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Continuous linear knapsack problems revisited

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Abstract

In this paper, the continuous linear knapsack problem is considered. Some preliminary results are formulated and proved, and theorems concerning the optimal solution of the considered problem are stated and proved.

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1. Introduction. Statement of the problem

Consider the continuous linear knapsack problem
(K)

$$\max\left\{ L(\mathrm{x}) = \sum_{j=1}^{n} c_j x_j \right\} \tag{1}$$

subject to

$$\sum_{j=1}^{n} a_j x_j \le a_0 \tag{2}$$

$$x_j \ge 0, \quad j = 1, \ldots, n. \tag{3}$$

An "economic" interpretation of the knapsack problem can be stated as follows.

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There is a set of n items $J = \{1, \dots, n\}$ and a knapsack with capacity (volume, weight restriction) a_0. Each item $j \in J$ has an associated value c_j and a volume (weight) $a_j, j = 1, \dots, n.$ The problem is to find a maximum volume subset of the set of items J whose total capacity (volume, weight) does not exceed the knapsack capacity a_0.

The knapsack problem and related problems are considered in references [1]–[31]. The solution of knapsack problems with arbitrary convex or concave objective functions is discussed in [Bitran and Hax 1], [Luss and Gupta 7], etc. Quadratic knapsack problems and related to them are studied in [Brucker 2], [Pardalos, Ye, and Han 10], [Robinson, Jiang, and Lerme 11], etc. Algorithms for the case of convex quadratic objective function are proposed in [Brucker 2], [Dussault, Ferland, and Lemaire 3], [Helgason, Kennington, and Lall 4], etc. The book of Kellerer, Pferschy and Pisinger [6] is in-depth study of knapsack problems. Algorithms for bound constrained quadratic programming problems are proposed in [Moré and Toraldo 8], [Pardalos and Kovoor 9], etc. A polynomial time algorithm for the resource allocation problem with a convex objective function and nonnegative integer variables is suggested in [Katoh, Ibaraki, and Mine 5]. Well-posedness and primal-dual analysis of convex separable optimization problems of the considered form are discussed in [Stefanov 22], etc. The solution of multidimensional convex separable continuous knapsack problem with bounded variables is studied, for example, in [Stefanov 23], and strictly convex separable optimization with linear equality constraints and bounded variables is considered in [Stefanov 25]. Separable programming problems and methods for solving them are studied in [Stefanov 15, 24, 29]. Characterization of the optimal solution of the convex generalized nonlinear transportation problem is given in [Stefanov 26]. Characterization of the optimal solution of the convex separable continuous knapsack problem and related problems is studied in [Stefanov 27], and the topic of numerical solution of separable stochastic inventory control problems is considered in [Stefanov 28]. A Lagrangian dual method and cutting plane methods for solving some box constrained variational inequalities are studied, for example, in [Stefanov 12, 13]. In [Stefanov 30], numerical solution of systems of nonlinear equations defined by convex functions is discussed. Review of some separable inventory models is presented in [Stefanov 29]. In [Yemelichev and Komlik 31], method for constructing sequence of feasible solutions for solving discrete optimization problems is proposed and studied.

Rest of the paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, theorems concerning the optimal solution of the considered problem are stated and proved. In Section 4, content of the paper is summarized and open problem for future work is formulated.

2. Preliminary results

In this section, some preliminary results are formulated and proved.

Proposition 1 : Let

$$x_1 \le x_2 \le \cdots \le x_n, \tag{4}$$

and

$$y_1 \ge y_2 \ge \cdots \ge y_n. \tag{5}$$

Then for each permutation $k_1, k_2, \ldots, k_n$ of integers $1, 2, \ldots, n,$ the following inequality is satisfied

$$\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} x_i y_{k_i}, \tag{6}$$

that is, the minimum value of a sum of the form $\sum_{i=1}^{n} x_{j_i} y_{k_i}$ is obtained when elements of one of the sequences are numbered in increasing order, and elements of another sequence are numbered in decreasing order.

