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Convergence results for stochastic split feasibility problem using Perturbed Mann iterative scheme in Hilbert space

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Abstract

The focus of this study is on the stochastic split feasibility problem as an extension to deterministic (classical) split feasibility problem. Stochastic convergence in the quadratic mean of an iteration based on Perturbed Mann-type iterative scheme for solving the stochastic split feasibility problem is proven. The Lipschitzian and Pseudo-contractive mapping of the operators in Hilbert space was adopted. A convergence in quadratic mean to a random fixed-point of a split feasibility problem was achieved.

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1. Introduction

The procedure for the fixed-point of a classical split feasibility problem which is usually considered deterministic in nature could have served as a good alternative for its stochastic equivalence (which is characterized by random influence and is believed to account for the uncertainties with which many real life events unfold), but for the reasons stated in [7, 10, 11,14], which can be summarized thus: (i) stochastic functions are contaminated with noise (errors), hence they cannot be fully specified, (ii) because of (i), the gradient of the stochastic function may not exist. This therefore gave rise to the study of random fixed-points theorem which are generalizations of classical fixed-points theorem applied to studies of stochastic feasibility problem, which is a system influenced by noise (i.e. a stochastic system). With the aforementioned circumstances, a new procedure that will take into consideration the stochastic nature of split feasibility problem is considered very expedient. Therefore, the study seeks to examine the random fixed-point of a split feasibility problem using a Perturbed Mann iterative scheme of a Lipschitzian and Pseudo-contractive mapping in Hilbert space and extend the same to its stochastic domain.

Split feasibility and split common problems as families of convex feasibility problems was first proposed by Censor and Effving[5] as well as Censor and Segal [6]. It is remarkable to state that while Censor and Effving [5] considered Directed operator, Censor and Segal worked on Inverse problems applied to phase retrieval and medical reconstructions of images [4]. Moudafi [9], deepened the work of [6] by considering relaxation at fixed-point equation for K-democontractive operators. Several literature like that of [3, 4, 10, 16-18] has made significant inputs on finding different fixed-point equation and iterative scheme for solving split common and feasibility problems, while their convergence was discussed by [1]. Nweke and Udom [10] investigated the random fixed-point algorithms for the stochastic split common problem of firmly non-expansive operators. Other researchers, like [13-14], have also made contributions on stochastic approximations with some families of optimization problems, while [8] specifically discussed on the viscosity iterative scheme using perturbation and non-expansive mapping for generalized equilibrium problems.

Let $C \subset H_1$ and $Q \subset H_2$, where C and Q are two non-empty closed convex subsets from two real Hilbert spaces H_1 and H_2 respectively with norm $\|.\|$ and corresponding inner product $\langle . \rangle$. Consider $T : \Omega \times H_1 \to H_1$, $N : \Omega \times H_2 \to H_2$ and $A : H_1 \to H_2$ (being a linear bounded operator with an adjoint $A^*: H_2 \to H_1$), then the classical split feasibility problem is to seek for an element

$$z \in C \text{ such that } Az \in Q \tag{1.1}$$

its solution set could be given thus:

$$\Gamma = \{z \in C : Az \in Q\} = C \cap A^{-1}Q \tag{1.2}$$

The stochastic split feasibility problem is to find a random element

$$z(\omega) \in C \text{ such that } Az(\omega) \in Q, \forall\, \omega \in \Omega \tag{1.3}$$

Similarly, its solution set is:

$$\Gamma = \{z(\omega) \in C : Az(\omega) \in Q\} = C \cap A^{-1}Q \tag{1.4}$$

2. Preliminaries and Definitions

Below are highlights of mathematical definitions and assumptions that will be used for the main result of this work.

Notation

- (Ω, Σ, P) - complete probability measure space
- $(H, B(H))$ - a measurable space
- $B(H)$ - a Borel sigma algebra of H
- H - a separable Hilbert space
- P - a probability measure on Σ
- Ω - Sample Space
- $X : \Omega \to H$ - H-valued random variable
- $T : \Omega \times H \to H$ - a random operator for each fixed $h \in H$

Definition 2.1: [10,15,18]. The random point $z(\omega) : \Omega \to H$ is known as a fixed-point of T if $T(\omega, z) = z(\omega)$ given that $(\Omega, \Sigma, \mathrm{P})$ is a complete probability measure space and $T : \Omega \times H \to H$ a random operator. Γ is donated as the set of all random fixed-points.

