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Energy and Wiener index of divisor graph of commutative ring

P. D. Khandare[§]
Department of Mathematics and Statistics
GES's Shri Bhausaheb Vartak College
Borivali 400091
Maharashtra
India

S. M. Jogdand [†]
Department of Mathematics
SSGM College, Loha
Nanded 431708
Maharashtra
India

R. A. Muneshwar*
Department of Mathematics
NES's Science College
Nanded 131602
Maharashtra
India

Abstract

In recent paper, P. D. Khandare et al. introduced a graphical structure of non empty finite ring R called as divisor graph $\mathbb{D}[R]$ and discussed some properties of the graph $\mathbb{D}[Z_n]$. In this article, we will continue the study of the divisor graph $\mathbb{D}[Z_n]$ and discuss some more properties of $\mathbb{D}[Z_n]$. If $n \in N$ then we derive general formula for total degree and stability number of $\mathbb{D}[Z_n]$. Moreover we find adjacency matrix of the graph $\mathbb{D}[Z_n]$ and derived formula for energy of $\mathbb{D}[Z_n]$ when $n = p^k, k \in N$ and $n = pq$, for primes p and q.

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[§] *E-mail:* **khandareprakash789@gmail.com**
[†] *E-mail:* **smjog12@gmail.com**
* *E-mail:* **muneshwarrajesh10@gmail.com (Corresponding Author)**

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1. Introduction

Let R be a finite commutative ring with 1, where $U(R), Z(R)$ denote a set of units and zero divisor respectively. The concept of zero divisor graph was initiated by I. Beck[2] in 1988. He introduced graphical structure by considering zero divisors as a vertex set and introduced concepts like girth, diameter, clique number etc. Later on Anderson and Livingston[1] has studied distinct properties of zero divisor graphs by considering only non-zero zero divisors of the ring. A concept of energy and Weiner index of zero divisor graph was introduced by Mohammad Reza and Reza Jahani[6]. Sarfraz Ahmad et al.[12], computed the Wiener index and Wiener polarity index of the linear and zigzag polyomino chains of the 4-cycle using the edge cut method. Also, they introduced another polyomino chain of 4k-cycle Zmn and calculate its Wiener index and Wiener polarity index for m = 3 and m = 4. In addition, they offer graphical representations for analyzing how these indexes behave. Also B. Surendranath Reddy et al. [11], determined the adjacency matrix and eigenvalues of the zero divisor graph $\Gamma[Z_n]$ for $n = p^3$ and $n = p^2q$, where p and q are primes. Moreover, they also determined the Wiener index of the zero divisor graph of Z_n. Philip D. Powell[8], discussed a method for expressing the determinant of an N × N complex block matrix in terms of its constituent blocks. As a result, the determinant of a matrix of N^2 blocks may be reduced to the product of the determinants of N unique combinations of single blocks. We were motivated by their work and we introduced a divisor graph $\mathbb{D}[R]$ of finite commutative ring R, which can be found in [9]. In this article, we determine Energy and Weiner index of divisor graph $\mathbb{D}[Z_n]$. The reader can be referred to [14], for further graph theory-related topics and notations.

2. Definition and Preliminaries

Definition 2.1 : Divisor Graph[9]

Let R be finite commutative ring, then the graph $\mathbb{D}[R]$ is divisor graph of R, where $V = \{u \in R \mid u \neq 0,1\}$ and for $u, v \in V, u \sim v$ if and only if $\exists\, w \in R$ such that $u = wv$ or $v = wu$. i.e. $u \sim v$ if and only if $u \mid v$ or $v \mid u$.

Definition 2.2 : Circulant Matrix[11]

A circulant matrix of size $n \times n$ with entries a, b and $a, b \in R$, denoted by $C(a, b, n)$ is of the form

$$A=\begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{bmatrix}$$

and $det(A)=[(a+(n-1)b](a-b)^{(n-1)}$

Definition 2.3 : Schur complement[8]

Let M,NP,Q be matrices and let M be invertible. Let

$$S=\begin{bmatrix} M & N \\ P & Q \end{bmatrix}$$

then $det(S)=det(M)\cdot det(Q-PM^{-1}N)$, Where $(Q-PM^{-1}N)$ is called the Schur complement of Q in S with respect to M.

Let R be ring of integer modulo n. Then the vertex set can be partition into disjoint subsets, $V=A\cup B$, where $A=U(R)-\{1\}$ and $B=Z(R)-\{0\}$. Here $U(R)$ and $Z(R)$ are sets of units and non zero zero divisors of ring Z_n. Divide set B such that, if $n=p_1p_2\cdots p_m$, $B=\cup W_{d_i}$, $W_{d_i}=\{k(d_i)\,|\,k\in R,\ k\notin\langle d_j\rangle,\ d_i\neq d_j\}$ and $W_{d_i}\cap W_{d_j}=\{0\}$ and if $n=p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$ then $B=\cup T_{d_i}$, where $T_{d_i}=\{k\cdot d_i\,|\,k\in R,\ k\notin\langle d_j\rangle,\ d_i^s\neq d_j,\ s\in N\}$. A loop at subset U of V denote vertices of U are mutually adjacent. Let V_1 and V_2 be subset of vertex set V, then $V_1\leftrightarrow V_2$ denote that each vertex of V_1 is adjacent to every vertex of V_2 and $V_1\nleftrightarrow V_2$ denote that no vertex of V_1 is adjacent to any vertex of V_2.

3. Total Degree of Divisor Graph of Z_n

Example 3.1 : If $n=2$, then total degree of $(\mathbb{D}[Z_n])$ is zero.

Since for $n=2,\mathbb{D}[R]$ is empty graph. Hence it is proved.

Theorem 3.2 : *If $n=p^k$, where p is prime number then total degree of $\mathbb{D}[Z_n]$ is* $(p^k-2)(p^k-3)$.

Proof: Let R is ring of integer modulo n. Since by theorem 4.2[9], if $n=p^k$ then $\mathbb{D}[Z_{p^k}]$ is complete graph of order p^k-2. Therefore $\forall u\in V,\ d(u)=p^k-3$. Hence the total degree of divisor graph $\mathbb{D}[Z_{p^k}]$ is $(p^k-2)(p^k-3)$.

Theorem 3.3 : *If $n=p_1p_2$ then total degree of graph $\mathbb{D}[Z_n]$ is* $deg(\mathbb{D}Z_n)=n^2-5n+6-2\phi(n)$.