Proof: From (4) we have $x_s - x_{s-1} \ge 0, s = 2, 3, \ldots, n.$ Then

$$[(y_s + y_{s+1} + \cdots + y_n) - (y_{k_s} + y_{k_{s+1}} + \cdots + y_{k_n})] \le 0, \quad s = 2, 3, \ldots, n \tag{7}$$

according to (5). Calculate

$$\begin{aligned} \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i y_{k_i} \equiv\ & x_1[(y_1 + \cdots + y_n) - (y_{k_1} + \cdots + y_{k_n})] + \\ & +(x_2 - x_1)[(y_2 + \cdots + y_n) - (y_{k_2} + \cdots + y_{k_n})] + \\ & +(x_3 - x_2)[(y_3 + \cdots + y_n) - (y_{k_3} + \cdots + y_{k_n})] + \\ & + \cdots\cdots\cdots\cdots + \\ & + (x_n - x_{n-1})[y_n - y_{k_n}] \le 0. \end{aligned} \tag{8}$$

In (8), we have used that

- $[(y_1 + \cdots + y_n) - (y_{k_1} + \cdots + y_{k_n})] = 0$ because both sums have the same terms but they are taken in different order;

- $x_2 - x_1 \geq 0$ according to (4) and $[(y_2 + \cdots + y_n) - (y_{k_2} + \cdots + y_{k_n})] \leq 0$ according to (7) with $s = 2$;
- $x_3 - x_2 \geq 0$ according to (4) and $[(y_3 + \cdots + y_n) - (y_{k_3} + \cdots + y_{k_n})] \leq 0$ according to (7) with $s = 3$;
- and so on,
- $x_n - x_{n-1} \geq 0$ according to (4) and $y_n - y_{k_n} \leq 0$ according to (5) (or according to (7) with $s = n$).

Therefore (8) implies

$$\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i y_{k_i},$$

that is, (6) holds true. □

Proposition 2 : Let $f_1(x), \ldots, f_n(x)$ be functions such that

$$x_1 \leq x_2 \quad \text{implies} \quad f_1(x_1) \geq f_1(x_2), \ldots, f_n(x_1) \geq f_n(x_2). \tag{9}$$

Then function $f(x) \stackrel{\text{def}}{=} \sum_{i=1}^{n} d_i f_i(x)$ with $d_i \geq 0, i = 1, \ldots, n,$ satisfies $f(x_1) \geq f(x_2)$.

Proof: From definition of $f(x)$, from (9) and from $d_i \geq 0, i = 1, \ldots, n$ it follows that

$$f(x_1) \equiv \sum_{i=1}^{n} d_i f_i(x_1) \stackrel{(9)}{\geq} \sum_{i=1}^{n} d_i f_i(x_2) \equiv f(x_2).$$

□

Recall that if $g(x)$ is a monotone nondecreasing function, that is,

$$x_1 \leq x_2 \quad \text{implies} \quad g(x_1) \leq g(x_2), \tag{10}$$

then $h(x) \stackrel{\text{def}}{=} \frac{1}{g(x)}$ is a monotone nonincreasing function.

Indeed, let $x_1 \leq x_2$. Then

$$h(x_1) \equiv \frac{1}{g(x_1)} \stackrel{(10)}{\geq} \frac{1}{g(x_2)} \equiv h(x_2).$$

Since x_1, x_2 with $x_1 \leq x_2$ have been arbitrarily chosen, then $h(x)$ is a monotone nonincreasing function by definition.

3. On the optimal solution of the continuous linear knapsack problem

Consider the continuous linear knapsack problem with box constraints.

