal{P}_n^i$ be the family of fair dominating sets of P_n with cardinality i. We shall investigate the fair dominating sets of paths.

A composition of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of summands is called the number of parts of the composition. In other words, a composition of an integer n is a way of writing n as the sum of a sequence of positive integers ([10]). Two sequences that differ in the order of their terms define different compositions of their sum. For example there are sixteen composition of the number 5.

To construct a fair dominating set of size k, we consider the composition of the number k. We do this based on being even or odd of $n-k$.

(i) If $n-k$ is odd and $n < 2k$. We construct the fair dominating sets of P_n based on two kind of composition of the number k.

(1) First we composite the number k to $\frac{n-k+1}{2}$ natural numbers. Suppose that $k = t_1 + t_2 + \ldots + t_{\frac{n-k+1}{2}}$, where t_i $(1 \leq i \leq \frac{n-k+1}{2})$ is a positive integer number. We define the family $\mathcal{A} \subseteq [n]$, based on the partition $k = t_1 + t_2 + \ldots + t_{\frac{n-k+1}{2}}$ as follows:

$$\mathcal{A} = A_1 \cup A_2 \cup \ldots \cup A_{\frac{n-k+1}{2}},$$

where the set A_1 contains t_1 consecutive numbers from $[n]$, the set A_2 contains t_2 consecutive numbers from $[n] \setminus A_1$ and finally the set $A_{\frac{n-k+1}{2}}$ contains $t_{\frac{n-k+1}{2}}$ consecutive numbers from $[n] \setminus (A_1 \cup A_2 \cup \ldots \cup A_{\frac{n-k+1}{2}-1})$ and $d(G[A_i], G[A_{i+1}]) = 3$. Note that in this case for a FD-set of path P_n with $V(P_n) = [n]$, say F in $\mathcal{A}$ we have two cases: (a) $1 \in F$ and $n \not\in F$ which we denote it by F_1, (b) $1 \not\in F$ and $n \in F$ that we denote it by F_2. Note that $|F_1| = |F_2|$.

(2) Now, we composite the number k to $n-k+1$ natural numbers. Suppose that $k = t_1 + t_2 + \ldots + t_{n-k+1}$, where t_i $(1 \leq i \leq n-k+1)$ is a non-negative integer number. We define the family $\mathcal{B} \subseteq [n]$, based on this partition, as follows:

$$\mathcal{B} = B_1 \cup B_2 \ldots \cup B_{n-k+1},$$

where the set B_1 contains t_1 consecutive numbers from $[n]$, the set B_2 contains t_2 consecutive numbers from $[n] \setminus B_1$ and finally the set B_{n-k+1} contains t_{n-k+1} consecutive numbers from $[n]\setminus(B_1\cup B_2\cup...\cup B_{n-k})$ and also $d(G[B_i],G[B_{i+1}])=2$.

And when $n\geq 2k$ we just can make FD-sets by $\mathcal{A}$. Now with these constructions and explanations, it is easy to see that the families of fair dominating sets of P_n is as follows;

(a) If $n<2k$, then $f\mathcal{P}_n^k=F_1\cup F_2\cup\mathcal{B}$.

(b) If $n\geq 2k$, then $f\mathcal{P}_n^k=F_1\cup F_2$.

(ii) If $n-k$ is even and $n<2k$, we composite the number k to $\lfloor\frac{n-k+1}{2}\rfloor$ natural numbers and we choose the vertices of FD-sets based on this partition, in the way that $d(G[C_i],G[C_{i+1}])=3$.