Proof: Let R be ring of integer modulo n, where $n = p_1 p_2$, then vertex set can be partition into disjoint subsets, $V = A \cup B$, where $A = U(R) - \{1\}$ and $B = Z(R) - \{0\}$. Here $U(R)$ and $Z(R)$ are sets of units and non zero zero divisors of ring Z_n. For $n = p_1 p_2$, set B be divided into two subset

$$W_{p_1} = \{k_1 p_1 \mid k_1 \in R, k_1 \notin \langle p_2 \rangle\}$$

$$W_{p_2} = \{k_2 p_2 \mid k_2 \in R, k_2 \notin \langle p_1 \rangle\}$$

Then $|A| = \phi(n) - 1, |W_{p_1}| = p_2 - 1 = x_2$ (say), $|W_{p_2}| = p_1 - 1 = x_1$ (say). For $x, y \in V(\mathbb{D}[R])$ following result are hold.

1. If $u \in A$ then vertex u is adjacent to all other vertices of the graph. 2. If $x \in W_{p_1}$ then it is adjacent to all other vertices of set W_{p_1} and A. 3. If $y \in W_{p_2}$ then it is adjacent to all vertices of the W_{p_2} and A. Then no. of pair of non adjacent vertices are

$$|W_{p_1}| \, |W_{p_2}| = (p_2 - 1)(p_1 - 1) = x_2 x_1 = \phi(n)$$

Since no. of possible edges in complete graph of order n are $\frac{n(n-1)}{2}$. Therefore, No.of edges in divisor graph when $n = p_1 p_2$

$$\begin{aligned} n(E) &= \frac{(n-2)(n-3)}{2} - \phi(n) \\ &= \frac{n^2 - 5n + 6 - 2\phi(n)}{2} \end{aligned}$$

We have, for vertex $u_i \in V, \sum_i d(u_i) = 2 \cdot n(E)$, where $n(E)$ denote no. of edges in the graph. Therefore a total degree of the graph $\mathbb{D}[Z_{p_1 p_2}]$ is $n^2 - 5n + 6 - 2\phi(n)$.

Corollary 3.4 : *If $n = p_1 p_2$, a stability number of $\mathbb{D}[Z_n]$ is $(p_1 - 1)(p_2 - 1) = \phi(n)$.*

Proof: The proof is followed by Theorem 3.3.

Theorem 3.5 : *If $n = p_1 p_2 p_3, p_1 < p_2 < p_3$, where p_i are primes then a total degree of the graph $\mathbb{D}[Z_n]$ is,*

$$\begin{aligned} deg(\mathbb{D}[Z_n]) = n^2 - 5n + 6 - 2\phi(n)[\,|W_{p_1 p_2}| + |W_{p_1 p_3}| + |W_{p_2 p_3}| + 3] \\ -2|W_{p_1}| - 2|W_{p_2}| - 2|W_{p_3}| \end{aligned}$$

Proof: Let vertex set of $\mathbb{D}[R]$, $V = A \cup B$, where $A = U(R) - \{1\}$, $B = Z(R) - \{0\}$. We have

$$W_{p_i} = \{k_i p_i \mid k_i \in R, k_i \notin \langle p_j \rangle\}, i \neq j, W_{p_1p_2} = \{s(p_1p_2) \mid s \in R, s \notin \langle p_3 \rangle\},$$

$$W_{p_1p_3} = \{s(p_1p_3) \mid s \in R, s \notin \langle p_2 \rangle\}, W_{p_2p_3} = \{s(p_2p_3) \mid \in R, s \notin \langle p_1 \rangle\},$$

where $|W_{p_1}| = (p_2 - 1)(p_3 - 1)$, $|W_{p_2}| = (p_1 - 1)(p_3 - 1)$, $|W_{p_3}| = (p_1 - 1)(p_2 - 1)$, $|W_{p_1p_2}| = (p_3 - 1)$, $|W_{p_1p_3}| = (p_2 - 1)$, $|W_{p_2p_3}| = (p_1 - 1)$.

Since, $p_1p_2 \nmid p_1p_3$, $p_1p_3 \nmid p_2p_3$, $p_1p_2 \nmid p_2p_3$, then we have $W_{p_i} \nleftrightarrow W_{p_j}$, $W_{p_1p_2} \nleftrightarrow W_{p_1p_3}$, $W_{p_1p_2} \nleftrightarrow W_{p_2p_3}$ and $W_{p_2p_3} \nleftrightarrow W_{p_1p_3}$. By theorem 3.7 and proposition then no. of non adjacent vertices of $\mathbb{D}[R]$ are

$$\begin{aligned}
&= |W_{p_1}|(|W_{p_2}| + |W_{p_3}| + |W_{p_2p_3}|) + |W_{p_2}|(|W_{p_3}| + |W_{p_1p_3}|) + |W_{p_3}||W_{p_1p_2}| \\
&+ |W_{p_1p_2}|[|W_{p_1p_3}| + |W_{p_2p_3}|] + |W_{p_2p_3}||W_{p_1p_3}| \\
&= (p_2 - 1)(p_3 - 1)[(p_1 - 1)(p_3 - 1) + (p_1 - 1)(p_2 - 1) + (p_1 - 1)] \\
&+ (p_1 - 1)(p_3 - 1)[(p_1 - 1)(p_2 - 1) + (p_2 - 1)] + (p_1 - 1)(p_2 - 1)(p_3 - 1) \\
&+ (p_3 - 1)[(p_2 - 1) + (p_1 - 1)] + (p_1 - 1)(p_2 - 1) \\
&= \phi(n)[x_3 + x_2 + x_1 + 3] + x_1x_2 + x_1x_3 + x_2x_3 \\
&= \phi(n)[|W_{p_1p_2}| + |W_{p_1p_3}| + |W_{p_2p_3}| + 3] + |W_{p_1}| + |W_{p_2}| + |W_{p_3}|.
\end{aligned}$$

Therefore, $n(E)$

$$= \frac{n^2 - 5n + 6}{2} - (\phi(n)[|W_{p_1p_2}| + |W_{p_1p_3}| + |W_{p_2p_3}| + 3] + |W_{p_1}| + |W_{p_2}| + |W_{p_3}|)$$

$$= \frac{n^2 - 5n + 6 - 2\phi(n)[|W_{p_1p_2}| + |W_{p_1p_3}| + |W_{p_2p_3}| + 3] - 2|W_{p_1}| - 2|W_{p_2}| - 2|W_{p_3}|}{2}$$

Hence, total degree of the graph $\mathbb{D}[Z_n]$ is,

$$deg(\mathbb{D}[Z_n]) = n^2 - 5n + 6 - 2\phi(n)[|W_{p_1p_2}| + |W_{p_1p_3}| + |W_{p_2p_3}| + 3] - 2|W_{p_1}| - 2|W_{p_2}| - 2|W_{p_3}|$$

Corollary 3.6 : *If* $n = p_1 \cdot p_2 \cdot p_3$, *a stability number of* $\mathbb{D}[Z_n]$ *is* $\phi(n)$ $[|W_{p_1p_2}| + |W_{p_1p_3}| + |W_{p_2p_3}| + 3] + |W_{p_1}| + |W_{p_2}| + |W_{p_3}|$.