Let variables $x_1, \ldots, x_n$ be ordered such that

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}, \tag{11}$$

(cf. [31]), and let the following box constraints (bounds on the variables) be satisfied instead of nonnegativity constraints (3):

$$0 \leq x_j \leq u_j, \quad j = 1, \ldots, n. \tag{12}$$

Theorem 1: *Let* (11) *be satisfied. Then the optimal solution of the box constrained continuous knapsack problem* (1), (2), (12) *is* $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ *with*

$$x_j^* = u_j, \quad j = 1, \ldots, s_0, \tag{13}$$

$$x_{s_0+1}^* = \frac{1}{a_{s_0+1}}\left(a_0 - \sum_{j=1}^{s_0} a_j u_j\right), \tag{14}$$

$$x_j^* = 0, \quad j = s_0 + 2, \ldots, n, \tag{15}$$

where

$$s_0 = \max\left\{s : \sum_{j=1}^{s} a_j u_j \leq a_0\right\}. \tag{16}$$

Proof: If $\sum_{j=1}^n a_j u_j \leq a_0$ then obviously $s_0 = n$ and $x_j^* = u_j, j = 1, \ldots, n.$

Otherwise, if $\sum_{j=1}^n a_j u_j > a_0$ then the optimal solution x^* of the considered problem (1), (2), (12) belongs to the set

$$X_0 \stackrel{\text{def}}{=} \left\{\mathrm{x} : \sum_{j=1}^{n} a_j x_j = a_0, \quad 0 \leq x_j \leq u_j, j = 1, \ldots, n\right\}.$$

In this case, we have to prove that each feasible solution $\mathrm{x}' = (x_1', \ldots, x_n') \in X_0$ for the considered problem (1), (2), (12) satisfies the following inequality $L(\mathrm{x}^*) \geq L(\mathrm{x}'),$ where components of x^* are defined by (13)–(15).

Denote $\delta_j = x_j^* - x_j', j = 1, \ldots, n.$ Then obviously

$$L(\mathrm{x}^*) - L(\mathrm{x}') = \sum_{j=1}^{n} c_j x_j^* - \sum_{j=1}^{n} c_j x_j' = \sum_{j=1}^{n} c_j (x_j^* - x_j') = \sum_{j=1}^{n} c_j \delta_j.$$

From definition of $\delta_j, j = 1, \dots, n$ and from (12), (13), (15), (16) it follows that $\delta_j \geq 0$ for $j = 1, \dots, s_0$ and $\delta_j \leq 0$ for $j = s_0 + 2, \dots, n,$ where $0 \leq s_0 \leq n - 1.$

Consider both possible cases for $\delta_{s_0+1}.$

Case 1: Let $\delta_{s_0+1} \geq 0.$ Then

$$L(\mathrm{x}^*) - L(\mathrm{x}') = \sum_{j=1}^{s_0+1} c_j \delta_j - \sum_{j=s_0+2}^{n} c_j \,|\, \delta_j \,|,$$

where δ_j is taken by absolute value in the second sum because $\delta_j \leq 0$ for $j = s_0 + 2, \dots, n.$

Since inequalities (11) are satisfied by assumption, then

$$c_j \geq \frac{c_{s_0+1}}{a_{s_0+1}} a_j, \quad j = 1, \dots, s_0 + 1,$$

$$c_j \leq \frac{c_{s_0+1}}{a_{s_0+1}} a_j, \quad j = s_0 + 2, \dots, n.$$

Therefore

$$L(\mathrm{x}^*) - L(\mathrm{x}') \geq \frac{c_{s_0+1}}{a_{s_0+1}} \left(\sum_{j=1}^{s_0+1} a_j \delta_j - \sum_{j=s_0+2}^{n} a_j \,|\, \delta_j \,| \right) =$$

$$= \frac{c_{s_0+1}}{a_{s_0+1}} \sum_{j=1}^{n} a_j \delta_j = \frac{c_{s_0+1}}{a_{s_0+1}} \left(\sum_{j=1}^{n} a_j x_j^* - \sum_{j=1}^{n} a_j x_j' \right) = 0,$$

where we have used that, as pointed out above, both x^* and x' belong to the set X_0, that is, both x^* and x' satisfy inequality constraint (2) as an equality.