Suppose that $k=t_1+t_2+...+t_{\lfloor\frac{n-k+1}{2}\rfloor}$, where t_i $(1\leq i\leq\lfloor\frac{n-k+1}{2}\rfloor)$ is a positive integer number. We define the family $\mathcal{C}\subseteq[n]$, based on composition $k=t_1+t_2+...+t_{\lfloor\frac{n-k+1}{2}\rfloor}$ as follows:

$$\mathcal{C}=C_1\cup C_2\cup...\cup C_{\lfloor\frac{n-k+1}{2}\rfloor},$$

where the set C_1 contains t_1 consecutive numbers from $[n]$, the set C_2 contains t_2 consecutive numbers from $[n]\setminus C_1$ and finally the set $C_{\lfloor\frac{n-k+1}{2}\rfloor}$ contains $t_{\lfloor\frac{n-k+1}{2}\rfloor}$ consecutive numbers from $[n]\setminus(C_1\cup...\cup C_{\lfloor\frac{n-k+1}{2}\rfloor-1})$ and $d(G[C_i],G[C_{i+1}])=3$.

Now suppose that $k=t_1+t_2+...+t_{\lfloor\frac{n-k+1}{2}\rfloor+1}$, where t_i $(1\leq i\leq\lfloor\frac{n-k+1}{2}\rfloor+1)$ is a non-negative integer number. We define the family $\mathcal{D}\subseteq[n]$, based on this composition, as follows:

$$\mathcal{D}=D_1\cup D_2\cup...\cup D_{\lfloor\frac{n-k+1}{2}\rfloor+1},$$

where the set D_1 contains t_1 consecutive numbers from $[n]$, the set D_2 contains t_2 consecutive numbers from $[n]\setminus D_1$ and finally the set $D_{\lfloor\frac{n-k+1}{2}\rfloor+1}$ contains $t_{\lfloor\frac{n-k+1}{2}\rfloor+1}$ consecutive numbers from $[n]\setminus(D_1\cup D_2\cup...\cup D_{\lfloor\frac{n-k+1}{2}\rfloor})$ and also $d(G[D_i],G[D_{i+1}])=3$.

Finally suppose that $k=t_1+t_2+...+t_{n-k+1}$, where t_i $(1\leq i\leq n-k+1)$ is a non-negative integer number. We define the family $\mathcal{E}\subseteq[n]$, based on this composition, as follows:

$$\mathcal{E} = E_1 \cup E_2 \cup \ldots \cup E_{n-k+1},$$

where the set E_1 contains t_1 consecutive numbers from $[n]$, the set E_2 contains t_2 consecutive numbers from $[n] \setminus E_1$ and finally the set E_{n-k+1} contains t_{n-k+1} consecutive numbers from $[n] \setminus (E_1 \cup E_2 \cup \ldots \cup E_{n-k})$ and also $d(G[E_i], G[E_{i+1}]) = 2$.

And when $n \geq 2k$ we just have the composition based on families of $\mathcal{C}$ and $\mathcal{D}$. So in this case the families of FD-sets of P_n is as follows;

(a) If $n < 2k$, then $f\mathcal{P}_n^k = \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$.

(b) If $n \geq 2k$, then $f\mathcal{P}_n^k = \mathcal{C} \cup \mathcal{D}$.

So we have the following result which is about the structures of fair dominating sets of P_n.

Theorem 2.1 : *The structure of the fair dominating sets of path P_n is as follows;*

(i) *If $n-k$ is odd, then* $f\mathcal{P}_n^k = \begin{cases} F_1 \cup F_2 \cup \mathcal{B}; & \text{if } n<2k, \\ F_1 \cup F_2; & \text{if } n \geq 2k. \end{cases}$

(ii) *If $n-k$ is even, then* $f\mathcal{P}_n^k = \begin{cases} \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}; & \text{if } n<2k, \\ \mathcal{C} \cup \mathcal{D}; & \text{if } n \geq 2k. \end{cases}$