Proof: The proof is clearly followed by the Theorem 3.6.

Theorem 3.7 : *If* $n = p_1 \cdot p_2 \cdots p_m$ *then total degree of the graph* $\mathbb{D}[Z_n]$

$$deg(\mathbb{D}[Z_n]) = n^2 - 5n + 6 - 2|W_{p_1}|(\sum_{p_1 \nmid d_i}|W_{d_i}|) - 2|W_{p_2}|(\sum_{p_2 \nmid d_i, p_1 \neq d_i}|W_{d_i}|) - \cdots$$
$$-2|W_{p_m}|(\sum_{p_m \nmid d_i, d_i \neq p_1, p_2, \cdots p_{(m-1)}}|W_{d_i}|) - 2|W_{p_1p_2}|(\sum_{p_1p_2 \nmid d_i, d_i \neq p_1, p_2}|W_{d_i}|)$$
$$-\cdots - 2|W_{p_1p_2\cdots p_{(m-2)}}||W_{p_1p_2\cdots p_{(m-1)}}|.$$

Proof: Let $d_1, d_2, \ldots, d_k$ be sequence of divisors of n. With the given partition of vertex set $V = A \cup B$, $A = U(R) - \{1\}$, $B = Z(R) - \{0\} = \cup W_{d_i}$, where $d_i \mid n$, $W_{d_i} = \{k(d_i) \mid k \in R, k \notin \langle d_j \rangle, d_i \neq d_j\}$, then by theorem 4.6[8], if $n = p_1 \cdot p_2 \cdots p_m$ then for $x \in W_{d_i}, y \in W_{d_j}, d_i \mid d_j$, we have $x \sim y$, and for each $x \in W_{d_i}, y \in W_{d_j}, d_i \nmid d_j$, we have $x \nsim y$.

Therefore, no. of pair of non adjacent vertices in the graph $\mathbb{D}[Z_{p_1p_2\cdots p_m}]$
$$= |W_{p_1}|(\sum_{p_1 \nmid d_i}|W_{d_i}|) + |W_{p_2}|(\sum_{p_2 \nmid d_i, p_1 \neq d_i}|W_{d_i}|) + \cdots + |W_{p_m}|(\sum_{p_m \nmid d_i, d_i \neq p_1, p_2, \cdots p_{(m-1)}}|W_{d_i}|)$$
$$+ |W_{p_1p_2}|(\sum_{p_1p_2 \nmid d_i, d_i \neq p_1, p_2}|W_{d_i}|) + \cdots + |W_{p_1p_2\cdots p_{(m-2)}}||W_{p_1p_2\cdots p_{(m-1)}}|$$

Thus,

$$n(E) = \frac{n^2 - 5n + 6}{2} - |W_{p_1}|(\sum_{p_1 \nmid d_i}|W_{d_i}|) - |W_{p_2}|(\sum_{p_2 \nmid d_i, p_1 \neq d_i}|W_{d_i}|) - \cdots$$
$$-|W_{p_m}|(\sum_{p_m \nmid d_i, d_i \neq p_1, p_2, \cdots p_{(m-1)}}|W_{d_i}|) - |W_{p_1p_2}|(\sum_{p_1p_2 \nmid d_i, d_i \neq p_1, p_2}|W_{d_i}|) - \cdots$$
$$-|W_{p_1p_2\cdots p_{(m-2)}}||W_{p_1p_2\cdots p_{(m-1)}}|$$

Therefore total degree of the graph,

$$deg(\mathbb{D}[Z_{p_1p_2\cdots p_m}]) = n^2 - 5n + 6 - 2|W_{p_1}|(\sum_{p_1 \nmid d_i}|W_{d_i}|) - 2|W_{p_2}|(\sum_{p_2 \nmid d_i, p_1 \neq d_i}|W_{d_i}|) - \cdots$$
$$-2|W_{p_m}|(\sum_{p_m \nmid d_i, d_i \neq p_1, p_2, \cdots p_{(m-1)}}|W_{d_i}|) - 2|W_{p_1p_2}|(\sum_{p_1p_2 \nmid d_i, d_i \neq p_1, p_2}|W_{d_i}|)$$
$$-\cdots - 2|W_{p_1p_2\cdots p_{(m-2)}}||W_{p_1p_2\cdots p_{(m-1)}}|.$$

Corollary 3.8 : *If* $n = p_1p_2p_3 \ldots p_m$, *then stability number of* $\mathbb{D}[Z_n]$ *is* $|W_{p_1}|(\sum_{p_1 \nmid d_i}|W_{d_i}|) + |W_{p_2}|(\sum_{p_2 \nmid d_i, p_1 \neq d_i}|W_{d_i}|) + \cdots + |W_{p_m}|(\sum_{p_m \nmid d_i, d_i \neq p_1, p_2, \ldots p_{(m-1)}}|W_{d_i}|) + |W_{p_1p_2}|(\sum_{p_1p_2 \nmid d_i, d_i \neq p_1, p_2}|W_{d_i}|) + \cdots + |W_{p_1p_2\cdots p_{(m-2)}}||W_{p_1p_2\cdots p_{(m-1)}}|$.

Proof: The proof is followed by the Theorem 3.7.

Theorem 3.9 : *If* $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, *then total degree of divisor graph is* $\frac{n^2-5n+6}{2} - n(E')$, *where n(E') denotes no. of non adjacent pair of vertices.*

Proof: Let the vertex set $V(\mathbb{D}[Z_n]) = A \cup B$, where $A = U(R) - \{1\}$, $B = Z(R) - \{0\}$. Notice that $A \leftrightarrow B$ and $A \leftrightarrow A$ but $B \nleftrightarrow B$. Therefore we consider the following cases.

Case I: If $n = p_1^{k_1} p_2^{k_2}$, then divisors of n are $p_1^{k_1}, p_2^{k_2}, p_1^{k_1} p_2^{k_2}$, for some $k_1, k_2 \geq 1$. Also observe that, $p_1 \mid p_1 p_2$ but $p_1^{k_1} \nmid p_1 p_2$. Let the partition of set B, $B = \cup T_{d_i}$, $T_{d_i} = \{k \cdot d_i \mid k \in R,\ k \notin \langle d_j \rangle,\ d_i^s \neq d_j,\ s \in N\}$.

$$T_{p_1} = \{kp_1 \mid k \in R, k \notin < p_2 >\},$$
$$T_{p_2} = \{kp_2 \mid k \in R, k \notin < p_1 >\},$$
$$T_{p_1 p_2} = \{kp_1 p_2 \mid k \in R\}$$

Also, $T_{p_1^{k_1}} \nleftrightarrow T_{p_2^{k_2}}, T_{p_1^{k_1}} \nleftrightarrow T_{p_1^{k_1-1} p_2} - T_{p_1^{k_1} p_2} \quad T_{p_2^{k_2}} \nleftrightarrow T_{p_1 p_2^{k_2-1}} - T_{p_1 p_2^{k_2}}$.