Hence, $L(\mathrm{x}^*) \geq L(\mathrm{x}'),$ that is, x^* with components, defined by (13)–(15), is an optimal solution of the considered problem (1), (2), (12) in Case 1.

Case 2: Let $\delta_{s_0+1} < 0.$ The proof in this case is similar to the proof in Case 1, taking into account that in this case, $1 \leq s_0 \leq n - 1$ instead of $0 \leq s_0 \leq n - 1$ because, if $s_0 = 0,$ then we would have $a_1 u_1 > a_0,$ that is, we would have

$$x_1^* = \frac{a_0}{a_1}, \quad x_j^* = 0, j = 2, \ldots, n.$$

The proof is complete. □

Remark 1: When $x_j \in \mathbb{R}_+, j = 1, \ldots, n,$ that is, $u_j = +\infty, j = 1, \ldots, n,$ then constraint (12) is transformed to constraint (3), and $s_0 = 0$ in this case.

Remark 2: The ordering of variables (11) takes $n \log_2 n$ operations.

Theorem 2 : *Let variables* $x_1, \ldots, x_n$ *be ordered such that* (11) *holds. Then the optimal solution of the continuous knapsack problem* $(1)-(3)$ *is*

$$x_1^* = \frac{a_0}{a_1}, \quad x_j^* = 0, j = 2, \ldots, n. \tag{17}$$

Proof: Statement of Theorem 2 follows from Theorem 1 with $x_j \in \mathbb{R}_+, j = 1, \ldots, n$ because, according to Remark 1, constraint (12) is transformed to constraint (3), $s_0 = 0$ in this case, and (13)–(15) imply (17). □

Remark 3: Note that the result of Theorem 2 is the same as the result obtained in Case 2 of Theorem 1 and the conclusion made in Remark 1.

Remark 4: The result of Theorem 1 can be applied to the continuous linear knapsack problem, in which constraint (12) is replaced by the constraint

$$0 \le x_j \le 1, \quad j = 1, \ldots, n, \tag{12'}$$

that is, constraint (12′) is obtained by constraint (12) with $u_j = 1, j = 1, \ldots, n.$

Then the optimal solution of the knapsack problem (1),(2),(12′) is of the form

$$x_1^* = 1, \ldots, x_{s_0}^* = 1, x_{s_0+1}^* = \frac{1}{a_{s_0+1}}\left(a_0 - \sum_{j=1}^{s_0} a_j\right), x_{s_0+2}^* = 0, \ldots, x_n^* = 0. \tag{18}$$

Consider the box constrained continuous linear knapsack problem (1), (2), (12) with *positive* constraint coefficients $a_j > 0, j = 1, \ldots, n$ in (2). Denote this problem by (K^+).

Reformulate problem (K^+) as the equivalent minimization problem as follows

(K^+)

$$\min\left\{-L(\mathrm{x}) = -\sum_{j=1}^{n} c_j x_j\right\} \tag{1′}$$

subject to

$$\sum_{j=1}^{n} a_j x_j \le a_0, \quad a_j > 0, j = 1, \ldots, n \tag{2′}$$

$$0 \le x_j \le u_j, \quad j = 1, \ldots, n. \tag{12}$$

Since the linear function is convex (and also concave) then (K^+) is a convex optimization problem.

The Lagrangian function for problem (K^+) is

$$L(\mathrm{x}, \lambda, \mathrm{p}, \mathrm{q}) = -\sum_{j=1}^{n} c_j x_j + \lambda\left(\sum_{j=1}^{n} a_j x_j - a_0\right) + \sum_{j=1}^{n} p_j(-x_j) + \sum_{j=1}^{n} q_j(x_j - u_j), \tag{19}$$

where $\lambda \ge 0$, $p_j \ge 0, q_j \ge 0, j = 1, \ldots, n$ are the Lagrange multipliers (dual variables), associated with the constraint $(2′)$ and both box constraints of (12), respectively.