Definition 2.2: [2-3, 13,15]. The following holds when $(\Omega, \Sigma, \mathrm{P})$ is a complete probability measure space and $T : \Omega \times H \to H$ is denoted as random operators:

(i) Lipschitzian, if for L>0

$$\int_{\Omega} \left\| T(\omega, x) - T(\omega, y) \right\| \, \mathrm{dP}(\omega) \le L \int_{\Omega} \left\| x(\omega) - y(\omega) \right\| \, \mathrm{dP}(\omega), \ \forall \ x(\omega), y(\omega) \in X$$

(ii) Pseudo-contractive, if

$$\int_{\Omega} \langle T(\omega, x) - T(\omega, y), x(\omega) - y(\omega) \rangle \, \mathrm{dP}(\omega) \ge \int_{\Omega} \left\| x(\omega) - y(\omega) \right\|^2 \mathrm{dP}(\omega),$$
$$\forall \ x(\omega), y(\omega) \in X$$

Definition 2.3: [3] Given that $P_C : H \to H$ is a metric projection onto C, with $C \subseteq H$ being a non-empty closed convex subset of a separable Hilbert space H and $(\Omega, \Sigma, \mathrm{P})$ a complete probability measure space where $x(\omega) \in H$, then the operator P_C is Idempotent, hence $Fix\, P_C = C$

Definition 2.4: [10,15]. Suppose that $\{X_n(\omega)\}_{n=1}^{\infty}$ and $z(\omega)$ are respectively random sequence and a random-point. Then, the sequence $\{X_n(\omega)\}_{n=1}^{\infty}$ is said to converge in quadratic mean to point $z(\omega)$ if $E \| X \|$ exist for all n and

$$\lim_{n\to\infty} E \left\| X_n(\omega) - z(\omega) \right\|^2 = \lim_{n\to\infty} \int_{\Omega} \left\| X_n(\omega) - z(\omega) \right\|^2 \mathrm{dP}(\omega) = 0$$

Remarks: Convergence in quadratic mean is a strong mode of convergence; hence a sequence that converges in quadratic also converges in probability. That is

$$\lim_{n\to\infty} \int_{\Omega} \| X_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) \Rightarrow \lim_{n\to\infty} \int_{\Omega} (\| X_n(\omega) - z(\omega) \| < \varepsilon) \mathrm{dP}(\omega) = 1, \forall \varepsilon > 0$$

Definition 2.5: [3]. Suppose that a stochastic linear operator $A : H_1 \to H_2$ is bounded and its adjoint is given by $A^* : H_2 \to H_1$ for a complete probability measure space $(\Omega, \Sigma, \mathrm{P})$, then

$$\int_{\Omega} \langle Ax, y \rangle_{H_2} \mathrm{dp}(\omega) = \int_{\Omega} \langle x, A^* y \rangle_{H_1} \mathrm{dp}(\omega), \ \forall \ x \in H_1 \text{ and } y \in H_2$$

where, $\langle . , . \rangle_{H_1}$ and $\langle . , . \rangle_{H_2}$ are inner products of H_1 and H_2 respectively.

Stochastic approximation based on projection method

Let y_n be an observation about an unknown parameter of a function $f(\cdot)$ at time n. Suppose that Z is the root of $f(\cdot) : f(z) = 0.$ Let x_n be the estimate for Z at time n, then

$$y_{n+1} = f(x_n) + \varepsilon_{n+1} \tag{2.1}$$

where, ε_{n+1} is the observation error. By considering y_{n+1} observation on $f(\cdot)$ at n with the observation error ε_{n+1}, the problem has been reduced to seeking the root z of $f(\cdot)$ based on y_n.

To solve (2.1), Robbins and Monro [11] had proposed a recursive algorithm known as RM algorithm which is given in (2.2) below as:

$$x_{n+1} = x_n - \alpha_n y_{n+1},\ a_n > 0 \tag{2.2}$$

where, α_n is the step size.

For convergence of (2.2), using approximation algorithm with expanding truncations by applying Trajectory Subsequence method for the root set Γ of $f(\cdot) : \Gamma = \{x \in \mathbf{R} : f(x) = 0\}$ with z being fixed in $\mathbf{R}^n$ and x_0 an arbitrary initial of the iterative scheme, then x_n being the estimate at n which serves as the n^{th} approximation to Γ is given in the form:

$$x_{n+1} = (x_n - \alpha_n y_{n+1}) I_{[\|x_n - \alpha_n y_{n+1}\| \le M_{\sigma(n)}]} + x^* I_{[\|x_n - \alpha_n y_{n+1}\| > M_{\sigma(n)}]} \tag{2.3}$$

where, $$\sigma(n) = \sum_{i=1}^{n-1} I_{[\|x_i - \alpha_i y_{i+1}\| > M_{\sigma(i)}]},\ \sigma(1) = 0 \tag{2.4}$$

$\mathbf{I}_{[inequality]}$ is an indicator function meaning that it equals 1 if the inequality indicated as subscript in RHS of (2.3) holds or 0 otherwise, $\sigma(n)$ is the number of truncations up-to time n, $M_{\sigma(n)}$ serves as the truncation bound when the $(n+1)^{th}$ estimate is generated. If $\| x_n - \alpha_n y_{n+1} \| \le M_{\sigma(n)}$ is fulfilled, then (2.3) evolves as the RM algorithm otherwise the estimate at $n+1$ is pulled back to the pre-specified point x^*, then the truncation bound is enlarged from $M_{\sigma(n)}$ to $M_{\sigma(n+1)}$. $M_{\sigma(n)}$ is chosen to be be a sequence of positive numbers increasingly diverging to infinity while x^* is a fixed point in R^n.