Hence the no. of pair of non adjacent vertices in the graph $\mathbb{D}[Z_n]$ are,

$$n(E') = |T_{p_1}|\,|T_{p_2}| + \sum_{j=2}^{k_1} |T_{p_1^j}| [\,|T_{p_1^{j-1} p_2}| - |T_{p_1^j p_2}|\,] + \sum_{j=2}^{k_2} |T_{p_2^j}| [\,|T_{p_2^{j-1} p_1}| - |T_{p_2^j p_1}|\,],$$

Then number of edges in the graph $\mathbb{D}[Z_n]$

$$n(E) = \frac{n^2-5n+6}{2} - n(E')$$

Case II: Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}, p_1 < p_2 < p_3, k_i \geq 1$

A divisors of n are $p_1^{k_1}, p_2^{k_2}, p_3^{k_3}, p_1^{k_1} p_2^{k_2}, p_2^{k_2} p_3^{k_3}, p_1^{k_1} p_3^{k_3}$. Consider a partition of set B as,

$$T_{p_1} = \{kp_1 \mid k_1 \in R, k_1 \notin < p_2 >, k_1 \notin < p_3 >\}, T_{p_2} = \{k_2 p_2 \mid k_2 \in R, k_2 \notin < p_1 >,$$
$$k_2 \notin < p_3 >\},$$
$$T_{p_3} = \{k_3 p_3 \mid k_3 \in R, k_3 \notin p_1, k_3 \notin < p_2 >\}, T_{p_1 p_2} = \{s_1 (p_1 p_2) \mid s_1 \in R,$$
$$s \notin < p_3 >\},$$
$$T_{p_1 p_3} = \{s_2 (p_2 p_3) \mid s_2 \in R, s_2 \notin < p_1 >\}, T_{p_1 p_3} = \{s_3 (p_1 p_3) \mid s_3 \in R, s_3 \notin < p_2 >\}.$$
$$T_{p_1^{k_1}} = \{w_1 (p_1^{k_1}) \mid w_1 \in R, w_1 \notin < p_2 >, w_1 \notin < p_3 >\}.$$

As we know, if $p_i^{k_s} \nmid p_j^{k_l}, i \neq j, s, l \in N$ and $p_1p_2 \nmid p_1p_3$, $p_1p_3 \nmid p_2p_3$, $p_1p_2 \nmid p_2p_3$, $p_1^k \nmid p_1p_2$, $p_2^k \nmid p_1p_2$, $\forall k \geq 2$ then

$$T_{p_i} \nleftrightarrow T_{p_j}, i \neq j, T_{p_1p_2} \nleftrightarrow T_{p_2p_3}, T_{p_1p_2} \nleftrightarrow T_{p_2p_3}, T_{p_1p_3} \nleftrightarrow T_{p_2p_3},$$

$$T_{p_1^k} \nleftrightarrow T_{p_1p_2} - T_{p_1^kp_2}, T_{p_2^k} \nleftrightarrow T_{p_1p_2} - T_{p_2^kp_2}.$$

Thus no. of pair of non adjacent vertices

$$\begin{aligned}
n(E') = {} & |T_{p_1}|(|T_{p_2}|+|T_{p_3}|+|T_{p_2p_3}|)+|T_{p_2}|(|T_{p_3}|+|T_{p_1p_3}|)+|T_{p_3}||T_{p_1p_2}| \\
& +|T_{p_1p_2}|(|T_{p_1p_3}|+|T_{p_2p_3}|)+|T_{p_1p_3}||T_{p_2p_3}|+\sum_{j=2}^{k_1}|T_{p_1^j}|(|T_{p_1^{j-1}p_2}| \\
& +|T_{p_1^{j-1}p_3}|+|T_{p_1^{j-1}p_2p_3}|-|T_{p_1^jp_2}|-|T_{p_1^jp_3}|-|T_{p_1^jp_2p_3}|) \\
& +\sum_{j=2}^{k_2}|T_{p_2^j}|(|T_{p_1p_2^{j-1}}+|T_{p_1p_2^{j-1}p_3}|+|T_{p_2^{j-1}p_3}|-|T_{p_1p_2^j}|-|T_{p_2^jp_3}-|T_{p_1p_2^jp_3}|) \\
& +\sum_{j=2}^{k_3}|T_{p_3^j}|(|T_{p_1p_3^{j-1}}|+|T_{p_2p_3^{j-1}}|+|T_{p_1p_2p_3^{j-1}}|-|T_{p_1p_3^j}|-|T_{p_2p_3^j}|-|T_{p_1p_2p_3^j}|) \\
& +\sum_{j=2}^{k_1}|T_{p_1^jp_2}|(|T_{p_1^{j-1}p_2p_3}|-|T_{p_1^jp_2p_3}|)+|T_{p_1^jp_3}|(|T_{p_1^{j-1}p_2p_3}|-|T_{p_1^jp_2p_3})| \\
& +\sum_{j=2}^{k_2}|T_{p_1p_2^j}|(|T_{p_1p_2^{j-1}p_3}|-|T_{p_1p_2^jp_3}|)+|T_{p_2^jp_3}|(|T_{p_1p_2^{j-1}p_3}|-|T_{p_1p_2^jp_3})| \\
& +\sum_{j=2}^{k_3}|T_{p_1p_3^j}|(|T_{p_1p_2p_3^{j-1}}|-|T_{p_1p_2p_3^j}|)+|T_{p_2p_3^j}|(|T_{p_1p_2p_3^{j-1}}|-|T_{p_1p_2p_3^j})|
\end{aligned}$$