The Karush-Kuhn-Tucker (KKT) optimality conditions for problem (K^+) are

$$-c_j + \lambda a_j - p_j + q_j = 0, \quad j = 1, \ldots, n \tag{20}$$

$$\sum_{j=1}^{n} a_j x_j^* \le a_0 \tag{21}$$

$$\lambda\left(\sum_{j=1}^{n} a_j x_j^* - a_0\right) = 0 \tag{22}$$

$$0 \le x_j^* \le u_j, \quad j = 1, \ldots, n \tag{23}$$

$$p_j x_j^* = 0, \quad j = 1, \ldots, n \tag{24}$$

$$q_j(x_j^* - u_j) = 0, \quad j = 1, \ldots, n \tag{25}$$

$$\lambda \ge 0, p_j \ge 0, q_j \ge 0, j = 1, \ldots, n. \tag{26}$$

Define the index set J as follows $J = \{1, \ldots, n\}$.

Theorem 3 : *A feasible solution* $\mathrm{x}^* = (x_1^*, \dots, x_n^*) \in X$, *where the feasible set* X *is defined by* $(2')$ *and* (12), *is an optimal solution to problem* (K^+), *defined by* $(1'), (2'), (12)$, *if and only if there exists a* $\lambda \geq 0$ *such that*

$$x_j^* = 0, \quad j \in J_0^\lambda = \left\{ j \in J : \lambda \geq \frac{c_j}{a_j} \right\}, \tag{27}$$

$$x_j^* = u_j, \quad j \in J_u^\lambda = \left\{ j \in J : \lambda \leq \frac{c_j}{a_j} \right\}, \tag{28}$$

$$x_j^* = \frac{1}{a_j}\left(a_0 - \sum_{J \setminus J_u^\lambda} a_j u_j \right), \quad j \in J^\lambda = \left\{ j \in J : \lambda = \frac{c_j}{a_j} \right\}. \tag{29}$$

Proof: *Necessity.* Let $\mathrm{x}^* = (x_1^*, \dots, x_n^*)$ be an optimal solution to problem (K^+). Therefore there exist scalars $\lambda, p_j, q_j, j = 1, \dots, n$, such that the KKT optimality conditions are fulfilled.

Consider the possible cases for $\lambda \geq 0$.

(1) Let $\lambda > 0$ strictly. Then the system of KKT conditions (20)–(26) is of the form (20), (23), (24), (25), (26) and

$$\sum_{j=1}^n a_j x_j^* = a_0, \tag{30}$$

that is, inequality constraint $(2')$ is satisfied with equality for the optimal solution x^* in this case.

(a) If $x_j^* = 0$ then $p_j \geq 0$ according to (26) and $q_j = 0$ according to (25). Hence (20) implies $-c_j + \lambda a_j - p_j = 0$, that is, $\lambda a_j = c_j + p_j \geq c_j$. Since $a_j > 0$ by assumption then

$$\lambda \geq \frac{c_j}{a_j}.$$

Denote the set of these indices j by J_0^λ.

(b) If $x_j^* = u_j$ then $p_j = 0$ according to (24) and $q_j \geq 0$ according to (26). Hence (20) implies $-c_j + \lambda a_j + q_j = 0$, that is, $\lambda a_j = c_j - q_j \leq c_j$. Since $a_j > 0$ by assumption then

$$\lambda \leq \frac{c_j}{a_j}.$$

Denote the set of these indices j by J_u^λ.

(c) If $0 < x_j^* < u_j$ then $p_j = 0$ according to (24) and $q_j = 0$ according to (25). Hence (20) implies $-c_j + \lambda a_j = 0$. Since $a_j > 0$ by assumption then

$$\lambda = \frac{c_j}{a_j}.$$

Because $x_j^* = 0$ for $j \in J_0^\lambda$ and $x_j^* = u_j$ for $j \in J_u^\lambda$ in accordance with subcases (a) and (b), then (30) implies

$$x_j^* = \frac{1}{a_j}\left(a_0 - \sum_{J \setminus J_u^\lambda} a_j u_j\right).$$

Denote the set of these indices j by J^λ.