Let denote the compositions of the number k in the method that we stated before of Theorem 2.1 by $C(k)$.. For the families discussed before, we have;

$$|\mathcal{A}| = \sum_{C(k)} \binom{\frac{n-k+1}{2}}{t_1, t_2, \ldots, t_{\frac{n-k+1}{2}}}, \; |\mathcal{B}| = \sum_{C(k)} \binom{n-k+1}{t_1, t_2, \ldots, t_{n-k+1}}.$$

$$|\mathcal{C}| = \sum_{C(k)} \binom{\lfloor \frac{n-k+1}{2} \rfloor}{t_1, \ldots, t_{\lfloor \frac{n-k+1}{2} \rfloor}}, \; |\mathcal{D}| = \sum_{C(k)} \binom{\lfloor \frac{n-k+1}{2} \rfloor + 1}{t_1, \ldots, t_{\lfloor \frac{n-k+1}{2} \rfloor + 1}}, |\mathcal{E}| = \sum_{C(k)} \binom{n-k+1}{t_1, \ldots, t_{n-k+1}}.$$

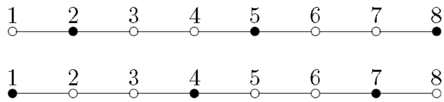
So we have the following theorem which is the main result of this paper:

Theorem 2.2 : *The number of the fair dominating sets of path P_n of size k is as follows;*

(i) *If* $n-k$ *is odd, then* $d_f(P_n,k)=\begin{cases}2|\mathcal{A}|+|\mathcal{B}|; & \text{if } n<2k,\\ 2|\mathcal{A}|; & \text{if } n\geq 2k.\end{cases}$

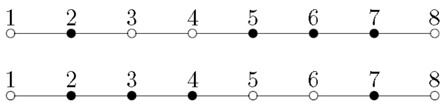
(ii) *If* $n-k$ *is even, then* $d_f(P_n,k)=\begin{cases}|\mathcal{C}|+|\mathcal{D}|+|\mathcal{E}|; & \text{if } n<2k,\\ |\mathcal{C}|+|\mathcal{D}|; & \text{if } n\geq 2k.\end{cases}$

Example 2.3 : Let apply Theorem 2.2 to obtain $d_f(P_8,k)$ for $3\leq k\leq 8$. First we obtain $d_f(P_8,3)$. Since $n-k=8-3$ is odd and $n=8>2k=6$, by explanation before Theorem 2.2, we have two kind of fair dominating sets, F_1 and F_2, where $1\not\in F_1$, $8\in F_1$ and $1\in F_2$, $8\not\in F_2$ and F_1 and F_2 constructed as follows. Composite the number $k=3$ to $\frac{n-k+1}{2}=3$ natural numbers, i.e., $3=1+1+1$. By explanation before Theorem 2.2 we have two cases for the fair dominating set. So, we have the *FD*-sets as follows (black vertices are in the fair dominating sets);



Therefore $d_f(P_8,3)=2$.

Now, we count $d_f(P_8,4)$. Since $n-k=8-4=4$ is even and $n=2k$, we have $f\mathcal{P}_8^4=\mathcal{C}\cup\mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are made as follows. First we construct $\mathcal{C}$. Composite the number $k=4$ to $\lfloor\frac{n-k+1}{2}\rfloor=2$ natural numbers. We have FD-sets based summands 1 and 3 as follows:

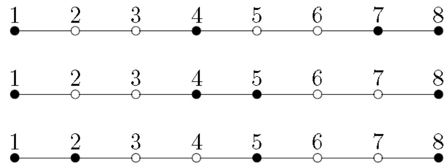


In this case we obtained $\frac{2!}{1!1!}=2$ FD-sets. The other case for $\mathcal{C}$, we consider composition of 4 to two numbers 2, so the FD-set is as follows;

We saw that in this case there is $\frac{2!}{2!}=1$ FD-set. Therefore;

$$|\mathcal{C}|=\sum_{C(k)}\binom{2}{t_1,t_2}=\frac{2!}{1!1!}+\frac{2!}{2!}=2+1=3.$$

Now we construct $\mathcal{D}$. Composite the number $k=4$ to $\lfloor\frac{n-k+1}{2}\rfloor+1=3$ natural numbers. FD-sets are as follows;



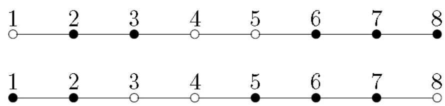
We saw that in composition $\mathcal{D}$ we have $\frac{3!}{1!1!2!} = 3$ FD-sets, also we obtain three FD-sets by formula;

$$|\mathcal{D}| = \sum_{C(k)} \binom{3}{t_1, t_2, t_3} = \frac{3!}{1!1!2!} = 3.$$