Hence no. of edges in the graph are $\frac{n^2-5n+6}{2} - n(E')$

Case III: Now we may generalised the result for $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots p_m^{k_m}$, $p_1 < p_2 < \ldots < p_m$. First we find no. of non adjacent pair of vertices,

$$\begin{aligned}
n(E') = {} & |T_{p_1}|(\sum_i|T_{d_i}|, p_1 \nmid d_i)+|T_{p_2}|(\sum_i|T_{d_i}|, p_2 \nmid d_i, d_i \neq p_1)+\cdots \\
& + |T_{p_m}|\sum_i(|T_{d_i}|, p_m \nmid d_i, d_i \neq p_1, p_2 \cdots p_{(m-1)}) \\
& + |T_{p_1p_2}|\sum_i|T_{d_i}|, p_1p_2 \nmid d_i, d_i \neq p_1, p_2) \\
& + |T_{p_1p_3}|(\sum_i|T_{d_i}|, p_1p_3 \nmid d_i, d_i \neq p_1, p_3) \\
& + \cdots+|T_{p_1p_2p_3\cdots p_{(m-1)}}|(|T_{p_1p_3\cdots p_{(m-1)}p_m}|+|T_{p_1p_2p_4\cdots p_{(m-1)}p_m}|+\cdots \\
& + |T_{p_1p_2p_3\cdots p_{(m-2)}p_m}|+|T_{p_2p_3\cdots p_{(m-1)}p_m}|)+|T_{p_1p_2p_3\cdots p_{(m-2)}p_m}||T_{p_2p_3\cdots p_{(m-1)}p_m}| \\
& + \sum_{j=2}^{k_1}|T_{p_1^j}|(|T_{p_1^{j-1}p_2}|+|T_{p_1^{j-1}p_3}|+\cdots|T_{p_1^{j-1}p_m}|+|T_{p_1^{j-1}p_2p_3}|+|T_{p_1^jp_2p_4}|+\cdots+|T_{p_1^{j-1}p_2p_3\cdots p_m}| \\
& - |T_{p_1^jp_2}|-|T_{p_1^jp_3}|-\cdots|T_{p_1^jp_m}|-|T_{p_1^jp_2p_3}|-|T_{p_1^jp_2p_4}|-\cdots-|T_{p_1^jp_2p_3\cdots p_m}|
\end{aligned}$$

$$+ \sum_{j=2}^{k_2} |T_{p_2^j}|(|T_{p_1p_2^{j-1}}|+|T_{p_2^{j-1}p_3}|+\cdots+|T_{p_2^{j-1}p_m}|+|T_{p_1p_2^{j-1}p_3}|+|T_{p_1p_2^{(j-1)}p_4}|+\cdots+|T_{p_1p_2^{j-1}p_3\cdots p_m}|$$
$$- |T_{p_1p_2^j}|-|T_{p_2^jp_3}|-\cdots|T_{p_2^jp_m}|-|T_{p_1p_2^jp_3}|-|T_{p_1p_2^jp_4}|-\cdots-|T_{p_1p_2^jp_3\cdots p_m}|)+\cdots$$
$$+ \sum_{j=2}^{k_m} |T_{p_m^j}|(|T_{p_1p_m^{j-1}}|+|T_{p_2p_m^{j-1}}|+\cdots+|T_{p_{(m-1)}p_m^{j-1}}|+|T_{p_1p_2p_m^{j-1}}|+|T_{p_1p_3p_m^{(j-1)}}|+\cdots$$
$$+ |T_{p_1p_2p_3\cdots p_m^{j-1}}|-|T_{p_1p_m^j}|-|T_{p_2p_m^j}|-\cdots|T_{p_{m-1}p_m^j}|-|T_{p_1p_2p_m^j}|-|T_{p_1p_3p_m^j}|-\cdots-|T_{p_1p_2p_3\cdots p_m^j}|)$$
$$+ \cdots \sum_{j=2}^{k_1} |T_{p_1^jp_2}|(|T_{p_1^{j-1}p_2p_3}|+|T_{p_1^{j-1}p_2p_4}|+\cdots|T_{p_1^{j-1}p_2p_m}|+|T_{p_1^{j-1}p_2p_3p_4}|+|T_{p_1^{j-1}p_2p_4p_5}|+\cdots$$
$$+ |T_{p_1^{j-1}p_2\cdots p_m}|-|T_{p_1^jp_2p_3}|-|T_{p_1^jp_2p_4}|-\cdots|T_{p_1^jp_2p_m}|-|T_{p_1^jp_2p_3p_4}|-|T_{p_1^jp_2p_4p_5}|-\cdots-|T_{p_1^jp_2p_3\cdots p_m}|$$
$$+ \sum_{j=2}^{k_1} |T_{p_1^jp_2p_3}|(|T_{p_1^{j-1}p_2p_3p_4}|+|T_{p_1^{j-1}p_2p_3p_5}|+\cdots+|T_{p_1^{j-1}p_2p_3p_4\cdots p_m}|-|T_{p_1^jp_2p_3p_4}|-|T_{p_1^jp_2p_3p_5}|-\cdots$$
$$- |T_{p_1^jp_2p_3p_4\cdots p_m}|+\cdots+\cdots+|T_{p_1^jp_2p_3p_4\cdots p_{m-1}}|(|T_{p_1^{j-1}p_2p_3p_4\cdots p_m}|-|T_{p_1^jp_2p_3p_4\cdots p_m}|)$$
$$+ \sum_{j=2}^{k_2} |T_{p_1p_2^j}|(|T_{p_1p_2^{j-1}p_3}|+|T_{p_1p_2^{j-1}p_4}|+\cdots|T_{p_1p_2^{j-1}p_m}|+|T_{p_1p_2^{j-1}p_3p_4}|+|T_{p_1p_2^{j-1}p_3p_5}|+\cdots$$
$$+ |T_{p_1p_2^{j-1}p_3\cdots p_m}|-|T_{p_1p_2^jp_3}|-|T_{p_1p_2^jp_4}|-\cdots-|T_{p_1p_2^jp_m}|-|T_{p_1p_2^jp_3p_4}|-|T_{p_1p_2^jp_3p_5}|-\cdots$$
$$- |T_{p_1p_2^jp_2p_3\cdots p_m}|+\cdots+\cdots$$
$$+ \sum_{j=2}^{k_1} |T_{p_1^jp_2p_2p_3\cdots p_{(m-1)}}|(|T_{p_1^{j-1}p_2p_3\cdots p_m}|-|T_{p_1^jp_2p_3\cdots p_m}|)+\sum_{j=2}^{k_m}|T_{p_1p_m^j}|(|T_{p_1p_2p_m^{j-1}}|$$
$$+ |T_{p_1p_3p_m^{j-1}}|+\cdots+|T_{p_1p_{(m-1)}p_m^{j-1}}|+|T_{p_1p_2p_3p_m^{j-1}}|+|T_{p_1p_3p_4p_m^{j-1}}|+\cdots+|T_{p_1p_2\cdots p_{(m-1)}p_m^{j-1}}|$$
$$- |T_{p_1p_2p_m^j}|-|T_{p_1p_3p_m^j}|-\cdots-|T_{p_1p_{(m-1)}p_m^j}|-|T_{p_1p_2p_3p_m^j}|-|T_{p_1p_3p_4p_m^j}|-\cdots-|T_{p_1p_2p_3\cdots p_m^{j-1}}|)+\cdots$$
$$+ \cdots+|T_{p_1p_2p_3\cdots p_{m-1}^j}|(|T_{p_1p_2p_3\cdots p_{(m-1)}^{j-1}p_m}|-|T_{p_1p_2p_3\cdots p_{(m-1)}^jp_m}|) \quad \ldots(1)$$

Therefore, if $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, then no. of edges $n(E)$ in the graph $\mathbb{D}[Z_n]$ are $\frac{n^2-5n+6-2n(E')}{2}$ and hence $\sum_{i=1,v\in V}^{n-2} d(v) = 2n(E) = n^2-5n+6-2n(E')$.