(2) Let $\lambda = 0$. Then the system of KKT conditions (20)–(26) is of the form

$$-c_j - p_j + q_j = 0, \quad j = 1, \ldots, n \tag{20'}$$

and (21), (23), (24), (25), (26).

(a) If $x_j^* = 0$ then $p_j \geq 0$ according to (26) and $q_j = 0$ according to (25). Hence (20′) implies $-c_j - p_j = 0$, that is, $c_j = -p_j \leq 0 \equiv \lambda a_j$. Since $a_j > 0$ by assumption then

$$\lambda \equiv 0 \geq \frac{c_j}{a_j}.$$

Denote the set of these indices j by J_0^λ.

(b) If $x_j^* = u_j$ then $p_j = 0$ according to (24) and $q_j \geq 0$ according to (26). Hence (20′) implies $-c_j + q_j = 0$, that is, $c_j = q_j \geq 0 \equiv \lambda a_j$. Since $a_j > 0$ by assumption then

$$\lambda \equiv 0 \leq \frac{c_j}{a_j}.$$

Denote the set of these indices j by J_u^λ.

(c) If $0 < x_j^* < u_j$ then $p_j = 0$ according to (24) and $q_j = 0$ according to (25). Hence (20′) implies $-c_j = 0$, that is, $c_j = 0 \equiv \lambda a_j$. Since $a_j > 0$ by assumption then

$$\lambda \equiv 0 = \frac{c_j}{a_j}.$$

Because $x_j^* = 0$ for $j \in J_0^\lambda$ and $x_j^* = u_j$ for $j \in J_u^\lambda$ in accordance with subcases (a) and (b), then

$$x_j^* = \frac{1}{a_j}\left(a_0 - \sum_{J \setminus J_u^\lambda} a_j u_j\right).$$

Denote the set of these indices j by J^λ.

Sufficiency. Let $\mathrm{x}^* \in X$ and components of x^* satisfy (27)–(29) with $\lambda \geq 0$.

(1) Let $\lambda > 0$ strictly. Set:

$$\lambda = \frac{c_j}{a_j}, p_j = q_j = 0, \quad j \in J^\lambda,$$

$$p_j = -c_j + \lambda a_j (\geq 0), q_j = 0, \quad j \in J_0^\lambda,$$

$$p_j = 0, q_j = c_j - \lambda a_j (\geq 0), \quad j \in J_u^\lambda.$$

(2) Let $\lambda = 0$. Set:

$$\lambda = \frac{c_j}{a_j} (= 0), p_j = q_j = 0, \quad j \in J^{\lambda=0},$$

$$p_j = -c_j, q_j = 0, \quad j \in J_0^{\lambda=0},$$

$$p_j = 0, q_j = c_j, \quad j \in J_u^{\lambda=0}.$$

It can be verified that KKT optimality conditions (20)–(26) for problem (K^+) are satisfied. Therefore x^* with components, defined by (27)–(29), is an optimal solution to problem (K^+).

Since the linear objective function of problem (K^+) is convex but not strictly convex, in the general case uniqueness of the optimal solution of problem (K^+) cannot be stated. □

Remark 5: Statement of Theorem 1, applied to problem (1),(2′),(12), follows from statement of Theorem 3 with

$$J_u^{\lambda} = \{1, \ldots, s_0\}, \quad J^{\lambda} = \{s_0 + 1\}, \quad J_0^{\lambda} = \{s_0 + 2, \ldots, n\}.$$

4. Conclusions

In this paper, the continuous linear knapsack problem is considered, some preliminary results are formulated and proved, and theorems concerning the optimal solution of the considered problem are stated and proved.

An open problem for future work is to obtain further results for the considered problems.

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