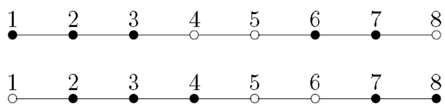
Therefore;

$$d_f(P_8, 4) = |\mathcal{C}| + |\mathcal{D}| = 3 + 3 = 6.$$

Now we shall obtain $d_f(P_8, 5)$. Since $n - k = 8 - 5 = 3$ is odd and $n < 2k$, we have $f\mathcal{P}_8^5 = F_1 \cup F_2 \cup \mathcal{B}$. First construct $\mathcal{A}$. Composite the number $k = 5$ to $\frac{n-k+1}{2} = 2$ natural numbers, i.e., with two summands 2 and 3. So FD-sets based the composition $5 = 2 + 3$ are as follows:



And the FD-sets of the composition $5 = 3 + 2$ are as follows;



In this case we have $2 \times 2 = 4$ FD-sets.

Now we consider the composition $5 = 1 + 4$. We know that $\frac{2!}{1!1!} = 2$ which we can make $2 \times 2 = 4$ FD-sets. So;

$$2|\mathcal{A}| = 2\sum_{P_j} \binom{2}{t_1, t_2} = 2 \times \left(\frac{2!}{1!1!} + \frac{2!}{1!1!} \right) = 2 \times (2 + 2) = 8.$$

For $\mathcal{B}$ composite the number $k = 5$ to $n - k + 1 = 4$ natural numbers with summands $1, 1, 1, 2$;

$$|\mathcal{B}| = \frac{4!}{1!3!} = 4.$$

So we have;

$$d_f(P_8,5)=2|\mathcal{A}|+|\mathcal{B}|=8+4=12.$$

Finally we obtain $d_f(P_8,6)$. Since $n-k=8-6$ is even and $n<2k$, we have $f\mathcal{P}_8^6=\mathcal{C}\cup\mathcal{D}\cup\mathcal{E}$. For $\mathcal{C}$, composite the number $k=6$ to $\lfloor\frac{n-k+1}{2}\rfloor=1$ natural numbers;

$$|\mathcal{C}|=\frac{1!}{1!}=1.$$

For $\mathcal{D}$, composite the number $k=6$ to $\lfloor\frac{n-k+1}{2}\rfloor+1=2$ natural numbers;

$$|\mathcal{D}|=\frac{2!}{1!1!}+\frac{2!}{1!1!}+\frac{2!}{2!}=2+2+1=5.$$

And for $\mathcal{E}$ we composite the number $k=6$ to $n-k+1=3$ natural numbers. So $|\mathcal{E}|=\frac{3!}{2!1!}+\frac{3!}{1!1!1!}+\frac{3!}{3!}=3+6+1=10$. Therefore;

$$d_f(P_8,6)=|\mathcal{C}|+|\mathcal{D}|+|\mathcal{E}|=1+5+10=16.$$

Obviously $d_f(P_8,7)=8$ and $d_f(P_8,8)=1$.

Using Theorem 2.2, we obtain $d_f(P_n,j)$ for $1\le n\le 13$ as shown in Table 1 (see [5]).

Table 1

$d_f(P_n,j)$, the number of fair dominating sets of P_n with cardinality j.

j	1	2	3	4	5	6	7	8	9	10	11	12	13
n													
1	1												
2	2	1											
3	1	3	1										
4	0	2	4	1									
5	0	2	4	5	1								
6	0	1	4	7	6	1							
7	0	0	3	7	11	7	1						
8	0	0	2	6	12	16	8	1					
9	0	0	1	6	11	20	22	9	1				
10	0	0	0	3	12	20	32	29	10	1			
11	0	0	0	2	6	21	36	49	37	11	1		
12	0	0	0	1	8	20	36	64	60	46	12	1	
13	0	0	0	0	5	20	36	63	106	102	56	13	1