The general conditions to establish convergence of RM algorithm using Trajectory sub-sequence method are fully shown in Chen [7].

3. Main Result

Theorem 3.1: *Let* $(\Omega, \Sigma, \mathrm{P})$ *be a complete probability measure space and* $C \subset H_1$ *and* $Q \subset H_2$ *be two non-empty convex subset of real Hilbert spaces* H_1 *and* H_2 *respectively. Suppose the stochastic split feasibility problem given in (1.3) is consistent and* Γ *denotes its nonempty solution set. Let* $P_C : C \to C$ *and* $P_Q : Q \to Q$ *be two Lipschitzian Pseudo-contractive projections onto C and*

Q respectively. Let $A: H_1 \to H_2$ *be a bounded linear random operator with an adjoint* $A^*: H_2 \to H_1$. *Then the solution of the stochastic split feasibility problem (1.3) generated by* $\{x_n(\omega)\}$ *and defined by*

$$x_{n+1}(\omega) = (1-\beta_n)x_n(\omega) + \beta_n[P_C(\omega, x_n) - \lambda A^*(1-P_Q)Ax_n(\omega)]$$
$$-\beta_n\theta_n(x_n(\omega) - x_1(\omega)),\ \forall x_1 \in H_1 \text{ and } \lambda \in \left(0, \tfrac{1}{\beta_n \|A\|^2}\right) \tag{3.1}$$

satisfying (i) $\lim_{n\to\infty}\theta_n = 0$ *(ii)* $\beta_n(1+\theta_n) \le 1$ *(iii)* $\sum \beta_n\theta_n = \infty$ *(iv)* $\lim_{n\to\infty}\frac{\beta_n}{\theta_n} = 0$ *(v)* $\lim_{n\to\infty}\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)\Big/\beta_n\theta_n = 0$, *where* β_n *and* θ_n *are sequences of real numbers in* $(0, 1)$, *converges in quadratic mean to the random fixed-point* $z(\omega) \in \Gamma$.

Proof:

$$\begin{aligned} x_{n+1}(\omega) =\ & (1-\beta_n)x_n(\omega) + \beta_n(P_C(\omega, x_n) - \lambda A^*(1-P_Q)Ax_n(\omega)) \\ & -\beta_n\theta_n(x_n(\omega) - x_1(\omega)) \\ x_{n+1}(\omega) =\ & x_n(\omega) - \beta_n(x_n(\omega) - P_C(\omega, x_n) + \lambda A^*(1-P_Q)Ax_n(\omega) \\ & +\theta_n(x_n(\omega) - x_1(\omega))) \end{aligned}$$

$$\therefore E\|x_{n+1}(\omega) - z(\omega)\|^2 = \int_\Omega \|x_{n+1}(\omega) - z(\omega)\|^2 \,\mathrm{dP}(\omega)$$

$$\begin{aligned} =\ & \int_\Omega \|x_n(\omega) - z(\omega)\|^2 \,\mathrm{dP}(\omega) + \beta_n^2 \int_\Omega \|x_n(\omega) - P_C(\omega, x_n) \\ & +\lambda A^*(1-P_Q)Ax_n(\omega) + \theta_n(x_n(\omega) - x_1(\omega)\|^2 \,\mathrm{dP}(\omega) \\ & -2\beta_n \int_\Omega \langle x_n(\omega) - z(\omega),\ x_n(\omega) - P_C(\omega, x_n) + \lambda A^*(1-P_Q)Ax_n(\omega) \\ & +\theta_n(x_n(\omega) - x_1(\omega)\rangle \mathrm{dP}(\omega) \end{aligned}$$

$$\begin{aligned} \le\ & \int_\Omega \|x_n(\omega) - z(\omega)\|^2 \,\mathrm{dP}(\omega) + 2\beta_n^2 \int_\Omega \|x_n(\omega) - P_C(\omega, x_n) \\ & +\lambda A^*(1-P_Q)Ax_n(\omega)\|^2 \,\mathrm{dP}(\omega) + 2\beta_n^2\theta_n^2 \int_\Omega \|(x_n(\omega) - x_1(\omega)\|^2 \,\mathrm{dP}(\omega) \\ & -2\beta_n \int_\Omega \langle x_n(\omega) - z(\omega),\ x_n(\omega) - P_C(\omega, x_n) \\ & +\lambda A^*(1-P_Q)Ax_n(\omega)\rangle \mathrm{dP}(\omega) \\ & -2\beta_n\theta_n \int_\Omega \langle x_n(\omega) - z(\omega),\ (x_n(\omega) - x_1(\omega)\rangle \mathrm{dP}(\omega) \end{aligned}$$