4. Weiner Index of divisor graph of Z_n

Theorem 4.1 : *If p is prime number and* $n = p^k, k > 0$ *then Weiner index of* $\mathbb{D}[Z_n]$ *is* $\frac{(n-3)(n-2)}{2}$.

Proof: Let $\mathbb{D}[Z_n]$ be a divisor graph of Z_n. Note that the vertex set is V = $Z_n - \{0,1\}$, then $|V| = n-2$. As we seen in [9], $\mathbb{D}[Z_n]$ complete graph of order p^k-2 when $n = p^k$, for positive integer *k*. i.e. vertex $x_i \in V$ is adjacent to each and every vertex x_j. Hence we get $d(x_i, x_j) = 1$

$$\begin{aligned}\sum_{x_i,x_j\in V} d(x_i,x_j) &= \sum_{j=2} d(x_1,x_j)+\sum_{j=3} d(x_2,x_j)+\sum_{j=4} d(x_3,x_j)+....\\ &+ \sum_{j=(n-3)} d(x_{n-3},x_j)\\ &= 1+1+....+1(n-3)times+(1+1+....+1)(n-4)times+....\\ &+ (1+1)+1\\ &= (n-3)+(n-4)+....+3+2+1\\ &= \frac{(n-3)(n-2)}{2}\end{aligned}$$

Hence Weiner index of the graph $\mathbb{D}[Z_n]$ is $\frac{(n-3)(n-2)}{2}$.

Theorem 4.2 : *If $n=pq$, where p and q are distinct primes then Weiner index of $\mathbb{D}[Z_n]$ is $\frac{n^2-5n+6+2\phi(n)}{2}$.*

Proof: We have partition of vertex set as $V=A\cup B$, where $A=U(R)-\{1\}$ and $B=Z(R)-\{0\}$. Let $n=pq$, then subset B can be rewrite as $B=W_p\cup W_q$, where

$$W_p=\{k_1p\,|\,k_1\neq 0\in R, k_1\notin\langle q\rangle\}, W_q=\{k_2q\,|\,k_2\neq 0\in R, k_2\notin\langle p\rangle\}.$$

Then $|A|=\phi(n)-1, |W_p|=q-1, |W_q|=p-1$. Since for $x,y\in V$, if $x\sim y, d(x,y)=1$ and for $x\nsim y, d(x,y)=2$.

Hence Weiner Index of the graph is

$$\begin{aligned}W(\mathbb{D}[Z_n]) &= \sum_{x\sim y} d(x,y)+\sum_{x\nsim y} d(x,y)\\ &= n(E)+n(E')\\ &= \frac{n^2-5n+6-2\phi(n)}{2}+2\phi(n)\\ &= \frac{n^2-5n+6+2\phi(n)}{2}\end{aligned}$$

Theorem 4.3 : *For $n=p_1p_2p_3\cdots p_m, p_i$ are prime numbers then Weiner index of the graph $\mathbb{D}[Z_n]$ is $\frac{n^2-5n+2n(E')}{2}$, where $n(E')$*

$$\begin{aligned}&=|W_{p_1}|\Big(\sum_{p_1\nmid d_i}|W_{d_i}|\Big)+|W_{p_2}|\Big(\sum_{p_2\nmid d_i, p_1\neq d_i}|W_{d_i}|\Big)+\cdots+|W_{p_m}|\Big(\sum_{p_m\nmid d_i, d_i\neq p_1,p_2,\cdots p_{(m-1)}}|W_{d_i}|\Big)\\ &+|W_{p_1p_2}|\Big(\sum_{p_1p_2\nmid d_i, d_i\neq p_1,p_2}|W_{d_i}|\Big)+\cdots+|W_{p_1p_2\cdots p_{(m-2)}}||W_{p_1p_2\cdots p_{(m-1)}}|\end{aligned}$$

Proof: Let R is ring of residue modulo n and $n=p_1p_2...p_m$. Let $d_1,d_2,\ldots,d_k$ be sequence of divisors of n. We have, $V=A\cup B$, $A=U(R)-\{1\}$,

$B = Z(R) - \{0\} = \cup_i W_{d_i}$, where $d_i \mid n$, $W_{d_i} = \{k(d_i) \mid k \in R,\ k \notin \langle d_j \rangle,\ d_i \neq d_j\}$ and $|W_{d_i}| = (p_{(k+1)} - 1)(p_{(k+2)} - 1) \cdots (p_m - 1)$. If $x \sim y$ then $d(x,y) = 1$ and for $x \nsim y, d(x,y) = 2$ and hence Weiner index of the graph $\mathbb{D}[Z_n]$ is $W(\mathbb{D}[Z_n])$

$$\begin{aligned} W(\mathbb{D}[Z_n]) &= \sum_{x \sim y} d(x,y) + \sum_{x \nsim y} d(x,y) \\ &= n(E) + 2n(E') = \frac{n^2 - 5n + 6}{2} - n(E') + 2n(E') \\ &= \frac{n^2 - 5n + 6 + 2n(E')}{2}, \end{aligned}$$

where

$$\begin{aligned} n(E') = {} & |W_{p_1}| \Big(\sum_{p_1 \nmid d_i} |W_{d_i}| \Big) + |W_{p_2}| \Big(\sum_{p_2 \nmid d_i, p_1 \neq d_i} |W_{d_i}| \Big) \\ & + \cdots + |W_{p_m}| \Big(\sum_{p_m \nmid d_i, d_i \neq p_1, p_2, \cdots p_{(m-1)}} |W_{d_i}| \Big) \\ & + |W_{p_1 p_2}| \Big(\sum_{p_1 p_2 \nmid d_i, d_i \neq p_1, p_2} |W_{d_i}| \Big) + \cdots + |W_{p_1 p_2 \cdots p_{(m-2)}}| \, |W_{p_1 p_2 \cdots p_{(m-1)}}| \end{aligned}$$

Theorem 4.4 : *If* $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 > p_z, z = 2,3,\cdots m$ *then Weiner Index of divisor graph is* $\mathbb{D}[Z_n]$) *is* $\frac{n^2 - 5n + 6 + 2n(E')}{2}$, *where* $n(E')$ *denotes no. of pair of non adjacent edges.*

Proof: Let R be ring of residue modulo n and $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. Then by theorem 3.9,4.2,4.3 Weiner Index is,

$$\begin{aligned} W(\mathbb{D}[Z_n]) &= \sum_{x \sim y} d(x,y) + \sum_{x \nsim y} d(x,y) \\ &= n(E) + 2n(E') \\ &= \frac{n^2 - 5n + 6 + 2n(E')}{2} \end{aligned}$$

where $n(E')$ is shown in equation(1).