A finite sequence of real numbers $(a_0, a_1, a_2, \ldots, a_n)$ is said to be

1. *unimodal if* $a_0 \le a_1 \le \cdots \le a_{k-1} \le a_k \ge a_{k+1} \ge \cdots \ge a_n$ *for some* $k \in \{0, 1, 2, \ldots, n\}$;
2. *logarithmically-concave* (or simply log-concave) if the inequality $a_k^2 \ge a_{k-1}a_{k+1}$ is valid for every $k \in \{1, 2, \ldots, n-1\}$.

A polynomial $\sum_{k=0}^{n} a_k x^k$ is said to be unimodal (or log-concave) if the coefficient sequence $\{a_k\}$ is unimodal (or log-concave). The unimodality problems of graph polynomials have always been of great interest to researchers in graph theory. For example in [14] it is conjectured that the chromatic polynomial of a graph is unimodal. In [11], this conjectured has been proved Also there is a famous conjecture due to Paul Erdös et al. on the unimodality of the independence polynomial of trees [2]. It is conjectured that the domination polynomial of every graph is unimodal [4]. Recently, Beaton and Brown [6] shown that paths, cycles and complete multipartite graphs are unimodal, and that the domination polynomial of almost every graph is unimodal with mode $\lceil \frac{n}{2} \rceil$. Also Lau and Alikhani [13] obtained sufficient conditions for this conjecture to hold. Also they studied the unimodality of graph G with $d(G, \gamma(G)) = \gamma(G) + k$, where k is an integer. Using Table 1, we close this paper with the following conjecture:

Conjecture 2.4 : The sequence of the number of fair-dominating sets of path P_n is unimodal.

References

[1] Akbari, S., Alikhani, S., Peng, Y.H., Characterization of graphs using domination polynomial, *Europ. J. Combin.*, 31 (2010) 1714-1724.

[2] Alavi, Y., Malde, P.J., Schwenk, A.J., Erdös, The vertex independence sequence of a graph is not constrained, *Congressus Numerantium* 58 (1987) 15-23.

[3] Alikhani, S., Akhbari, M.H., Eslahchi, C., Hasni, R., On the number of outer connected dominating sets of graphs, *Utilitas Math.* 91 (2013) 99-107.

[4] Alikhani, S., Peng, Y.H., Introduction to domination polynomial of a graph, *Ars Combin*. 114 (2014) 257-266.

[5] Alikhani, S., Safazadeh, M., On the number of fair dominating sets of graphs, submitted. Available at https://arxiv.org/abs/2107.10671.

[6] Beaton,I., Brown, J.I., On the unimodality of domination polynomials, Available at https://arxiv.org/abs/2012.11813.

[7] Brown, J.I., Tufts, J., On the roots of domination polynomials, *Graphs Combin.* 30, (2014) 527-547.

[8] Caro, Y., Hansberg, A., Henning, M., Fair domination in graphs, *Discrete Appl. Math.* 312 (2012) 2905-2914.

[9] Haynes, T.W., Hedetniemi, S.T., Slater, P.J. (1998) Fundamentals of domination in graphs. Marcel Dekker, NewYork.

[10] Heubach, S., Mansour, T., Compositions of n with parts in a set, *Congressus Numerantium* 168 (2004) 33-51.

[11] Huh, J., Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, *J. Amer. Math. Soc*. 25 (2012) 907-927.

[12] Kotek, T., Preen, J., Simon, F., Tittmann, P., Trinks, M., Recurrence relations and splitting formulas for the domination polynomial, *Elec. J. Combin.* 19(3) (2012) 27 pp.

[13] Lau, G.C., Alikhani, S., More on the unimodality of domination polynomial of a graph, *Discrete Math. Alg. Appl.*, in press. https://doi.org/10.1142/S179383092150138X.

[14] Read, R.C., An introduction to chromatic polynomials, *J. Combin. Theory* 4 (1968) 52-71.

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