$$\leq \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, dP(\omega) + 2\beta_n^2 \left[\int_\Omega 2 \| x_n(\omega) - P_C(\omega, x_n) \|^2 \, dP(\omega) \right.$$
$$\left. + 2\lambda^2 \| A \|^2 \int_\Omega \| (1 - P_Q) A x_n(\omega) \|^2 \, dP(\omega) \right] + 2\beta_n^2 \theta_n^2 \int_\Omega \| (x_n(\omega)$$
$$- x_1(\omega) \|^2 \, dP(\omega) - 2\beta_n \int_\Omega \langle x_n(\omega) - z(\omega),\, x_n(\omega) - P_C(\omega, x_n) \rangle dP(\omega)$$
$$- 2\lambda\beta_n \int_\Omega \langle A x_n(\omega) - A z(\omega),\, (1 - P_Q) A x_n(\omega) \rangle dP(\omega)$$
$$- 2\beta_n \theta_n \int_\Omega \langle x_n(\omega) - z(\omega),\, x_n(\omega) - x_1(\omega) \rangle dP(\omega)$$

Appealing to pseudo-contractiveness of P_C and P_Q

$$\leq \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) + 4\beta_n^2 \int_\Omega \| x_n(\omega) - P_C(\omega, x_n) \|^2 \, \mathrm{dP}(\omega)$$
$$+ 4\lambda^2 \beta_n^2 \| A \|^2 \int_\Omega \| (1 - P_Q)(\omega, x_n) \|^2 \, \mathrm{dP}(\omega) + 2\beta_n^2 \theta_n^2 \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 \, \mathrm{dP}(\omega)$$
$$- 2\beta_n \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\lambda\beta_n \| A \|^2 \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega)$$
$$- 2\beta_n \theta_n \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega)$$
$$\leq (1 - 2\beta_n - 2\beta_n \theta_n - 2\lambda\beta_n \| A \|^2) \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega)$$
$$+ 4\beta_n^2 \int_\Omega \| x_n(\omega) - P_C(\omega, x_n) \|^2 \, \mathrm{dP}(\omega) + 4\lambda^2 \beta_n^2 \| A \|^2 \int_\Omega \| (1 - P_C)(\omega, x_n) \|^2 \, \mathrm{dP}(\omega)$$
$$+ 2\beta_n^2 \theta_n^2 \int_\Omega \| (x_n(\omega) - x_1(\omega) \|^2 \, \mathrm{dP}(\omega) \tag{3.2}$$

Appealing to approximate fixed points sequence for Lipschitzian Pseudo-contractive mapping in Hilbert space [2]. We have:

$$\left.\begin{aligned} &\lim_{n\to\infty} \int_\Omega \left\| x_n(\omega) - P_C(\omega, x_n) \right\|^2 \mathrm{dP}(\omega) = 0 \\ &\lim_{n\to\infty} \int_\Omega \left\| (1 - P_Q)(\omega, x_n) \right\|^2 \mathrm{dP}(\omega) = 0 \\ &\lim_{n\to\infty} \int_\Omega \left\| x_n(\omega) - x_1(\omega) \right\|^2 \mathrm{dP}(\omega) = 0 \end{aligned}\right\} \tag{3.3}$$

Hence satisfying β_n, λ and θ_n, we have

$$\lim_{n\to\infty}\int_{\Omega}\| x_{n+1}(\omega)-z(\omega)\|^2 \,\mathrm{dP}(\omega) =$$
$$\lim_{n\to\infty}(1-2\beta_n-2\beta_n\theta_n-2\lambda\beta_n \| A\|^2)\int_{\Omega}\| x_n(\omega)-z(\omega)\|^2 \,\mathrm{dP}(\omega)=0$$

Hence, $\{x_n(\omega)\}$ converges in quadratic mean.

This completes the proof.

Theorem 3.2: *Let* $(\Omega,\Sigma,\mathrm{P})$ *be a complete probability measure space and* $C\subset H_1$ *and* $Q\subset H_2$ *be two non-empty convex sets of real Hilbert spaces* H_1 *and* H_2 *respectively. Suppose the stochastic split feasibility problem given in (1.3) is consistent and* Γ *denotes its non-empty solution set. Let* $P_C : C\to C$ *and* $P_Q : Q\to Q$ *be two Lipschitzian Pseudo-contractive projections onto C and Q respectively. Let* $A: H_1\to H_2$ *be a bounded linear random operator with an adjoint* $A^*: H_2\to H_1$, *then the solution of the stochastic split feasibility problem (1.4) generated by* $\{x_n(\omega)\}$ *and defined by*

$$x_{n+1}(\omega)=(1-\beta_n)x_n(\omega)+\beta_n[x_n(\omega)-\lambda(x_n(\omega)-P_Cx_n(\omega)+A^*(1-P_Q)Ax_n(\omega))]$$
$$-\beta_n\theta_n(x_n(\omega)-x_1(\omega)),\forall x_1\in H_1 \text{ and } \lambda\in\left(0,\, 1\big/\beta_n\| A\|^2\right) \qquad (3.4)$$

converges in quadratic mean to the random fixed point $z(\omega)\in\Gamma$.