5. Adjacency Matrix of Divisor Graph:

In this section we find a standard form of adjacency matrix of $\mathbb{D}[R]$ when $n = p^k$ and $n = pq$. We have $V(\mathbb{D}[R]) = A \cup B$, $A = U(R) - \{1\}$, $B = Z(R) - \{0\}$ and $|B| = n - \phi(n) - 1$, $|A| = \phi(n) - 1$. In particular, we find an adjacency matrix when $n = 2^3$, where $V = \{2,3,4,5,6,7\}$. As we know $\mathbb{D}[R]$ is a complete graph if and only if $n = p^k$. So we get the standard form for adjacency matrix as

$$M(\mathbb{D}[Z_8]) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Here observe that M is a circulant matrix of size $(n-2)\times(n-2)$ with 0 is on the diagonal entries and $1's$ at all non diagonal entries. As the graph is complete then for $n = 3^3, 2^4 ... p^k$ we will obtain a similar form of adjacency matrix. This leads us to derive a general form of adjacency matrix of $\mathbb{D}[Z_{p^k}]$).

Theorem 5.1 : *If $n = p^k, k \in N$ and p is prime number then adjacency matrix of the graph $\mathbb{D}[Z_n]$, has the form*

$$M(\mathbb{D}[Z_n]) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

is matrix of size $(n-2)\times(n-2)$. It is clear that M is circulant matrix with entries 0 and 1.

Proof: Let $R = Z_n$, where $n = p^k$. By theorem 4.2, $\mathbb{D}[R]$ is complete graph if and only if $n = p^k$. Hence $a_{ij} = 1, i \neq j$ and $a_{ij} = 0, i = j$. Thus we get the required adjacency matrix.

Adjacency matrix of Z_{pq}.

Consider $n = 2\times 5$.

Then $V = \{2,3,4,5,6,7,8,9\}$. Here *p, q* are only positive divisors of *n*. With our partition, $V = A \cup B$, then $A = \{3,7,9\}, B = \{2,4,5,6,8\}$. By considering vertices of graph in ascending order we obtain adjacency matrix as follows.

$$M(\mathbb{D}[Z_{10}]) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Instead of doing this, we rearrange vertices as, write units at first then multiples of 2, followed by multiples of 5.

$$M(\mathbb{D}[Z_{10}]) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we get interesting structure of adjacency matrix where A is circulant matrix of size $\phi(n)+q-2$ and O is zero matrix of size $(p-1)\times(p-1)$.

Theorem 5.2 : *For* $n = pq, p < q,$ *where p and q are odd primes then adjacency matrix* $(\mathbb{D}[Z_{pq}])$ *has the form,*

$$M(\mathbb{D}[Z_{pq}]) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Where, $C = B^t$ and A, D are circulant matrices of size $(\phi(n)+q-2) \times (\phi(n)+q-2)$ and $(p-1)\times(p-1)$ respectively.

Proof: Let R is ring of order $n = pq,$ Since $V = A \cup B,$ where $B = W_p \cup W_q$ and for $x_i \in W_p, y_i \in W_q$ we have $x_i \nsim y_i$ and $x_i \sim x_j, y_i \sim y_j,$ for $x_j \in W_p, y_j \in W_q$. Hence we get adjacency matrix by considering units as first, then multiples of p and multiples of q as

$$M(\mathbb{D}[Z_{pq}]) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Where, $C = B^t$ and A, D are circulant matrices of size $(\phi(n)+q-2) \times (\phi(n)+q-2)$ and $(p-1)\times(p-1)$ respectively.

6. Energy of Divisor Graph:

In this section, we calculate energy of divisor graph $\mathbb{D}[Z_n]$ for $n = p^k$ and $n = pq$.

Theorem 6.1 : *If* $n = p^k, k \in N$ *and p is prime number then Energy of* $\mathbb{D}[Z_{p^k}]$ *is* $2(n-3)$.

Proof: Let p be prime number. According to above result $\mathbb{D}[Z_{p^k}] = K_{p^k-2}$. Therefore

$$M(\mathbb{D}[Z_{p^k}]) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

We know, Characteristic polynomial $f(\lambda) = |M - \lambda I| = 0,$

$$|M_{(p^k-2)} - \lambda I_{(p^k-2)}| = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{vmatrix}$$

$$= (\lambda+1)^{(p^k-3)}(\lambda-(p^k-3))$$

Clearly given adjacency matrix M has eigen value -1 with multiplicity $|V|-1$ and $|V|-1$ with multiplicity 1. Hence the energy of the graph $\mathbb{D}[Z_{p^k}]$ is $\sum_i |\lambda_i| = 2|V|-1$. Hence the energy of the graph $\mathbb{D}[Z_{p^k}]$ is $\sum_{i=1}^{n-2} |\lambda_i| = 2|V|-1$

Theorem 6.2 : *If $n = pq$ where p,q be prime numbers then characteristic polynomial of adjacency matrix of $\mathbb{D}[Z_{pq}]$ is*

$$(-\lambda+(pq-p-2))(-\lambda-1)^{(pq-p-2)}(p-2)(1-\lambda-2\frac{(-\lambda+q-2)(pq-p-q)}{p(x)})\times(-\lambda-1)^{(p-2)}$$

where $p(x) = \lambda^2 - (pq-p-3)\lambda-(pq-p-2).$

Proof: Let R be ring of integer modulo n and $n = p_1p_2$, where p_1 and p_2 are primes. Then by theorem 5.2, we get the following block adjacency matrix M and eigen value of M are roots of the equation $|M - \lambda I| = 0.$ We have,

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Then,

$$|M-\lambda I|=\begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots\cdots & 1 \\ 1 & -\lambda & 1 & \cdots & 1 & 1 & 1 & \cdots\cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & -\lambda & 1 & 0 & \cdots\cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & -\lambda & 0 & \cdots\cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & -\lambda & 1 & \cdots 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots 1 & -\lambda \end{vmatrix}$$