Proof:

$$E\| x_{n+1}(\omega)-z\| =\int_{\Omega}\| x_{n+1}(\omega)-z\| \,\mathrm{dP}(\omega)$$
$$=\int_{\Omega}\| (1-\beta_n)x_n(\omega)+\beta_n[x_n(\omega)-\lambda(x_n(\omega)-P_C(\omega,x_n)+A^*(1-P_Q)Ax_n(\omega))]$$
$$-\beta_n\theta_n(x_n(\omega)-x_1(\omega))-z(\omega)\| \,\mathrm{dP}(\omega)$$
$$=\int_{\Omega}\| x_n(\omega)-z(\omega)-\lambda\beta_n[x_n(\omega)-P_C(\omega,x_n)+A^*(1-P_Q)Ax_n(\omega)]$$
$$-\beta_n\theta_n(x_n(\omega)-x_1(\omega))\| \,\mathrm{dP}(\omega)$$

$$\therefore\quad E\| x_{n+1}(\omega)-z(\omega)\|^2=\int_{\Omega}\| x_{n+1}(\omega)-z(\omega)\|^2 \,\mathrm{dP}(\omega)$$
$$=\int_{\Omega}\| x_n(\omega)-z(\omega)\|^2 \,\mathrm{dP}(\omega)+\beta_n^2\int_{\Omega}\| \lambda[x_n(\omega)-P_C(\omega,x_n)+A^*(1-P_Q)Ax_n(\omega)]$$
$$-\theta_n(x_n(\omega)-x_1(\omega))\|^2 \,\mathrm{dP}(\omega)-2\beta_n\langle x_n(\omega)-z(\omega),\lambda[x_n(\omega)-P_C(\omega,x_n)$$
$$+A^*(1-P_Q)Ax_n(\omega)]-\theta_n(x_n(\omega)-x_1(\omega))\rangle \mathrm{dP}(\omega)$$

$$
\begin{aligned}
&\leq \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\quad + \beta_n^2 \left\{ \int_\Omega 2\lambda^2 \| x_n(\omega) - P_C(\omega, x_n) + A^*(1 - P_Q) A x_n(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\theta_n^2 \right. \\
&\quad \left. \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 \, \mathrm{dP}(\omega) \right\} - 2\lambda\beta_n \langle x_n(\omega) - z(\omega), x_n(\omega) - P_C x_n(\omega) \\
&\quad + A^*(1 - P_Q) A x_n(\omega) \rangle \mathrm{dP}(\omega) - 2\theta_n \beta_n \langle x_n(\omega) - z(\omega),\ x_n(\omega) - x_1(\omega) \rangle \mathrm{dP}(\omega)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\quad + 2\lambda^2 \beta_n^2 \int_\Omega \| x_n(\omega) - P_C(\omega, x_n) + A^*(1 - P_Q) A x_n(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\theta_n^2 \beta_n^2 \\
&\quad \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\lambda\beta_n \langle x_n(\omega) - z(\omega), x_n(\omega) - P_C(\omega, x_n) \\
&\quad + A^*(1 - P_Q) A x_n(\omega) \rangle \mathrm{dP}(\omega) - 2\theta_n \beta_n \langle x_n(\omega) - z(\omega),\ x_n(\omega) - x_1(\omega) \rangle \mathrm{dP}(\omega)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\quad + 2\lambda^2 \beta_n^2 \left[2 \int_\Omega \| x_n(\omega) - P_C x_n(\omega) \|^2 \, \mathrm{dP}(\omega) + 2 \| A \|^2 \int_\Omega \| (1 - P_Q) A x_n(\omega) \|^2 \, \mathrm{dP}(\omega) \right] \\
&\quad - 2\theta_n^2 \beta_n^2 \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\lambda\beta_n \langle x_n(\omega) - z(\omega), x_n(\omega) - P_C(\omega, x_n) \rangle \\
&\quad \mathrm{dP}(\omega) - 2\lambda\beta_n \langle A x_n(\omega) - A z(\omega), (1 - P_Q) A x_n(\omega) \rangle \mathrm{dP}(\omega) \\
&\quad - 2\theta_n \beta_n \langle x_n(\omega) - z(\omega),\ x_n(\omega) - x_1(\omega) \rangle \mathrm{dP}(\omega)
\end{aligned}
$$

Appealing to Lipschitzian Pseudo-contractiveness of P_C and P_Q

$$
\begin{aligned}
&\leq \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\lambda\beta_n \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\quad - 2\lambda\beta_n \| A \|^2 \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\theta_n \beta_n \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, dp(\omega) \\
&\quad + 4\lambda^2 \beta_n^2 \int_\Omega \| x_n(\omega) - P_C x_n(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\quad + 4\lambda^2 \beta_n^2 \| A \|^2 \int_\Omega \| (1 - P_Q) A x_n(\omega) \|^2 \, \mathrm{dP}(\omega) - 2\theta_n^2 \beta_n^2 \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\leq (1 - 2\lambda\beta_n - 2\theta_n \beta_n - 2\lambda\beta_n \| A \|^2) \int_\Omega \| x_n(\omega) - z(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\quad + 4\lambda^2 \beta_n^2 \int_\Omega \| x_n(\omega) - P_C x_n(\omega) \|^2 \, \mathrm{dP}(\omega) + 4\lambda^2 \beta_n^2 \| A \|^2 \int_\Omega \| (1 - P_Q) A x_n(\omega) \|^2 \, \mathrm{dP}(\omega) \\
&\quad - 2\theta_n^2 \beta_n^2 \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 \, \mathrm{dP}(\omega)
\end{aligned}
$$

Appealing to approximate fixed points sequence for Lipschitzian Pseudo-contractive mapping in Hilbert space [2]. We have:

$$\left.\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega}\left\|x_n(\omega)-P_C(\omega,x_n)\right\|^2 \mathrm{dP}(\omega)=0\\
&\lim_{n\to\infty}\int_{\Omega}\left\|(1-P_Q)(\omega,x_n)\right\|^2 \mathrm{dP}(\omega)=0\\
&\lim_{n\to\infty}\int_{\Omega}\left\|(x_n(\omega)-x_1(\omega)\right\|^2 \mathrm{dP}(\omega)=0
\end{aligned}\right\} \tag{3.5}$$

Hence satisfying β_n, λ and θ_n, we have

$$\lim_{n\to\infty}\int_{\Omega}\| x_{n+1}(\omega)-z(\omega)\|^2 \mathrm{dP}(\omega)=\lim_{n\to\infty}(1-2\lambda\beta_n-2\theta_n\beta_n-2\lambda\beta_n\| A\|^2)$$
$$\int_{\Omega}\| x_n(\omega)-z(\omega)\|^2 \mathrm{dP}(\omega)=0$$

Hence, $\{x_n(\omega)\}$ converges in quadratic mean.
This completes the proof.

Theorem 3.3: *Let* $(\Omega,\Sigma,\mathrm{P})$ *be a complete probability measure space and* $C\subset H_1$ *and* $Q\subset H_2$ *be two nonempty convex sets of real Hilbert spaces* H_1 *and* H_2 *respectively. Suppose the split feasibility problem given in (1.3) is consistent and* Γ *denotes its nonempty solution set. Let* $P_C: C\to C$ *and* $P_Q: Q\to Q$ *be two Lipschitzian Pseudo-contractive projections onto C and Q respectively. Let* $A: H_1\to H_2$ *be a bounded linear random operator with an adjoint* $A^*: H_2\to H_1$. *Then the solution of the stochastic split feasibility problem (1.3) generated by* $\{x_n(\omega)\}$ *and defined by*

$$x_{n+1}(\omega)=(1-\beta_n)x_n(\omega)+\beta_n P_C\left(x_n(\omega)+\frac{\lambda_k}{\| A\|^2}A^*(P_Q(Ax_n(\omega))-Ax_n(\omega))\right)$$
$$-\beta_n\theta_n(x_n(\omega)-x_1(\omega)),\forall x_1\in H_1 \text{ and } \lambda_k\in[0,2] \tag{3.6}$$

converges in quadratic mean to the random fixed point $z(\omega)\in\Gamma$.

Proof: Sequence (10) is derived from Projected Landweber method of iteration [3].

$$E\| x_{n+1}(\omega)-z\|^2=\int_{\Omega}\| x_{n+1}(\omega)-z\|^2\, dP(\omega)$$

$$=\int_{\Omega}\left\|(1-\beta_n)x_n(\omega)+\beta_n P_C\left(x_n(\omega)+\frac{\lambda_k}{\|A\|^2}A^*(P_C(Ax_n(\omega))-Ax_n(\omega))\right)-\beta_n\theta_n(x_n(\omega)-x_1(\omega))-z\right\|^2 dP(\omega)$$

$$= \int_\Omega \left\| x_n(\omega) - z - \beta_n \left(x_n(\omega) - P_C x_n(\omega) - \frac{\lambda_k}{\|A\|^2} A^*(P_C(Ax_n(\omega)) - Ax_n(\omega)) + \theta_n(x_n(\omega) - x_1(\omega)) \right) \right\|^2 dP(\omega)$$