Notice that A^{-1} is invertible matrix of size $(pq-p-1)$ and

$$A^{-1}=\begin{bmatrix} \frac{-\lambda+(pq-p-3)}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} & \frac{-1}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} & \cdots & \frac{-1}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} \\ \frac{-1}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} & \frac{-\lambda+(pq-p-3)}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} & \cdots & \frac{-1}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} \\ \vdots & \vdots & & \vdots \\ \frac{-1}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} & \frac{-1}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} & \cdots & \frac{-\lambda+(pq-p-3)}{\lambda^2-(pq-p-3)\lambda-(pq-p-2)} \end{bmatrix}$$

Let $p(x)=\lambda^2-(pq-p-3)\lambda-(pq-p-2)$ Then

$$BA^{-1}C=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} A^{-1} \begin{bmatrix} 1 & 1 & \cdots 1 \\ 1 & 1 & \cdots 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & \cdots 1 \\ 0 & 0 & \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots 0 \end{bmatrix}$$

$$BA^{-1}C=B\begin{bmatrix} \frac{-\lambda+q-2}{p(x)} & \frac{-\lambda+q-2}{p(x)} & \cdots & \frac{-\lambda+q-2}{p(x)} \\ \frac{-\lambda+q-2}{p(x)} & \frac{-\lambda+q-2}{p(x)} & \cdots & \frac{-\lambda+q-2}{p(x)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-\lambda+q-2}{p(x)} & \frac{-\lambda+q-2}{p(x)} & \cdots & \frac{-\lambda+q-2}{p(x)} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} \\ \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} \end{bmatrix}$$

Hence

$$D-CA^{-1}B=\begin{bmatrix}-\lambda & 1 & 1 & \cdots & 1\\ 1 & -\lambda & 1 & \cdots & 1\\ 1 & 1 & -\lambda & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & -\lambda\end{bmatrix}-\begin{bmatrix}\frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)}\\ \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)}\\ \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)}\\ \vdots & \vdots & \vdots\\ \frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & \frac{(-\lambda+q-2)(pq-p-q)}{p(x)}\end{bmatrix}$$

$$|D-CA^{-1}B|=\begin{vmatrix}-\lambda-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & 1-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & 1-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)}\\ 1-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & -\lambda-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & 1-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)}\\ \vdots & \vdots & \ddots & \vdots\\ 1-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & 1-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)} & \cdots & -\lambda-\frac{(-\lambda+q-2)(pq-p-q)}{p(x)}\end{vmatrix}$$

Hence, the characteristic polynomial of M is

$$\begin{aligned}|M-\lambda I| &= |A|\cdot|D-CA^{-1}B|\\ &= (-\lambda+(pq-p-2))(-\lambda-1)^{(pq-p-2)}\\ &\times (p-2)(1-\lambda-2\frac{(-\lambda+q-2)(pq-p-q)}{p(x)})\times(-\lambda-1)^{(p-2)},\end{aligned}$$

where $p(x) = \lambda^2-(pq-p-3)\lambda-(pq-p-2)$

Example 6.3 : If R is ring of integer modulo n and $n=3\times 5,$ then energy of divisor graph $\mathbb{D}[Z_{35}]$ is 20.

Solution: By theorem 6.2, characteristic polynomial of adjacency matrix of $\mathbb{D}[Z_{15}]$ is

$$f(\lambda)=(\lambda+1)^{10}[\lambda^3-10\lambda^2-15\lambda+52]=0.$$

A sum of roots of factor $(\lambda+1)^{10}$ is -10 and of $(\lambda^3-10\lambda^2-15\lambda+52)$ is 10. Hence energy of graph $\mathbb{D}[Z_{35}]$ is 20.

7. Conclusion

In this paper, we continued the study of divisor graph $\mathbb{D}[R]$ of commutative ring Z_n and discuss some properties like total degree, stability number, Weiner index and energy of the graph $\mathbb{D}[Z_n]$. We

derived formula for total degree, stability number,Weiner index of the graph $\mathbb{D}[Z_n]$ for all natural values of n. Moreover, if $n = p^k$ and $n = pq$, where p, q prime number then we find adjacency matrix and energy of the graph $\mathbb{D}[Z_n]$.

References

[1] Anderson. D. F and Livingston P. S: The zero-divisor graph of a commutative ring, *Journal of Algebra*, 217 (1999), 434-447.

[2] Beck. I, Colouring of a commutative ring, *J. Algebra* 116 (1988) 208â€"226.

[3] F. Zhang : The Schur Complement and its application, Springer(2005).

[4] I Gutman : The energy of a graph, Ber. Math-Stat. Sekt. Forschungszent Graz,10(1978)1-22.

[5] I Gutman : The energy of graph: old and new result, Algebraic Combinatorics and Applications, Springer-verlag. Berlin(2001),196-211.

[6] Mohammad Reza Ahmadi and Reza Jahani-Nezhad: Energy and Weiner Index of Zero divisor graphs, *Iranian Journal of Mathematical Chemistry*, 2(1), (2011), 45-51.

[7] P. M. Magi, SR. Maghe Joshe, Anjaly Kishore: Adjacency Matrix and Eigen Value of the Zero Divisor Graph, *J. Math. Comput. Sci.* 10 (2020), No. 4, 1285-1297.

[8] Philip D. Powell: Calculating Determinants of Block Matrices, arXiv:1112.4379v1, (2011).

[9] P. D. Khandare, S. M. Jogdand, R. A. Muneshwar: On the Divisor Graph of Finite Commutative Ring, *J. Math. Comput. Sci.*,12:169(2022).

[10] S.Akbari, A.Mohammadian: On the Zero divisor graph of a commutative rings, *J.Algebra*, 274, 847-855 (2004).

[11] S. Reddy, R.S. Jain, N. Laxmikanth : Eigenvalues and Wiener index of the zero-divisor graph $\Gamma(Z_n)$, arXiv preprint arXiv:1707.05083(2017).

[12] Sarfraz Ahmad, Hafiz Muhammad Afzal Siddiqui, Arfan Ali, Mohammad R., Farahani, Muhammad Imran and Ismail Naci Cangul:

On Wiener index and Wiener polarity index of some polyomino chains, *Journal of Discrete Mathematical Sciences and Cryptography*, 22:7, 1151-1164(2019).

[13] Sandeep S Joshi, Kishor F. Pawar: Energy, Weiner Index and Line Graph of prime Graph of Ring, *International Journal of Math. Combin*. Volume 3(2018), 74-80.

[14] West. D. B: Introduction to Graph Theory, Prentice Hall, 2001.

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