$$= \int_\Omega \|x_n(\omega) - z\|^2 dP(\omega)$$

$$+\beta_n^2 \int_\Omega \left\| x_n(\omega) - P_C(\omega, x_n) - \frac{\lambda_k}{\|A\|^2} A^*(P_C(Ax_n(\omega)) - Ax_n(\omega)) + \theta_n(x_n(\omega) - x_1(\omega)) \right\|^2 dP(\omega)$$

$$-2\beta_n \int_\Omega \left\langle x_n(\omega) - z, x_n(\omega) - P_C x_n(\omega) - \frac{\lambda_k}{\|A\|^2} A^*(P_C(Ax_n(\omega)) - Ax_n(\omega)) + \theta_n(x_n(\omega) - x_1(\omega)) \right\rangle dP(\omega)$$

$$\le \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega) + 2\beta_n^2 \int_\Omega \|x_n(\omega) - P_C(\omega, x_n)\|^2 dp(\omega) - 4\beta_n^2\lambda_k^2 \int_\Omega \|P_C(P_Q(\omega, x_n) - x_n(\omega))\|^2 \mathrm{dP}(\omega)$$

$$-4\beta_n^2\lambda_k^2\theta_n^2 \int_\Omega \|x_n(\omega) - x_1(\omega)\|^2 \mathrm{dP}(\omega) - 2\beta_n \int_\Omega \langle x_n(\omega) - z, x_n(\omega) - P_C(\omega, x_n)\rangle \mathrm{dP}(\omega)$$

$$-2\beta_n\lambda_k \int_\Omega \langle x_n(\omega) - z, P_C(P_Q(\omega, x_n) - x_n(\omega))\rangle dp(\omega) - 2\beta_n\lambda_k\theta_n \int_\Omega \langle x_n(\omega) - z, x_n(\omega) - x_1(\omega)\rangle \mathrm{dP}(\omega)$$

Appealing to Lipschitzian Pseudo-contractiveness of P_C and P_Q

$$\le \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega) + 2\beta_n^2 \int_\Omega \|x_n(\omega) - P_C(\omega, x_n)\|^2 \mathrm{dP}(\omega)$$

$$-4\beta_n^2\lambda_k^2 \int_\Omega \|P_C(P_Q(\omega, x_n) - x_n(\omega))\|^2 \mathrm{dP}(\omega)$$

$$-4\beta_n^2\lambda_k^2\theta_n^2 \int_\Omega \|x_n(\omega) - x_1(\omega)\|^2 \mathrm{dP}(\omega) - 2\beta_n \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega)$$

$$-2\beta_n\lambda_k \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega) - 2\beta_n\lambda_k\theta_n \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega)$$

$$\le (1 - 2\beta_n - 2\beta_n\lambda_k - 2\beta_n\lambda_k\theta_n) \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega)$$

$$-2\beta_n^2 \int_\Omega \|P_C(\omega, x_n) - x_n(\omega)\|^2 \mathrm{dP}(\omega) - 4\beta_n^2\lambda_k^2 \int_\Omega \|P_C(P_Q(\omega, x_n) - x_n(\omega))\|^2 \mathrm{dP}(\omega)$$

$$-4\beta_n^2\lambda_k^2\theta_n^2 \int_\Omega \|x_n(\omega) - x_1(\omega)\|^2 \mathrm{dP}(\omega) \tag{3.7}$$

$$\le \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega) \tag{3.8}$$

Hence $\{x_n\}$ is stochastically bounded.

From (3.7)

$$(2\beta_n + 2\beta_n\lambda_k + 2\beta_n\lambda_k\theta_n) \int_\Omega \|x_n(\omega) - z\|^2 \mathrm{dP}(\omega)$$

$$+2\beta_n^2 \int_\Omega \|P_C(\omega, x_n) - x_n(\omega)\|^2 \mathrm{dP}(\omega) + 4\beta_n^2\lambda_k^2$$

$$\int_\Omega \| P_C(P_Q(\omega, x_n) - x_n(\omega)) \|^2 dP(\omega) + 4\beta_n^2 \lambda_k^2 \theta_n^2 \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 dP(\omega) \quad (3.9)$$

$$\leq \int_\Omega \| x_n(\omega) - z \|^2 dP(\omega) - \int_\Omega \| x_{n+1}(\omega) - z \|^2 dP(\omega)$$

$$= \left[\int_\Omega \| x_n(\omega) - z \| dP(\omega) - \int_\Omega \| x_{n+1}(\omega) - z \| dP(\omega) \right]$$

$$\left[\int_\Omega \| x_n(\omega) - z \| dP(\omega) + \int_\Omega \| x_{n+1}(\omega) - z \| dP(\omega) \right]$$

Appealing to monotone of $\{x_n\}$, from (3.9)

$$\lim_{n\to\infty} \int_\Omega \| P_C(\omega, x_n) - x_n(\omega) \|^2 dP(\omega) = \lim_{n\to\infty} \int_\Omega \| P_C(P_Q(\omega, x_n) - x_n(\omega)) \|^2 dP(\omega)$$

$$= \lim_{n\to\infty} \int_\Omega \| x_n(\omega) - x_1(\omega) \|^2 dP(\omega) = 0$$

$$\therefore \lim_{n\to\infty} \int_\Omega \| x_n(\omega) - z \| dP(\omega) = \lim_{n\to\infty} \int_\Omega \| x_{n+1}(\omega) - z(\omega) \| dP(\omega) = 0$$

This completes the proof.

4. Conclusion

This work has extended the convergence results for convex split feasibility problem to stochastic convex split feasibility problem using Perturbed Mann iteration scheme. It has established that the solution of convex split feasibility problem generated by these random iterative schemes converges in mean square (quadratic mean) to a random fixed-point using Lipschitzian and Pseudo-contractive mapping in Hilbert.

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Declaration of Interest

There is no conflict of interest.

References

[1] Beg, I. and Abbas, M.: Convergence of iterative algorithms to common random fixed points of random operators. *Journal of Applied Mathematics and Stochastic Analysis*, 1-16 (2006). DOI: 10.1155/JAMSA/2006/89213.

[2] Berinde, V: Iterative Approximations of fixed points. Springer-Verlag, New York (2007).

[3] Cegielski, A.: Iterative methods for fixed point problems in Hilbert spaces. Springer (2012). DOI: 10.1007/978-3-642-30901-4.

[4] Ceng, L, C., Ansari, Q. H and Yao, J.C.: An extragradient method for solving split feasibility and fixed point problems. *Computational Mathematics and Application*, 64: 633–642 (2012).

[5] Censor, Y., Elfving, T.: A multi-projection algorithm using Bregman projections in a Product space. *Numerical Algorithms*, 8: 221–239(1994).

[6] Censor, Y. and Segal, A.: The split common fixed point problem for directed operators. *Journal of Convex Analysis*, 16:587–600 (2009).

[7] Chen, H. F.: Stochastic approximations and its applications. Kluwer academic publishers New York (2012).

[8] Mahdi T. and Mahdi A.: Viscosity method for generalized equilibrium problems with perturbation and nonexpansive mappings in Hilbert spaces. *Journal of Information and Optimization Sciences*, 40(7):1367-1390 (2019). DOI:10.1080/02522667.2018.1481616.

[9] Moudafi, A.: The split common fixed-point problem for demicontractive mappings, *Inverse Problems*, 26(5): 50-57(2010). DIO: 10.1088/0266-5611/26/5/055007.

[10] Nweke, C. J. and Udom, A. U: Random fixed-point algorithms for the stochastic split common problem of firmly non-expansive operators. *Journal of Interdisciplinary Mathematics* (2022). DOI: 10.1080/09720502.2021.1978684.

[11] Robbins, H., and Monro, S.: A stochastic optimization method. The Annals of Mathematical Statistics 22 (3):400–407(1951). DOI: 10.1214/aoms/1177729586.

[12] Udom, A. U. and Nweke, C. J.: Convergence results for stochastic convex feasibility problem using random Mann and simultaneous projection iterative algorithms in Hilbert space. *Communications in Statistics-Theory and Methods* (2021). DOI: 10.1080/03610926.2021.1990956.

[13] Udom A. U. and Ossai E. O.: Stochastic approximation results for random variational inequality problem with monotone operator in Hilbert space and its application to optimal control. *Journal of Statistics and Management Systems*, 22(8):1539-1554 (2019). DOI:10.1080/09720510.2019.1637160.

[14] Udom, A. U and Ugah, T. E.: Strong convergence result for solution of stochastic convex minimization problem generated by generalized random iterative scheme. *Journal of Information and Optimization Sciences*, 41(5):1207-1221(2019). DOI: 10.1080/02522667.2019.1641269.

[15] Udom, A. U.: Stochastic approximation result for variational inequality problem using random-type iterative schemes. *Communications in Statistics- Theory and Methods.* 47(22): 5435-5444 (2018). DOI:/10.1080/03610926.2017.1395046.

[16] Wittmann, R: Approximation of fixed points of nonexpansive mappings. *Archieves of Mathematics*. (Basel),58:486–491(1992).

[17] Xu, H.-K: Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. *Inverse Problems*, 26:10, 50-68, (2010). DOI:10.1088/0266-5611/26/10/105018.

[18] Zhang, S. S., Wang, X.R and M. Liu, M.: Almost sure T-stability and convergence for iterative algorithms. *Applied Mathematics and Mechanics*, 32(6):805-810 (2011).

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